Twins in permutations

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Joint work with **Andrzej Dudek** and **Andrzej Ruciński**

Similarity of permutations

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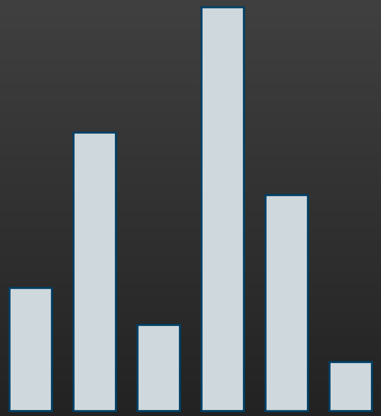

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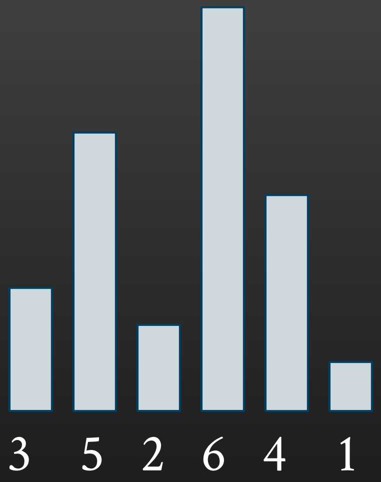

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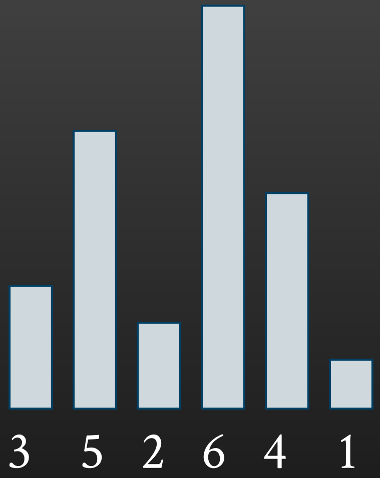

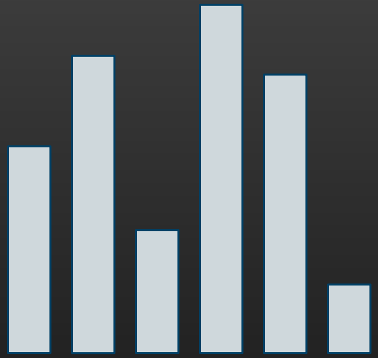

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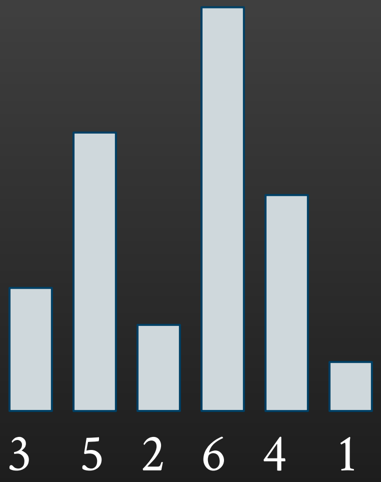

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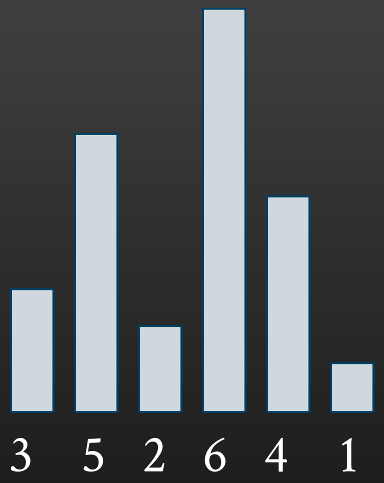
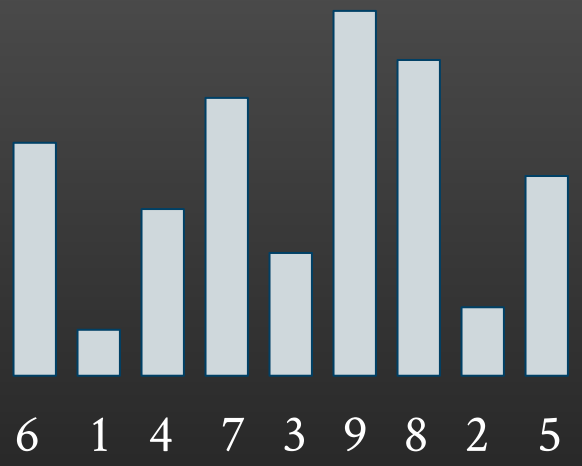


