Random substitutions

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One World Combinatorics on Words Seminar

Random substitutions - quick summary

- Substitutions but flip a coin every time you substitute a letter
- Systematic study began $\sim$ 2016. Still lots of open questions!
- Language built similar way to language of a substitution
- Somewhere between substitutions (long range order) and SFTs (local disorder)
- Many other names: multi/set-valued substitutions, stochastic substitutions, $0L$-systems, ...

Recently explored aspects *(various authors)*

- Complexity/Entropy
- Periodicity
- Measure theoretic entropy
- C-balancedness
- Rauzy fractals
- Automorphism groups
- Topological mixing
- Diffraction
- Cohomology groups

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Happy to discuss any of the other aspects after the talk!

Notation: Symbolic dynamics

- $\mathcal{A} = \{a_1, a_2, \ldots, a_d\}$ — **Finite alphabet** on d letters
- $\mathcal{A}^+ = \bigcup_{n\geq 1} \mathcal{A}^n$, $\mathcal{A}^* = \mathcal{A}^+ \cup \{\varepsilon\}$ — **Words** in $\mathcal{A}$
- $\mathcal{A}^{\mathbb{Z}} = \{\cdots x_{-2}x_{-1} \cdot x_0x_1x_2 \cdots \mid x_i \in \mathcal{A}\}$ — **Full shift** on $\mathcal{A}$
- $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \colon x_i \mapsto x_{i+1}$ — **Left shift** map
- $X \subseteq \mathcal{A}^{\mathbb{Z}}$, closed, σ-invariant — **Subshift**
- **Notation:** $u \triangleleft w$ mean u is a subword/factor of w (either finite or (bi)infinite)

Substitutions

$$\theta\colon \mathcal{A} \to \mathcal{A}^{+}$$ — **substitution**

$$\mathcal{L} = \left\{ u \in \mathcal{A}^{*} \mid u \triangleleft \theta^{k}(a),\ \ a \in \mathcal{A},\ \ k \geq 0 \right\}$$ — **language**

$$X_{\theta} := \left\{ x \in \mathcal{A}^{\mathbb{Z}} \mid u \triangleleft x \implies u \in \mathcal{L} \right\}$$ — **subshift**

$$\theta \colon \mathcal{A} \to \mathcal{A}^+ \quad \text{— substitution}$$

$$\mathcal{L} = \left\{ u \in \mathcal{A}^* \mid u \triangleleft \theta^k(a), \;\; a \in \mathcal{A}, \;\; k \geq 0 \right\} \quad \text{— language}$$

$$X_\theta := \left\{ x \in \mathcal{A}^{\mathbb{Z}} \mid u \triangleleft x \implies u \in \mathcal{L} \right\} \quad \text{— subshift}$$

Basic properties:
- X_θ – Cantor set
- linear complexity
- aperiodic
- minimal
- uniquely ergodic

Holds whenever θ is *primitive* and *recognisable* (definitions to come)

Abelianisation

Define M_θ by $m_{ij} =$ [number of times a_j appears in $\theta(a_i)$]

$$\theta \colon \begin{cases} a & \mapsto & ab \\ b & \mapsto & aa \end{cases}$$

$$M := M_\theta = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \qquad \lambda = 2, -1 \qquad \mathbf{R} = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)^T$$

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θ **primitive** $\iff$ $M^k > 0$ for k large enough

$\lambda_{PF} > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq 0$ (**Perron–Frobenius**)

R — right PF-eigenvector encodes letter-frequencies $\mathbf{R}_i = \mathrm{freq}(a_i)$

Complexity function

$$\mathcal{L}^n := \mathcal{L} \cap \mathcal{A}^n$$

$$p(n) := \#\mathcal{L}^n$$ — **complexity function**

$$h_{\text{top}} = \lim_{n \to \infty} \frac{\log p(n)}{n}$$ — **entropy**

$$\theta \text{ primitive} \implies p(n) \sim Cn \implies h_{\text{top}} = 0$$

Periodicity

A substitution θ is called **recognisable** if for all $x \in X_\theta$, there is a unique $y \in X_\theta$ and a unique $0 \leq i < |\theta(y_0)|$ such that $\sigma^{-i}(x) = \theta(y)$.

