Self-induced systems

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Joint work with Nic Ormes, Samuel Petite

A possible motivation for this talk

Question of Christian Mauduit (Ferenczi, 2006) :

What can be said of the following substitution on $A = \mathbb{Z}$?

- Drunken man substitution : $n \mapsto (n-1)(n+1)$

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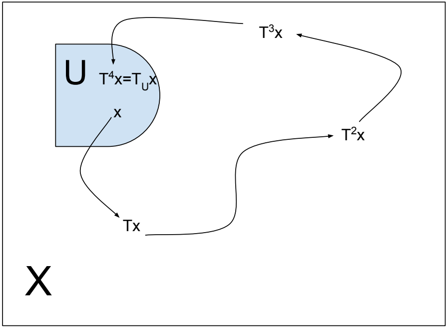
We will come back to these substitutions later

The question of this talk

What are the self-induced systems ?

Induced map

$T : X \to X,\ U \subset X$



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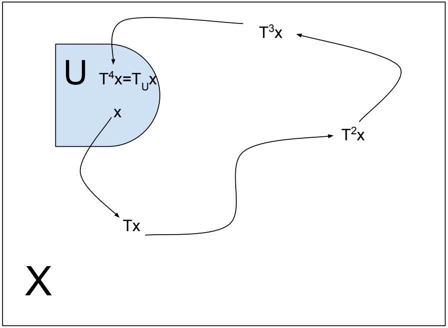

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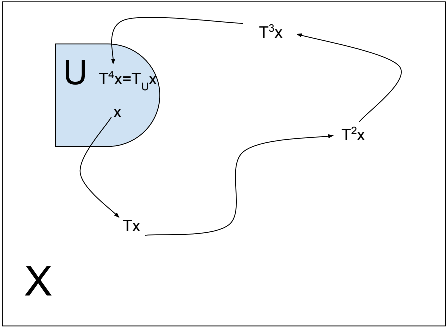

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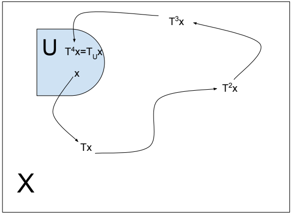
(U, T_U) : induced system

The framework

Dynamical system : (X, T)



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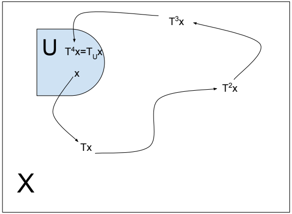

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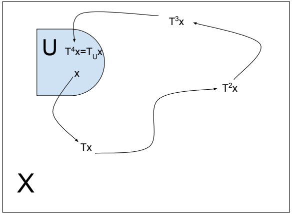

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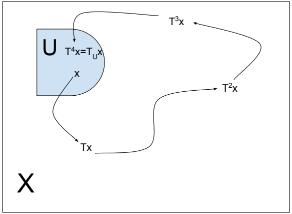

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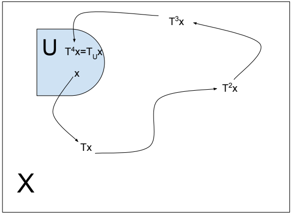

(Moving) definition : (X, T) is **self-induced** if (X, T) if there exists $U \subsetneq X$ (???) such that (X, T) is isomorphic (???) to (U, T_U) (???).

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- Poincaré recurrence theorem : T_U defined μ-almost everywhere (for some fixed T-invariant measure μ with $\mu(U) > 0$)
- (X, T) minimal and U open.
- or any way to have (U, T_U) well defined

Some examples: minimal substitutions subshifts

Theorem. (Mossé 1992) Let (X, S) be a subshift generated by the primitive substitution $\tau : A^* \to A^*$. Then,

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Proof: $U = \tau(X)$ is a clopen set.

A too classical minimal substitutions subshifts

$\tau : 0 \mapsto 01, 1 \mapsto 0$, $\tau(x) = x \in \{0,1\}^{\mathbb{Z}}$, S the shift,

$$X = \overline{\{S^n x | n \in \mathbb{Z}\}}$$

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- (X, S) is isomorphic to $([0], S_{[0]})$ and is self-induced.