Two permutations $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are *similar* if they have the same *relative order*:

$$x_i < x_j \text{ iff } y_i < y_j.$$

General twins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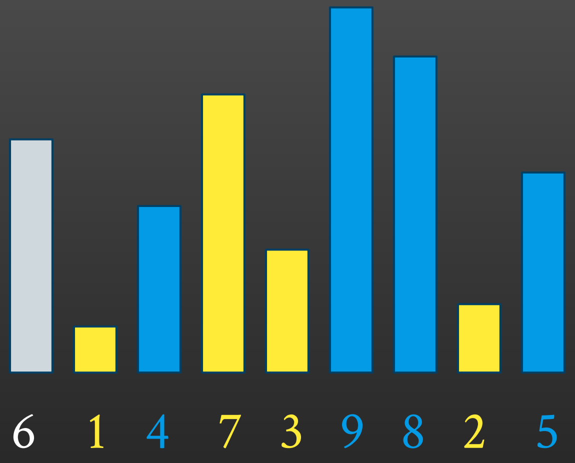

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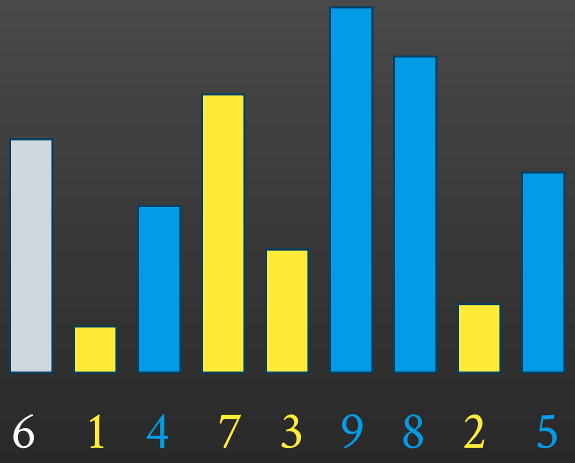

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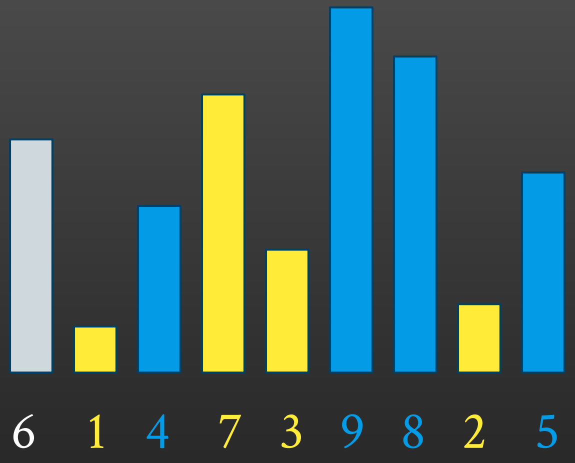

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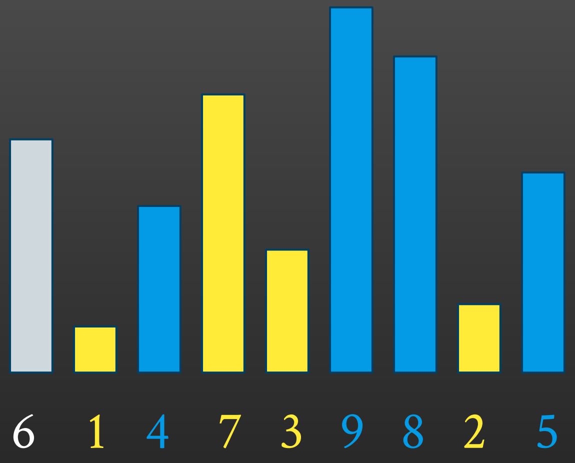

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$t(n) =$ the largest k such that *every* permutation of length n contains twins of length k.


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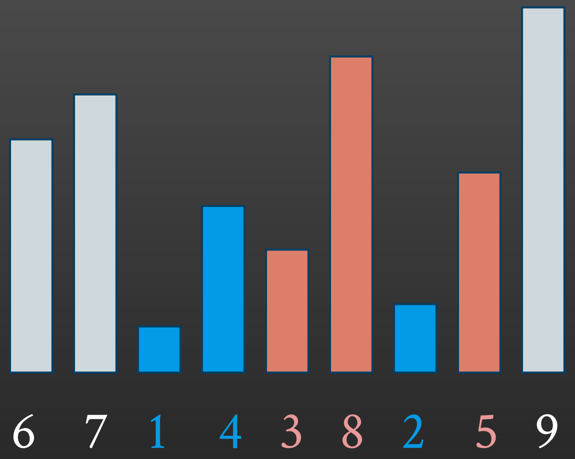
Tight twins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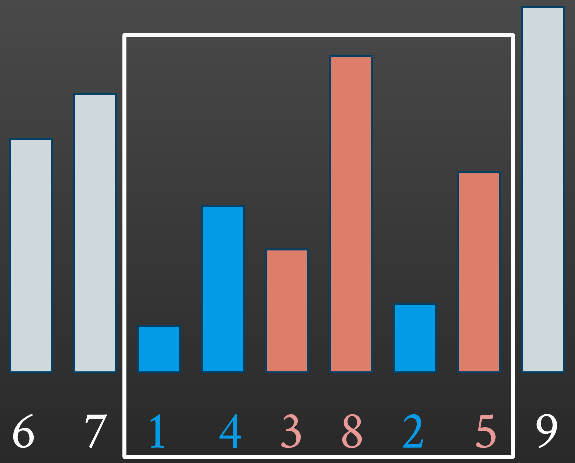
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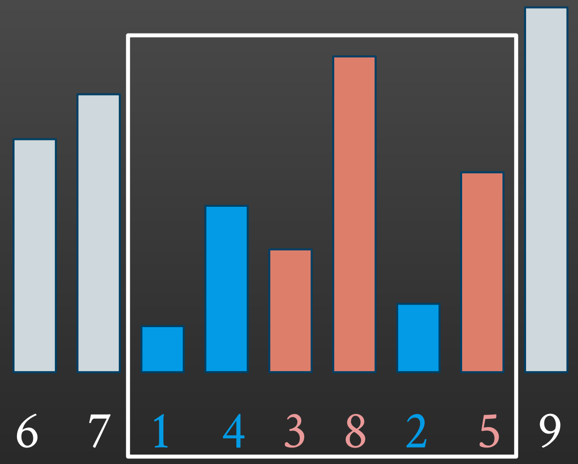
Twins in a permutation π are called *tight* if they jointly form a connected *segment* of π.