"every $x \in X_\theta$ has a unique decomposition into inflation words"

Theorem (Mossé, '92)

Let θ be primitive. Then,

$$\theta \quad \textit{recognisable} \quad \Longleftrightarrow \quad X_\theta \quad \textit{aperiodic.}$$

$\Longrightarrow$ easy

$\Longleftarrow$ hard

Random substitutions

"Letters have choices for how they are substituted"

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(Random period doubling)

$$\vartheta\colon \begin{cases} a & \mapsto & \{ab, ba\} \\ b & \mapsto & \{aa\} \end{cases}$$

Choices are independent for each appearance of a letter:

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Choices are independent for each appearance of a letter:

$$a \mapsto ab \mapsto \overbrace{ba}^{\vartheta(a)} aa \mapsto aa \overbrace{ab}^{\vartheta(a)} \overbrace{ba}^{\vartheta(a)} \overbrace{ba}^{\vartheta(a)} \mapsto baabbaaaaaabaaab \mapsto \cdots$$

$$\mathcal{L} = \left\{ u \in \mathcal{A} \mid u \triangleleft v,\;\; v \in \vartheta^k(a),\;\; k \geq 0 \right\}$$

$$X_\vartheta := \left\{ x \in \mathcal{A}^{\mathbb{Z}} \mid u \triangleleft x \implies u \in \mathcal{L} \right\}.$$

$$\vartheta(a) = \{ab, ba\}$$

$$\vartheta^2(a) = \left\{ \overbrace{abaa, baaa}^{\vartheta(ab)}, \overbrace{aaab, aaba}^{\vartheta(ba)} \right\}$$

$$\vartheta^3(a) = \left\{ \begin{array}{l} \left.\begin{array}{l} abaaabab, abaaabba, abaabaab, baaaabab, \\ abaababa, baaaabba, baaabaab, baaababa, \end{array}\right\} \vartheta(abaa) \\ \left.\begin{array}{l} aaabbabab, aaababba, aaabbaab, aabaabab, \\ aaabbaba, aabaabba, aababaab, aabababa, \end{array}\right\} \vartheta(baaa) \\ \left.\begin{array}{l} abababaa, ababbaaa, abbaabaa, baababaa, \\ abbabaaa, baabbaaa, babaabaa, bababaaa, \end{array}\right\} \vartheta(aaab) \\ \left.\begin{array}{l} ababaaab, ababaaba, abbaaaab, baabaaab, \\ abbaaaba, baabaaba, babaaaab, babaaaba \end{array}\right\} \vartheta(aaba) \end{array} \right\}$$

Notation: $\psi(u) = (|u|_{a_1}, \ldots, |u|_{a_d})$ is the abelianisation of the word u.

Main assumption

- Assume random substitutions are **compatible**:

$$u, v \in \vartheta(a) \implies \psi(u) = \psi(v)$$

- $M = M_\vartheta$ makes sense
- ϑ **primitive** if M is primitive *(can be defined for non-compatible too)*
- Letter-frequencies exist uniformly, encoded by **R** [R, Spindeler '18]
- Word-frequencies do **not** necessarily exist

Random period doubling

$$\vartheta\colon \begin{cases} a & \mapsto & \{ab, ba\} \\ b & \mapsto & \{aa\} \end{cases}$$

ϑ is **compatible**

$$M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$M^2 > 0$ so ϑ is primitive

$$\vartheta\colon \begin{cases} a & \mapsto & \{abbbbc, bbcabb\} \\ b & \mapsto & \{aac, aca, caa\} \\ c & \mapsto & \{b\} \end{cases}$$

ϑ is **compatible**

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$M^3 > 0$ so ϑ is primitive

Dynamics

Theorem (R., Spindeler '18)

Properties of X_ϑ for primitive ϑ:

- *Cantor set or finite*
- *Either no periodic points or periodic points are dense*
- *Uncountably many minimal components*
- *Almost all orbits are dense (in particular topologically transitive)*
- $h_{top} > 0$
- *Canonical measure μ_p induced by production probabilities (shown to be ergodic [Gohlke, Spindeler '20])*

A compatible random substitution ϑ is called **recognisable** if for all $x \in X_\vartheta$, there is a unique $y \in X_\vartheta$ and a unique $0 \leq i \leq |\vartheta(y_0)|$ such that $\sigma^{-i}(x) \in \vartheta(y)$.

Theorem (R., '20)

Let ϑ be primitive. Then,

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Converse fails! (So periodicity is hard to study!)