An other classical minimal substitutions subshifts

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- (Y, S) is not isomorphic to $([0], S_{[0]})$ but is self-induced.

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- Let (X, S) be a minimal substitution subshift. For all clopen set $U \subset X$ there is a clopen set $V \subset U$ such that (X, S) is topologically conjugate to (V, S_V).
- This property is equivalent to the self-induction property on clopen sets.

Some examples: Rotations on the torus

Theorem. (Ornstein-Rudolph-Weiss 1982) Let $(\mathbb{T}, R_\alpha)$ and $(\mathbb{T}, R_\beta)$ be two **non periodic rotations**. There exists a Lebesgue set U, with $\mathrm{Leb}(U) > 0$, such that $(U, R_{\alpha,U})$ is (measure theoretically) isomorphic to $(\mathbb{T}, R_\beta)$.

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But, Sturmian subshifts are self-induced on clopen sets if, and only if, ... (can you expect?) the "slope" is quadratic.

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Example. If (p_n) is the sequence of prime numbers then $(\mathbb{Z}_{(p_n)}, z \mapsto z+1)$ is not self-induced (on a clopen set)

Some examples: the full shift

Exercise. (Example of G. Vigny) The **full shift** is (topologically) **self-induced** (on a closed set).

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Exercise. (Example of G. Vigny) The **full shift** is (topologically) **self-induced** (on a closed set).

Proof: $U = \{(x_n)_{n\in\mathbb{Z}} | x_{2n}x_{2n+1} \in \{00, 11\}\}$ (closed set)

Observation: Abramov formula

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$$h(U, T_U, \mu_U) = \frac{h(X, T, \mu)}{\mu(U)}$$

Consequence: if (U, T_U, μ_U) and (X, T, μ) are isomorphic then

$$h(X, T, \mu) \in \{0, +\infty\}$$

Some answers: expansive case (subshifts)

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What about non-expansive and non-equicontinuous systems?

Expansiveness and equicontinuity

$T : X \to X$ on a compact metric space (X, d) is

- *expansive* if :

$$\exists\delta > 0,\ \forall x, y \in X, x \neq y, \exists n \in \mathbb{Z}, d(T^n x, T^n y) > \delta.$$

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- *equicontinuous* if :

$$\forall \epsilon > 0,\ \exists \delta > 0,\ (d(x,y) \leq \delta \Longrightarrow d(T^n x, T^n y) < \epsilon\ \forall n).$$

Some answers: all ergodic systems are self-induced

Definition. Let (X, T) be a minimal Cantor system. We say that a nonempty closed set C is a **Poincaré section** for (X, T) if the induced map $T_C : C \to C$ is a well-defined homeomorphism.

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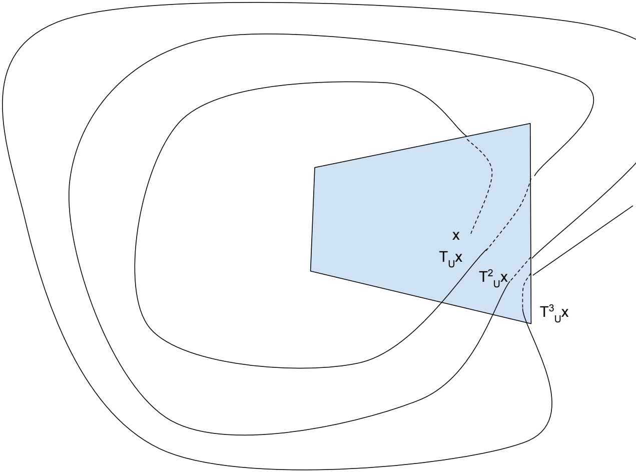