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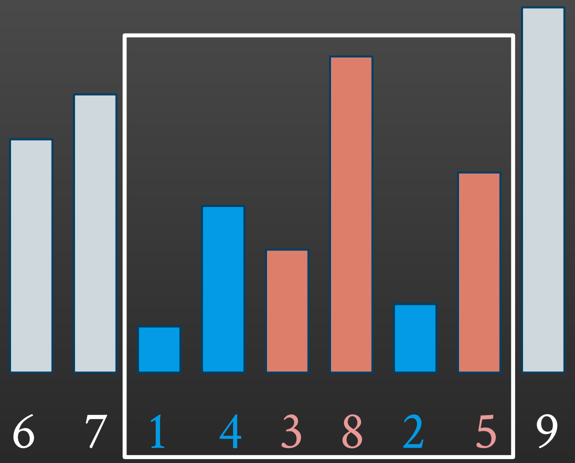
Let $tt(n)$ be the largest k such that every permutation of length n contains tight twins of length k.



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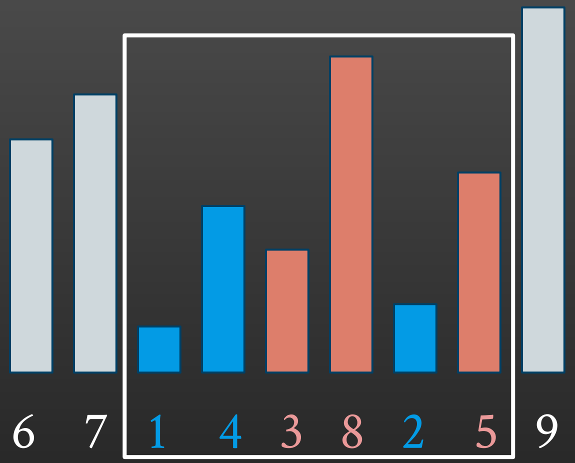

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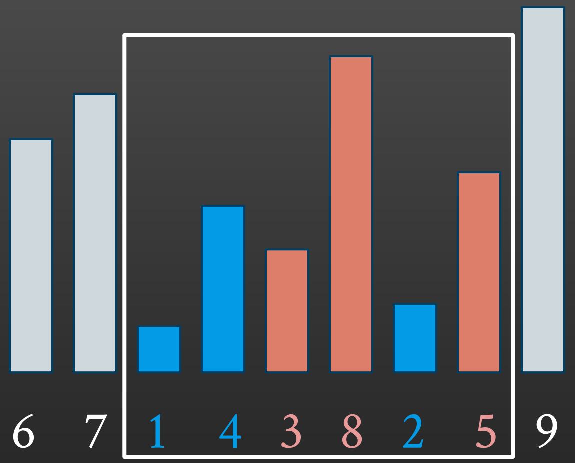

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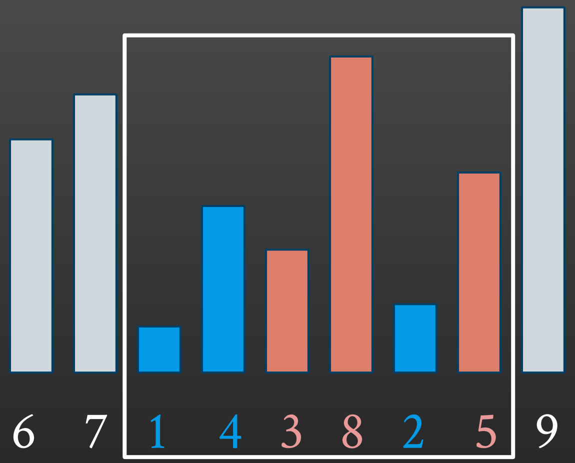

