Abelian Repetition Threshold Revisited

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Repetitions: Integral Powers

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  = d-powers are k-avoidable except for the case $d = k = 2$

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- Keränen, 1992:
  - ⋆ A-squares are 4-avoidable
  - an ultimate answer to the question by Erdös (1960)

Repetitions: Fractional Powers and Repetition Threshold

- For a rational number $\alpha > 1$, a word v is an α-power (u^α) if
  - $|v| = \alpha|u|$
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  - ⋆ easy lower bounds
  - ⋆ complicated constructions for $k \geq 4$ (based on Pansiot's encoding)

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  - compatibility with d-A-powers
    - ⋆ for integer $\alpha = d$, α-A-powers are exactly d-A-powers defined earlier
    - ⋆ α-A-power of length αn is a factor of $\lceil\alpha\rceil$-A-power of length $\lceil\alpha\rceil n$

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- What are candidates for the case $\alpha > 2$?
  1. $u_1 \cdots u_d u'$: right tail u' is "less" than some u_i in some "Abelian" sense
  2. $u' u_1 \cdots u_d$: same for the left tail
  3. $u' u_1 \cdots u_d u''$: same for two tails

1,2—no symmetry, 3—no compatibility with d-A-powers

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  - ★ α-A-power is a word $u_1 \cdots u_d u'$ such that $d = \lfloor\alpha\rfloor$, $u_1 \sim \cdots \sim u_d$, u' is A-equivalent to a prefix of u_1 and $\frac{|u_1 \cdots u_d u'|}{|u_1|} = \alpha$
  - dual α-A-power is the reversal of an α-A-power

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- Cassaigne & Currie (1999):
  - for any $\varepsilon > 0$, $\alpha = (1+\varepsilon)$ is k-A-avoidable for $k = 2^{\mathrm{poly}(\varepsilon^{-1})}$
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- Samsonov & S. (2010):
  - definition of ART (and some variations)
  - analytic lower bounds: $\frac{k-2}{k-3}$ is not k-A-avoidable for $k \geq 5$
  - lower bounds by search (easy): $\frac{9}{5}$ is not 4-A-avoidable, $\frac{11}{3}$ is not 2-A-avoidable
  - upper bounds for growth rates of avoiding languages
  - conjecture $\Longrightarrow$

Abelian Repetition Threshold (2)

Conjecture (Samsonov & S., 2010)

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- ⋆ No case of this conjecture is proved
- ⋆ No proof that the threshold is between $ART(k)$ and $ART(k)^+$
  - can it happen that $ART(4) = 2$?
- ⋆ morphisms built by Dekking and Keränen (and their modifications) avoid d-A-powers but do not avoid $(d - \varepsilon)$-A-powers for any $\varepsilon > 0$

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- Why so bad?

Equality vs A-equivalence

- Equality is inherited by prefixes and suffixes, A-equivalence is not
  - To prove a word α-free, it suffices to show that there is no pair of equal factors among $\Theta(n^2)$ pairs
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- For two random k-ary words of length n, the probability of equality is k^{-n}, while the probability of A-equivalence is $\theta(n^{1-k})$
  - if a word is not α-free, it almost surely contains a short α-power
  - if a word is not α-A-free, it contains only long α-A-powers with a non-negligible probability

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! To detect α-A-freeness one may need to study very long words

Prefix Trees

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  - languages $F(k, \alpha)$ of k-ary α-free words and $AF(k, \alpha)$ of k-ary α-A-free words are factorial

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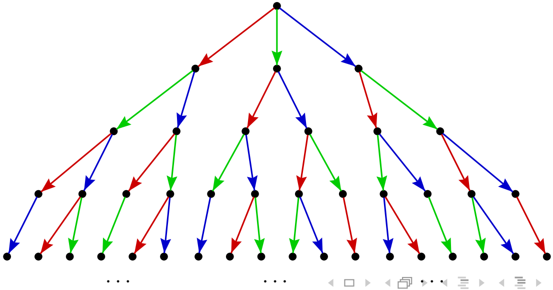
  - edges have the form $u \xrightarrow{a} ua$, where a is a letter
  - ⋆ path from the root to the node u spells u
  - ⋆ u is an ancestor of v in $\mathcal{T}_L \iff u$ is a prefix of v

$\mathcal{T}_{F(3,2)}$:



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- Random walk (a Markov chain) is constructed by depth-first search
  - start from the root
  - visiting a node u, try a new letter a chosen uniformly at random:
    - exclude a from $new(u)$
    - if ua is in L, visit ua next, setting $new(ua) = \{0, \ldots, k-1\}$
    - if not, repeat the choice from the current set $new(u)$
    - if $new(u)$ is empty, return to the parent of u
  - keep the track of
    - the number C of visited nodes
    - the maximum depth M reached in the tree ($= \max |u|$)
  - stop if
    - $M = N$ (random walk of length N built)
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- Do some science:
  - Repeat search multiple times, analyse statistics, formulate conjectures