Random Fibonacci: $\vartheta\colon a \mapsto \{ab, ba\}, b \mapsto \{a\}$

Irrational letter-frequencies $\implies$ No periodic points.

Fixed point of $\theta\colon a \mapsto ab, b \mapsto a$ can also be decomposed into just the words ba, a (because Fibonacci and reflected Fibonacci have same language).

Entropy for random substitutions

Topological entropy

$$h_{\text{top}} = \lim_{n\to\infty} \frac{1}{n} \log p(n) \qquad \geq \qquad \lim_{k\to\infty} \frac{1}{|\vartheta^k(a_i)|} \log \#\vartheta^k(a_i)$$

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Estimates:

$$\begin{aligned} |\vartheta^k(a_i)|_{a_j} &= (M^k \mathbf{e}_i)_j \overset{\text{PF}}{\simeq} (\lambda^k \mathbf{R}\mathbf{L}^T \mathbf{e}_i)_j = \lambda^k (\mathbf{R} L_i)_j = R_j L_i \lambda^k \\ |\vartheta^k(a_i)| &\simeq L_i \lambda^k \end{aligned}$$

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$$\begin{aligned} \#\vartheta^k(a_i) &= \#\vartheta^{k-1}(a_i) \prod_{j=1}^{d} \#\vartheta(a_j)^{|\vartheta^{k-1}(a_i)|_{a_j}} = \cdots \\ &= \prod_{s=1}^{k} \prod_{j=1}^{d} \#\vartheta(a_j)^{|\vartheta^{k-s}(a_i)|_{a_j}} \end{aligned}$$

$$|\vartheta^k(a_i)|_{a_j} \simeq R_j L_i \lambda^k, \qquad |\vartheta^k(a_i)| \simeq L_i \lambda^k, \qquad \#\vartheta^k(a_i) = \prod_{s=1}^{k} \prod_{j=1}^{d} \#\vartheta(a_j)^{|\vartheta^{k-s}(a_i)|_{a_j}}$$

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Putting together

$$\begin{aligned}
\frac{1}{|\vartheta^k(a_i)|} \log \#\vartheta^k(a_i) &\simeq \frac{1}{L_i\lambda^k} \log \prod_{s=1}^{k} \prod_{j=1}^{d} \#\vartheta(a_j)^{R_j L_i \lambda^{k-s}} \\
&= \frac{1}{L_i\lambda^k} L_i \sum_{s=1}^{k} \lambda^{k-s} \sum_{j=1}^{d} R_j \log \#\vartheta(a_j) \\
\text{(Notice now independent of } i\text{)} \quad &= \frac{1-\lambda^{-k}}{\lambda - 1} \sum_{j=1}^{d} R_j \log \#\vartheta(a_j) \\
&\to \frac{1}{\lambda - 1} \sum_{j=1}^{d} R_j \log \#\vartheta(a_j)
\end{aligned}$$

In ‘good’ cases, these estimates work!

Theorem (Gohlke, ’20)

If ϑ is a compatible, primitive random substitution and if

$$u \neq v \in \vartheta(a_i) \implies \vartheta^k(u) \cap \vartheta^k(v) = \varnothing, \tag{$*$}$$

then

$$h_{top} = \frac{1}{\lambda - 1} \sum_{j=1}^{d} R_j \log \#\vartheta(a_j).$$

Why do we need $(*)$?

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Why do we need $(*)$?

Because we implicitly assumed it when we wrote

$$\#\vartheta^k(a_i) = \#\vartheta^{k-1}(a_i) \prod_{j=1}^{d} \#\vartheta(a_j)^{|\vartheta^{k-s}(a_i)|_{a_j}}.$$

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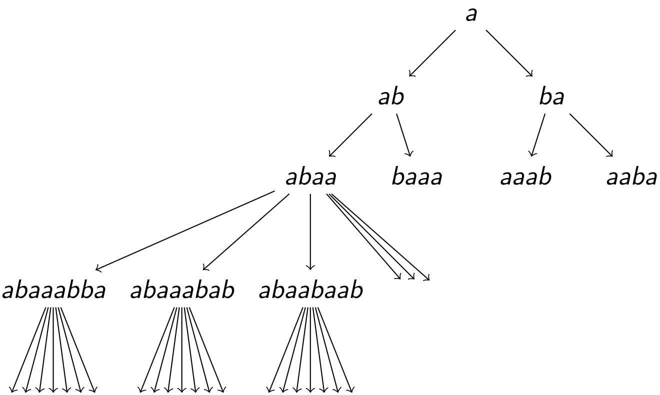
Why do we need $(*)$?