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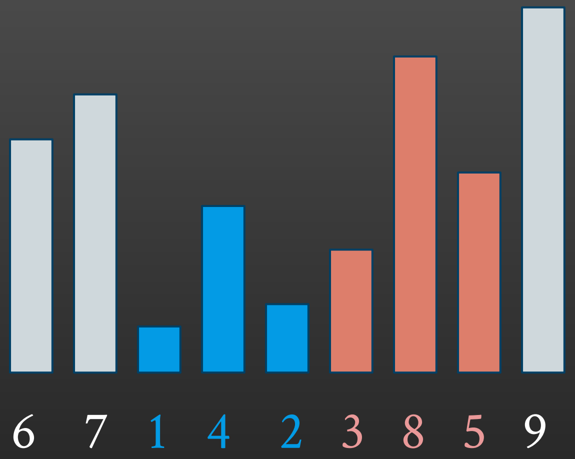
Problem: What is the number of tight twins of length n?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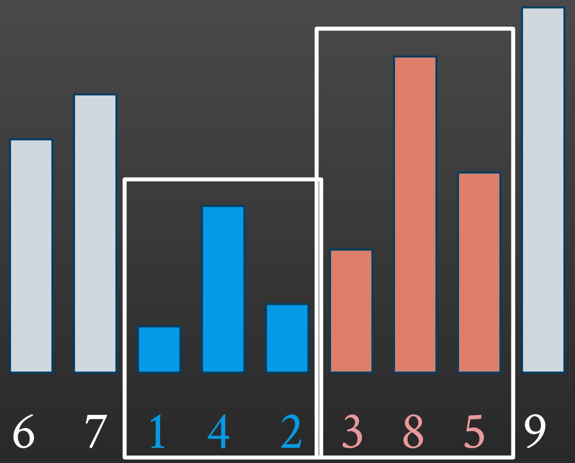
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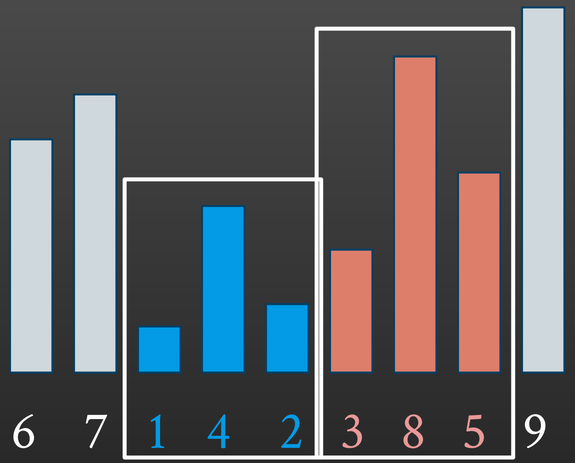
Twins in a permutation π are called *block-tight* if they are tight and each forms a connected *segment* of π.



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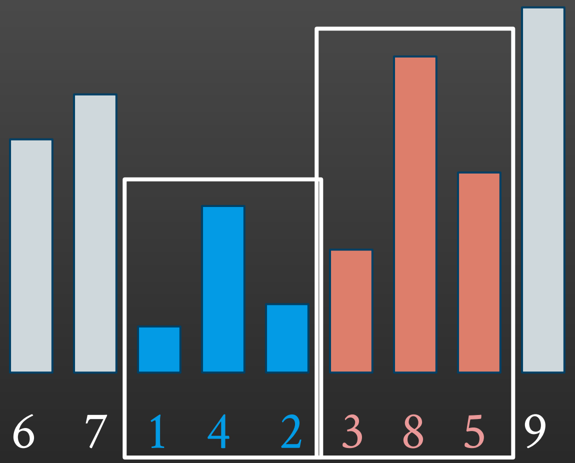

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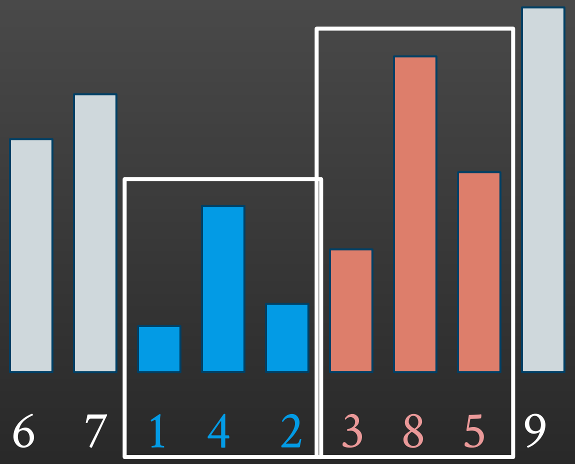

Theorem (Avgustinovich, Kitaev, Pyatkin, Valyuzhenich, 2011): $btt(n) = 1$, for all $n \geq 1$.

