Avoiding large squares in trees and planar graphs

Daniel Gonçalves, Pascal Ochem, and Matthieu Rosenfeld

CNRS, LIRMM, Montpellier

One World Combinatorics on Words Seminar

Avoiding all squares

- Thue number $\pi(G)$: smallest number of colors in a square-free coloring of G.
- $\pi(\mathcal{C}) = \max\{\pi(G) \ : \ G \in \mathcal{C}\}$.
- $\pi(path) = 3$ [Thue 1906]
- $\pi(tree) = 4$ [Grytczuk 2007]
- $\pi(tw_t) \leqslant 4^t$ [Kündgen and Pelsmajer 2008]
- $7 \leqslant \pi(outerplanar) \leqslant 12$ [Barát and Varjú 2007]
- $11 \leqslant \pi(planar \cap tw_3) \leqslant \pi(planar) \leqslant 768$ [O. 2011; Dujmovic, Esperet, Joret, Walczak, and Wood 2020]

Avoiding not all squares

Avoiding only small squares:

- Avoiding only squares of period 1: proper coloring.
- Avoiding only squares of period 1 and 2: star coloring.

Avoiding only large squares:

- Grytczuk proposed to avoid only squares of period $\geqslant k$.
- $\pi_k(G)$: smallest number of colors in a coloring of G avoiding squares of period $\geqslant k$.
- $\pi(G) = \pi_1(G) \geqslant \pi_2(G) \geqslant \pi_3(G) \geqslant \cdots$.
- $\pi_1(path) = \pi_2(path) = 3; \quad \pi_3(path) = 2$
- $\lambda(AA) = \lambda(ABAB) = 3; \qquad \lambda(ABCABC) = 2.$

Our results

Avoiding large squares in trees:

Theorem

- $\pi_1(tree) = 4$
- $\pi_2(tree) = \pi_3(tree) = \pi_4(tree) = 3$
- $\pi_5(tree) = 2$

Avoiding large squares in planar graphs:

Theorem

For every fixed k, $\pi_k(planar \cap tw_3) \geqslant 11$.

This disproves a conjecture of Grytczuk that $\pi_k(planar) = 4$ for some k.

Trees I

W.l.o.g, a tree is *level colored*.

Just to confuse people, we consider the word W with reading direction towards the root.

If W contains ms, then the tree contains msm^R.
So if msm^R contains a large square, then W cannot contain ms.

d-directed word:
if w contains f and $|f| \geqslant d$, then w does not contain f^R.

Trees II

W is the morphic image of any ternary $\left(\frac{7}{4}^+\right)$-free word.

- A 12-uniform morphism to prove $\pi_2(tree) \leqslant 3$:

$$\begin{aligned} 0 &\rightarrow \texttt{011220012201} \\ 1 &\rightarrow \texttt{122001120012} \\ 2 &\rightarrow \texttt{200112201120} \end{aligned}$$

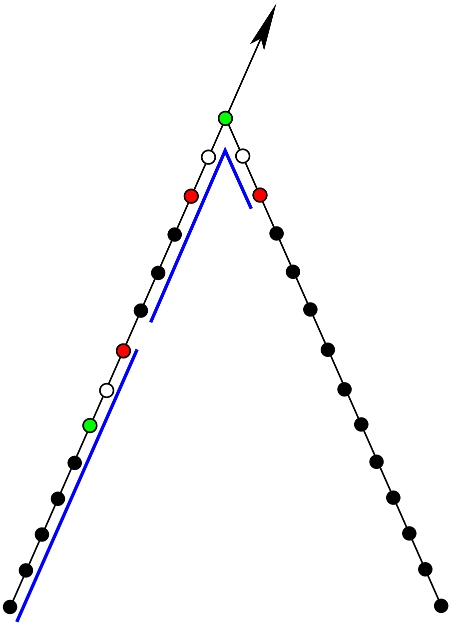
  W is 3-directed and $\left(\frac{19}{10}^+, 2\right)$-free.

- A 21-uniform morphism to prove $\pi_5(tree) \leqslant 2$:

$$\begin{aligned} 0 &\rightarrow \texttt{001101110001010110010} \\ 1 &\rightarrow \texttt{001101110001001110101} \\ 2 &\rightarrow \texttt{001101110001001101010} \end{aligned}$$

  W is 20-directed and $\left(\frac{83}{42}^+, 5\right)$-free.