Actually,

$$\#\vartheta^k(a_i) \;\leq\; \#\vartheta^{k-1}(a_i) \prod_{j=1}^{d} \#\vartheta(a_j)^{|\vartheta^{k-s}(a_i)|_{a_j}}.$$

$a \mapsto \{ab, ba\}, b \mapsto \{aa\}$



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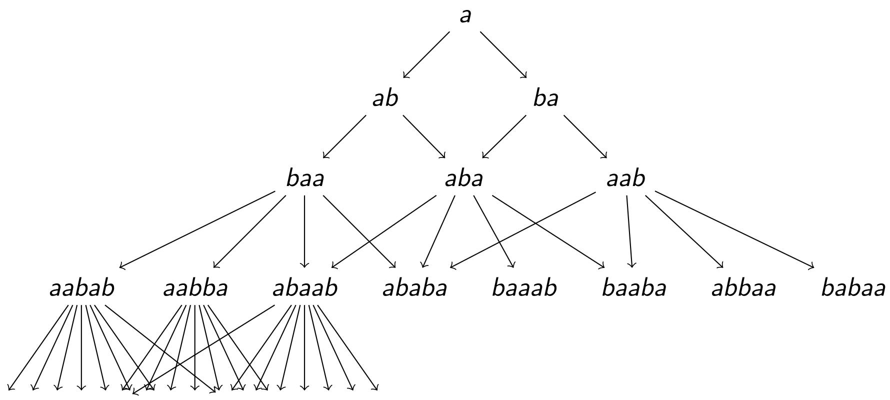

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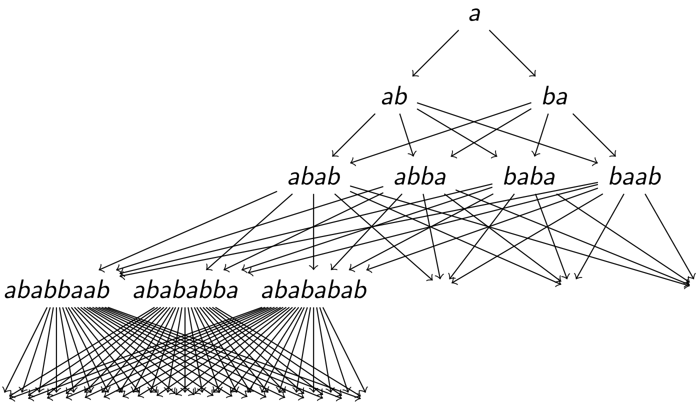

Theorem (Gohlke '20)

Let ϑ be a primitive, compatible random substitution. Then

$$\frac{1}{\lambda}\sum_{j=1}^{d} R_j \log \#\vartheta(a_j) \;\leq\; h_{\mu_p}(X_\vartheta) \;\leq\; \frac{1}{\lambda - 1}\sum_{j=1}^{d} R_j \log \#\vartheta(a_j).$$

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- *Equality for upper bound iff for all $a \in \mathcal{A}$, $q \geq 1$ and $u \neq v \in \vartheta(a)$, we have $\vartheta^k(u) \cap \vartheta^k(v) = \varnothing$.* ***Disjoint Set Condition***
- *Equality for lower bound iff for all $a \in \mathcal{A}$, $k \geq 1$ and $u, v \in \vartheta(a)$, we have $\vartheta^k(u) = \vartheta^k(v)$.* ***Identical Set Condition***

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- *Equality for lower bound iff for all $a \in \mathcal{A}$, $k \geq 1$ and $u, v \in \vartheta(a)$, we have $\vartheta^k(u) = \vartheta^k(v)$.* ***Identical Set Condition***

By taking powers, ϑ^k, the bounds converge — $\frac{1}{\lambda^k} \simeq \frac{1}{\lambda^k - 1}$.
Allows us to numerically approximate entropy.

Theorem (Gohlke, Mitchell, R., Samuel '21)

Same but for measure theoretic entropy (with a condition on production probabilities). + conditions for unique m.m.e. (uses recogn.)