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Algorithm	Powers	Update time	Query time	Space
Naive	any	$O(1)$	$O(n^3)$	$O(n)$
Greedy	$\alpha < 2$	$O(k)$	$O(kn^{3/2})$ on average	$O(kn)$
Dictionary	$\alpha < 2$	$O(n)$	$O(n)$	$O(n^2)$
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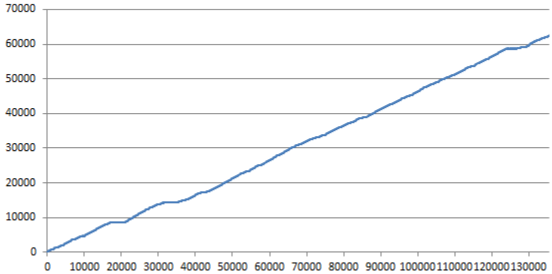
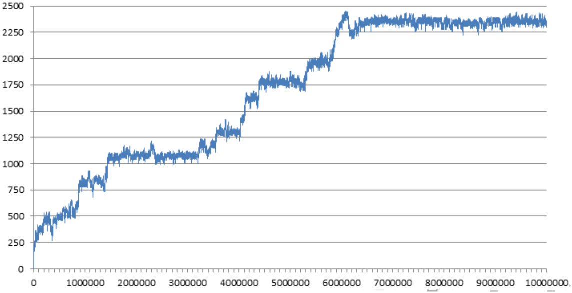
Two graphs: Current Depth vs Number of Visited Nodes

$AF(5, \frac{3}{2}^+)$:
Infinite-like
random walk



$AF(4, \frac{9}{5}^+)$:
Finite-like
random walk,
forced backtrack



Big Alphabets: Main Result

Experimental results for $AF(k, \frac{k-2}{k-3}^+)$, $k = 6, 7, 8, 9, 10$:

Alphabet size	Avoided power	$N = 10^6$, 100 runs			$N = 2{\cdot}10^6$, 100 runs		
		M_{max}	M_{av}	M_{med}	M_{max}	M_{av}	M_{med}
6	$(4/3)^+$	112	98.9	98	114	101.1	101
7	$(5/4)^+$	116	100.3	100	124	103.9	102
8	$(6/5)^+$	103	94.8	95	102	96.2	96
9	$(7/6)^+$	108	95.6	96	107	98.8	99
10	$(8/7)^+$	121	107.7	108	128	111.6	111

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Theorem

For $k = 6, 7, 8, 9, 10$, $ART(k) > \frac{k-2}{k-3}$

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Alphabet size	Avoided power	$N = 10^6$, 100 runs			$N = 2{\cdot}10^6$, 100 runs			Maximum length
		M_{max}	M_{av}	M_{med}	M_{max}	M_{av}	M_{med}	
6	$(4/3)^+$	112	98.9	98	114	101.1	101	116
7	$(5/4)^+$	116	100.3	100	124	103.9	102	125
8	$(6/5)^+$	103	94.8	95	102	96.2	96	105
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For $k = 6, 7, 8, 9, 10$, $ART(k) > \frac{k-2}{k-3}$

Proof of the Main Result

The proof is by exhaustive search enhanced by the lemma below

- *m*-permutation is a length-m word with m distinct letters

Lemma

Let L_1, L_2, and L_3 be the subsets of $L = AF(k, \frac{k-2}{k-3}^+)$ defined as follows:

- L_1 is the set of all $w \in L$ such that w has the prefix $01 \cdots (k-3)$ and contains no $(k-1)$-permutations
- L_2 is the set of all $w \in L$ such that w has the prefix $01 \cdots (k-2)$ and contains no k-permutations
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Then L is finite $\iff L_1, L_2, L_3$ are all finite.

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Then L is finite $\iff L_1, L_2, L_3$ are all finite.

- The total number of searched nodes ranges from < 0.5 billions for $k = 8$ to > 500 billions for $k = 10$
- For $k = 6, 7, 8, 9$, we found the maximum length of a word in L
- For $k = 10$, we know only the maximum length in $L_1 \cup L_2 \cup L_3$

Other Important Experimental Findings

- The languages that should be finite
  - $AF(k, \frac{k-3}{k-4})$ for $k \geq 7$ ($M_{max} = 510$ for $k = 7$)
  - $AF(4, \frac{9}{5}^+)$ ($M_{max} = 3152$)
  - $AF(3, 2^+)$ ($M_{max} = 5449$)
  - $AF(2, \frac{11}{3}^+)$ ($M_{max} = 775$)
- The languages that should be infinite
  - $AF(k, \frac{k-3}{k-4}^+)$ for $k \geq 7$
  - $AF(6, \frac{3}{2})$
  - $AF(5, \frac{3}{2}^+)$
  - $AF(3, \frac{5}{2}^+)$
- Uncertain
  - $AF(4, \frac{11}{6}^+)$

New Conjecture

Conjecture

- $ART(2) > \frac{11}{3}$
- $2 < ART(3) \leq \frac{5}{2}$
- $ART(4) > \frac{9}{5}$
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Details: see the preprint at arXiv:2109.09306

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Future work: a lot of...