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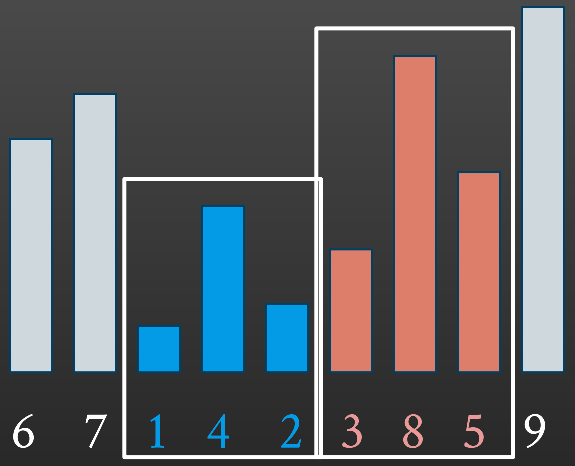

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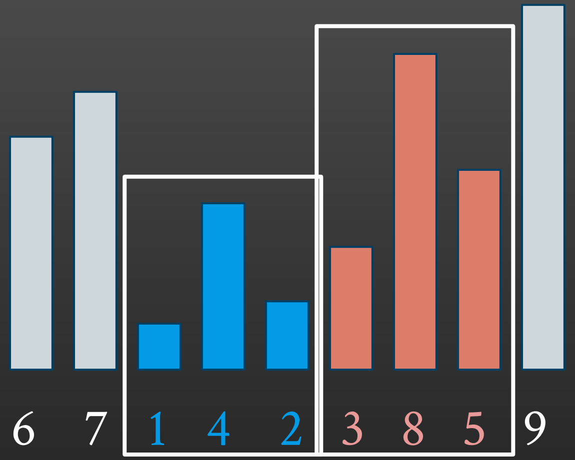

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Conjecture (Grytczuk, 2021): Every *pattern* avoidable on words is *avoidable* on permutations.

Weak twins


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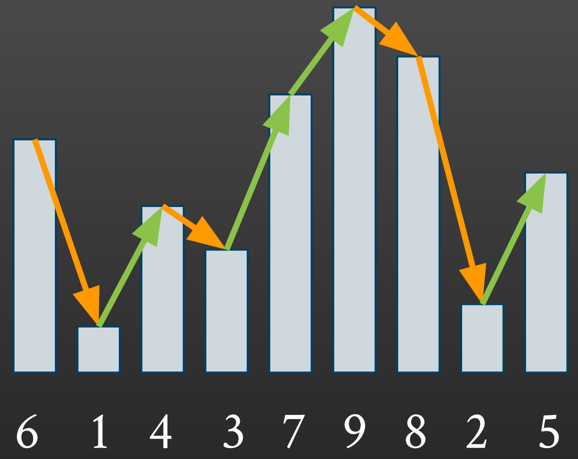
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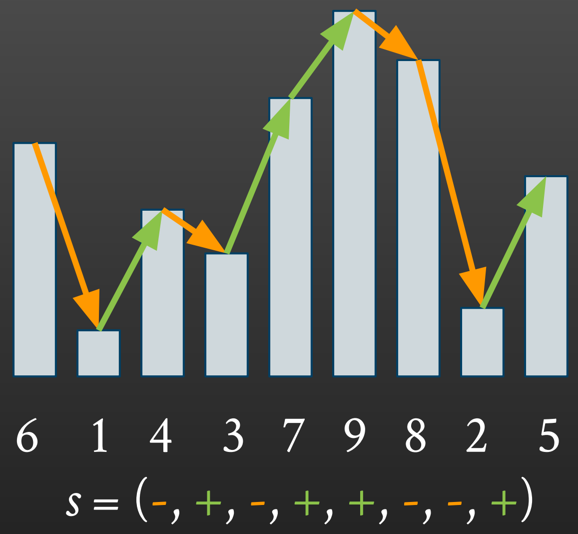
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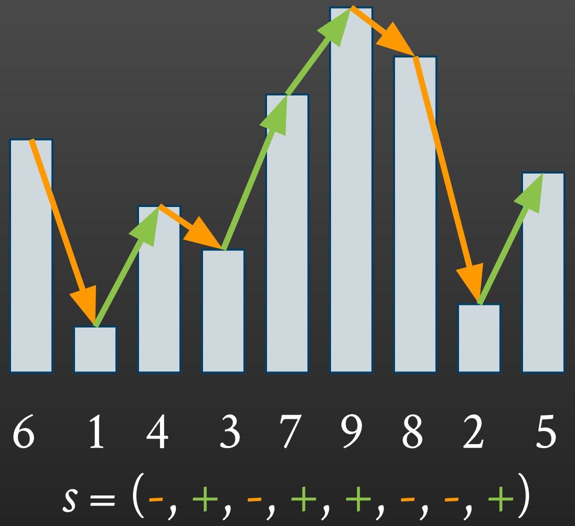
The *shape* of a permutation $\pi = (x_1, \ldots, x_n)$ is a sequence $s(\pi) = (s_1, \ldots, s_{n-1})$ of signs {+, -}:

$$s_i = \mathrm{sign}(x_{i+1} - x_i).$$



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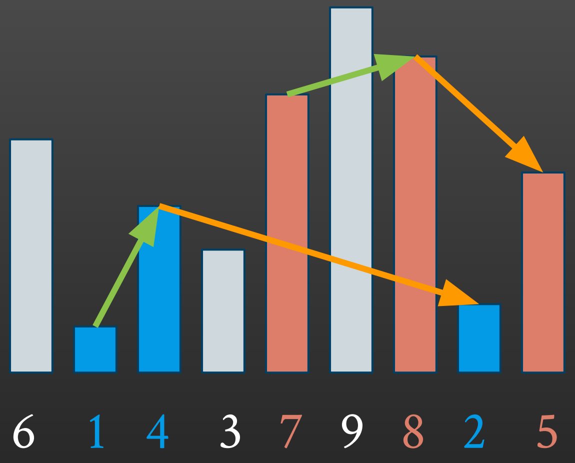