Trees III

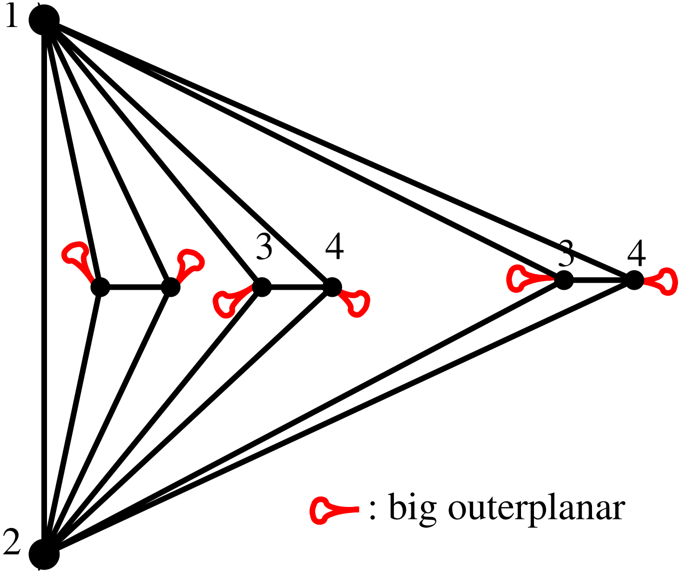


Planar graphs

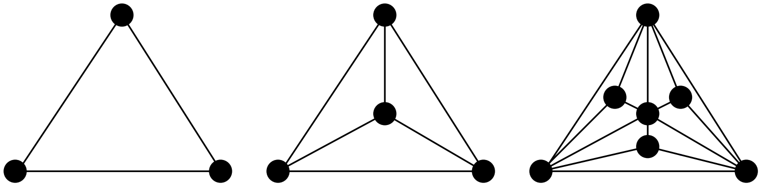
Theorem

For every fixed k, $\pi_k(\mathit{planar} \cap \mathit{tw}_3) \geqslant 11$.

$\pi(\textit{planar} \cap tw_3) \geqslant \pi(\textit{outerplanar}) + 4 \geqslant 11$

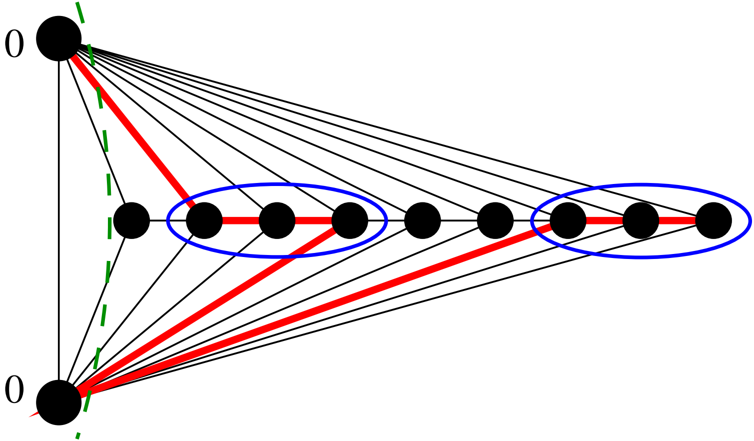


A universal family for *planar* $\cap$ tw_3: G_i



Restriction to proper colorings

WLOG, a coloring of G_i avoiding large squares is proper.



Colored paths

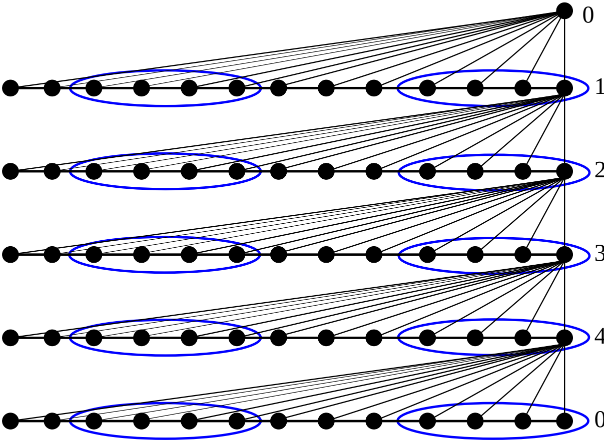
Lemma

Let k be a fixed integer and let P be a path. In every proper coloring of P avoiding squares of period at least k, every subpath of P with $4k$ vertices contains at least 3 colors.

Colored outerplanar graphs

Lemma

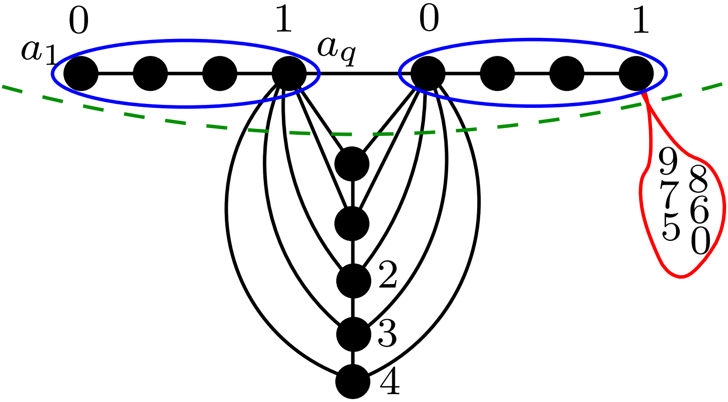
For every fixed k, there exists an outerplanar graph that admits no proper 5-coloring avoiding squares of period at least k.



For contradiction

- The G_i's can be properly colored with $\{0, \ldots, 9\}$ such that every square has period at most q.
- q is minimal.
- $q \geqslant 2$.
- Some G_i contains a square of period q.

Trying to extend the 10-coloring



The contradiction

- Our square of period q extends to a repetition of period q and exponent $\frac{2q+1}{q}$.
- This further extends to a repetition of period q and exponent $\frac{2q+2}{q}$.
- After $2q$ such extensions, we get a repetition of period q and exponent $\frac{4q}{q}$.
- This is a square of period $2q$, contradicting the minimality of q.

Open problems

- $\pi(tw_t) \leqslant 4^t$
- $7 \leqslant \pi(\mathit{outerplanar}) \leqslant 12$
- $11 \leqslant \pi(\mathit{planar}) \leqslant 768$
- For every fixed k, $11 \leqslant \pi_k(\mathit{planar} \cap tw_3) \leqslant 64$

Thank you!