Examples

Upper bound - Random period doubling

$$\vartheta\colon a \mapsto \{ab, ba\},\ b \mapsto \{aa\}$$

Disjoint Set Condition ✓ , **Recognisable** ✗

$h_{\text{top}} \ = \ \frac{2}{3}\log 2 \ \approx \ 0.4621$

Unknown if X_ϑ has a unique m.m.e. — can't use theorem, as ϑ is not recognisable; $(aab)^\infty \in X_\vartheta$.

Examples

Inbetween - Random Fibonacci

$$\vartheta\colon a \mapsto \{ab, ba\},\ b \mapsto \{a\}$$

DSC/ISC ✗ , Recognisable ✗

$$\frac{1}{\lambda^k} L_k \;<\; h_{\mathrm{top}} \;<\; \frac{1}{\lambda^k - 1} L_k, \qquad L_k \;:=\; \frac{1}{\lambda} \# \vartheta^k(a) \;+\; \frac{1}{\lambda^2} \# \vartheta^k(b)$$

[Nilsson, '12] $$h_{\mathrm{top}} \;=\; \sum_{i=1}^{\infty} \frac{\log(i)}{\lambda^{i+2}} \;\approx\; 0.444399$$

We have no idea if it has a unique m.m.e.

Examples

Lower bound - Random Thue–Morse

$$\vartheta\colon a \mapsto \{ab, ba\},\; b \mapsto \{ab, ba\}$$

Identical Set Condition ✓ , **Recognisable** ✗

$h_{\text{top}} \;=\; \frac{1}{2}\log 2 \;\approx\; 0.3466$

X_ϑ is actually a coded shift with finite code $C = \{ab, ba\}$.
So, there is the unique m.m.e., even though can't use theorem as not recognisable

Periodicity

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Already know that recognisable $\implies$ aperiodic.

Also know irrational letter-frequencies $\implies$ aperiodic. Can we do better?

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Theorem (R., '20)

Let ϑ be a primitive random substitution of constant length n satisfying disjoint set condition. If every element of X_ϑ contains a word from the set $\mathcal{L}^n \setminus \vartheta(\mathcal{A})$, then X_ϑ is aperiodic.

$$\vartheta \colon a \mapsto \{aabba, ababa\}, b \mapsto \{aaaaa\}$$

The word $aaaab \notin \mathcal{L}^5$ appears in every element of X_ϑ so aperiodic.

Random period doubling — $\vartheta\colon a \mapsto \{ab, ba\},\ b \mapsto \{aa\}$

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Let ϑ be a random substitution satisfying disjoint set condition. There is a decidable procedure to check if a word $u \in \mathcal{L}$ is a periodic block for X_ϑ.

Hence, we can enumerate periodic points!

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$$aab \mapsto baabaa \mapsto aabaabaabaab \mapsto \cdots \mapsto (aab)^\infty$$

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Hence, we can enumerate periodic points!

p	$\lvert \mathrm{Per}_p \rvert$	$\lvert \mathrm{Orb}_p \rvert$	p	$\lvert \mathrm{Per}_p \rvert$	$\lvert \mathrm{Orb}_p \rvert$
3	3	1	24	176,391	7,334
6	15	2	27	1,533	56
9	21	2	30	216,030	7,179
12	375	30	33	10,992	333
15	108	7	36	19,375,935	538,143
18	2,427	133	39	24,612	631
21	402	19	42	13,106,514	312,050

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Questions:

- For every $3|p$, is there a word with prime period p?
- Looks like if p has a high power of 2 as a factor, then $\lvert \mathrm{Per}_p \rvert$ is large. How to formalise this?
- Have $\limsup_{p\to\infty} \frac{1}{p} \log \lvert \mathrm{Per}_p \rvert = h_{\mathrm{top}}$. How about $\lim_{k\to\infty} \frac{1}{3.2^k} \log \lvert \mathrm{Per}_{3.2^k} \rvert$?

Other open questions

- Is h_{top} the log of an irrational for random Fibonacci? $\sum \frac{\log(i)}{\lambda^{i+2}}$
- Palindromic complexity?
- Can we classify when $\mathrm{Per}(X_\vartheta) = \varnothing$?
- Can we enumerate Per_p without DSC?
- 'Power-free' properties?
- What kinds of subshifts can appear as RS-subshifts? We know that we can get all mixing SFTs [Gohlke, R., Spindeler, '19]
- Essentially nothing has been done in $\mathbb{Z}^2$ or higher.