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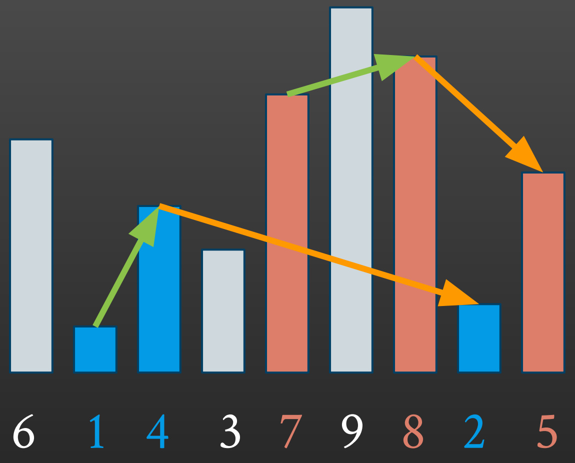

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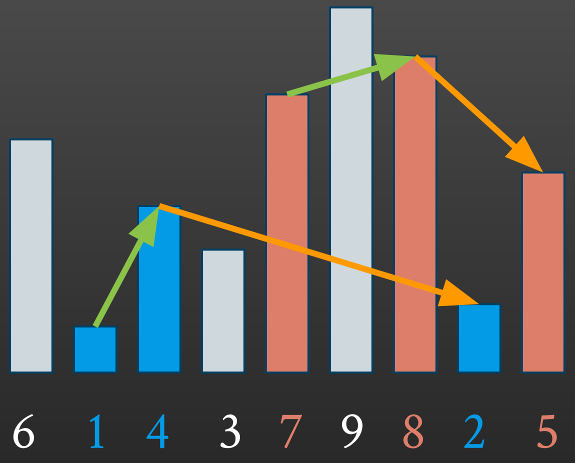

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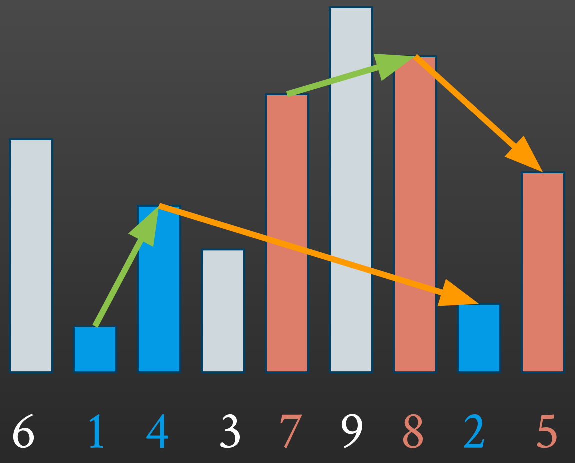

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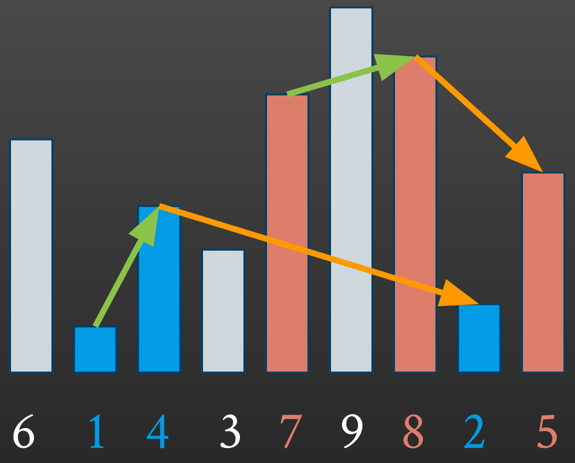

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Problem: What is the number of weak twins of length n?