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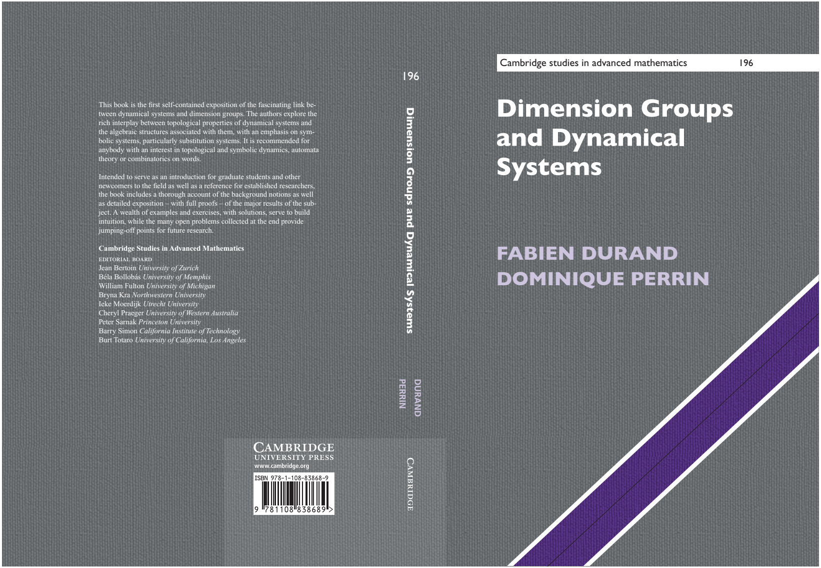
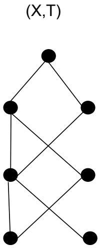
Theorem. (Dahl-Molberg 2009) Let (X, T) and (Y, R) be minimal Cantor systems. Then, there exists a Poincaré section C in (Y, R) such that (C, R_C) is topologically conjugate to (X, T).

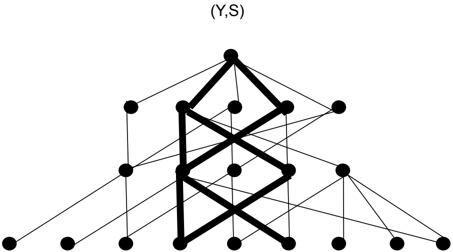
Proof: easy with Bratteli diagrams

Bratteli diagrams



The idea for Dahl-Molberg theorem





Some corollaries

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Some corollaries

Corollary. Minimal Cantor systems are self-induced (on Poincaré section). (Even those with finite positive entropy)

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Proof: Vershik representation of ergodic systems

Main result

Theorem. Let (X, T) be a minimal Cantor system. It is a self-induced (on a clopen set) if, and only if, (X, T) is conjugate to a recognizable, primitive, aperiodic, **generalized substitution** subshift.

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- $\pi_j \circ \sigma$ **is continuous on the set** $\{a \in K : |\sigma(a)| \geq j\}$.

Proof of the main result

Proof: Set $A = X$, define $\sigma : A \to A^+$ by

$$\sigma(x) = \varphi(x) T(\varphi(x)) T^2(\varphi(x)) \cdots T^{r_U(\varphi(x))-1}(\varphi(x)),$$

where $r_U(\varphi(x))$ is the return time of $\varphi(x)$ to $U = \varphi(X)$ where φ is the conjugacy. $\square$

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Question : Does there exist something else than "classical substitutions"?

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- σ is **primitive** if given any non-empty open set $V \subset K$, there is j such that for any letter $a \in K$ and any $k \geq j$, one of the letters of $\sigma^k(a)$ is in the set V.
- σ is **recognizable** if for every $z \in X$, there is a unique set of integers $n_k : k \in \mathbb{Z}$ and a unique $x \in X$ such that $\sigma(x_k) = z[n_k, n_{k+1} - 1]$ for all $k \in \mathbb{Z}$.

Infini-Bonacci substitution

$A = \{1, \dots\} \cup \{\infty\}$

$$n \mapsto 1(n+1), \infty \mapsto 1\infty$$

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- $\mathcal{A}_i = \{0, 1, \ldots, i\}$ and

$$\sigma_i : 0 \mapsto 01,\ i \mapsto i0,\ \text{and } a \mapsto (a+1)0,\ \text{for } a \neq 0, i,$$
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- The system it generates is not expansive nor equicontinuous

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$$\sigma(x) = \begin{cases} S(x)z & \text{if } x_0 = 0, \\ S(x)zz & \text{if } x_0 = 1, \end{cases}$$

or more concisely,

$$\sigma(x) \mapsto S(x)zz^{x_0}.$$

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- (X, T) is self-induced of the clopen set $\varphi(X)$.

THANK YOU FOR YOUR ATTENTION