Twins in other structures

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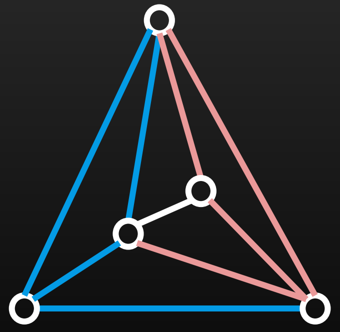

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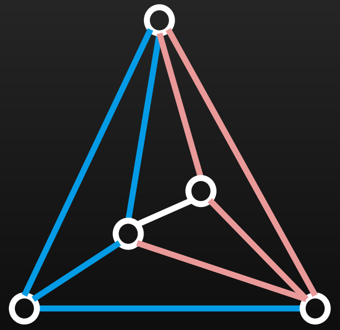

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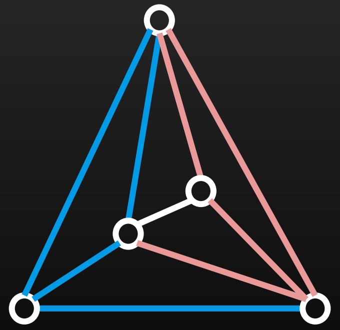

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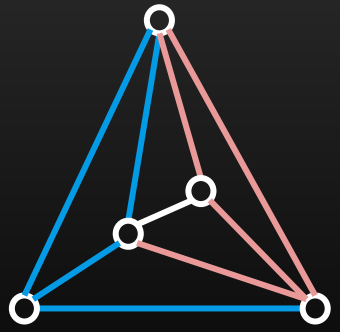

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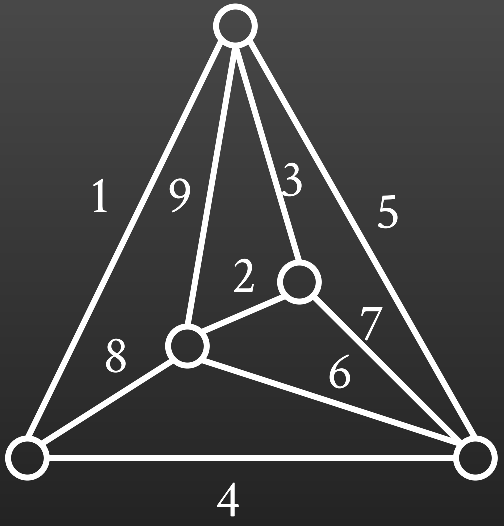
Digraphs, Posets, Hypergraphs, Matroids, Banach Spaces,...

Twins in edge-ordered graphs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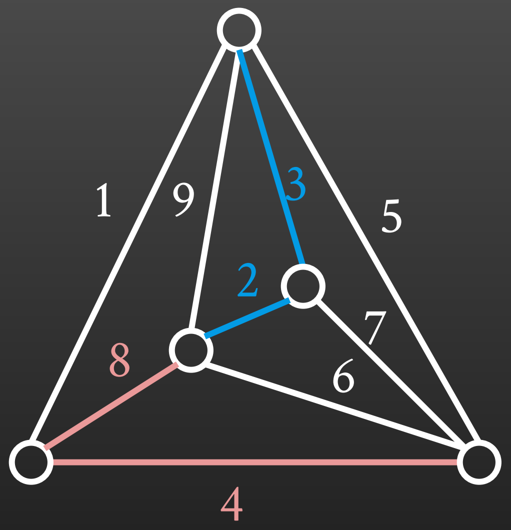
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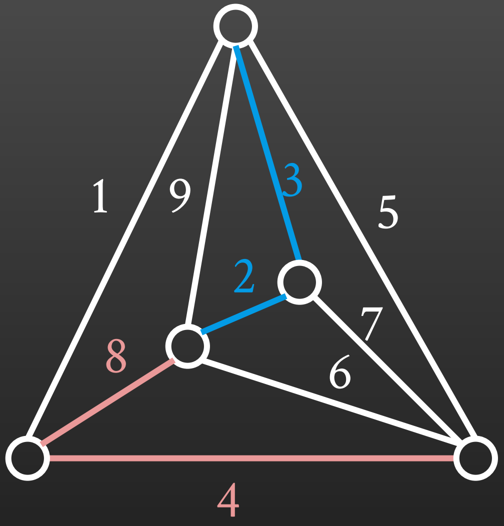

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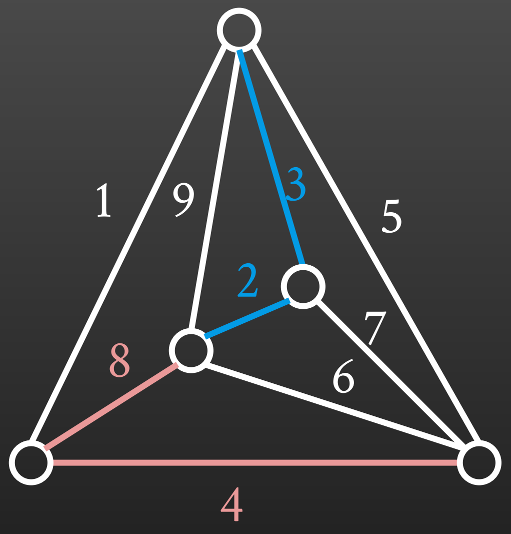

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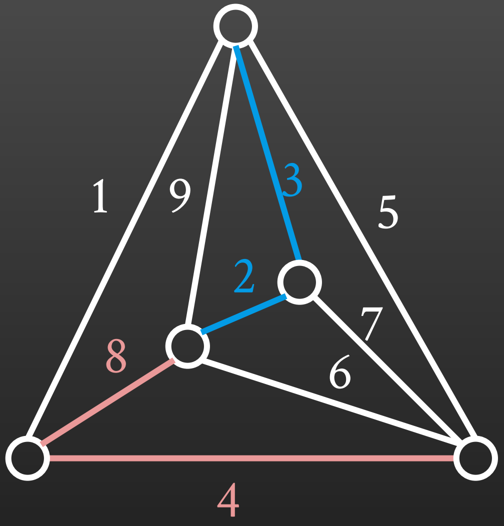
Let $t(G)$ be the largest k such that every edge-ordering of G has a twin path of length k.



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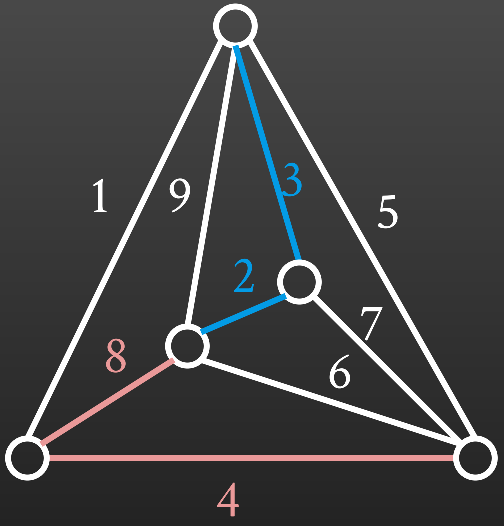

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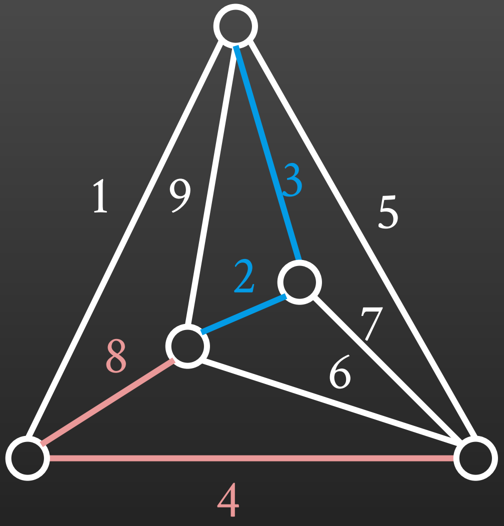

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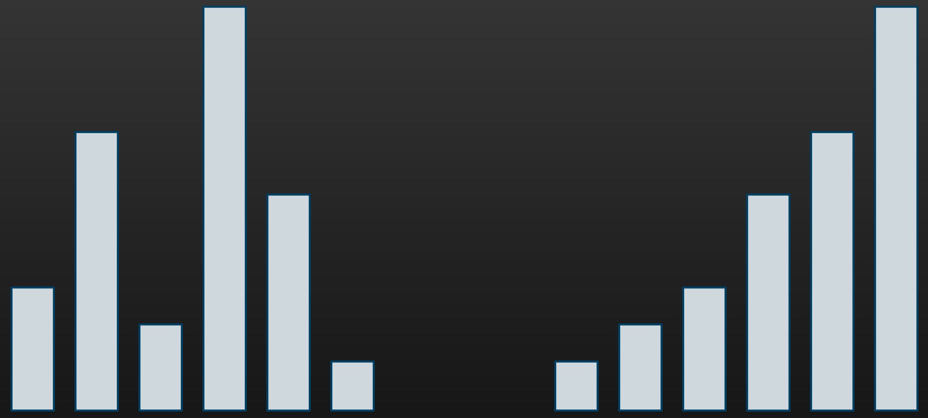
A problem of Ulam

Ulam's Problem: Given two *structures, A* and *B*, what is the least number $k = U(A,B)$ such that each of these structures can be decomposed into k substructures that can be matched into k *isomorphic* pairs?



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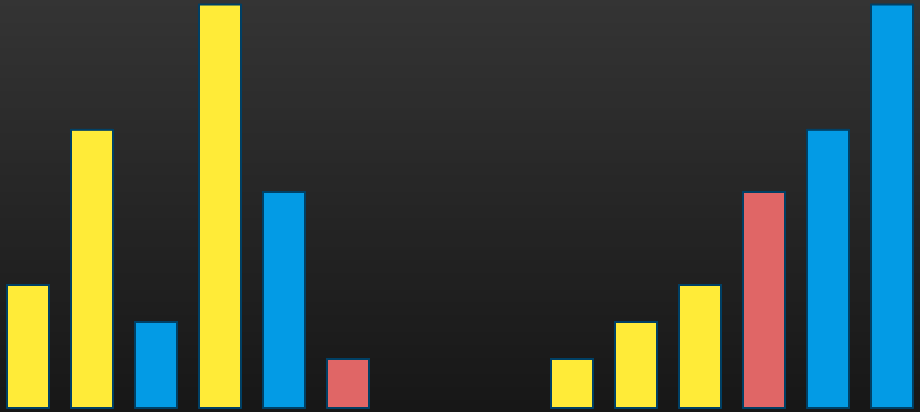

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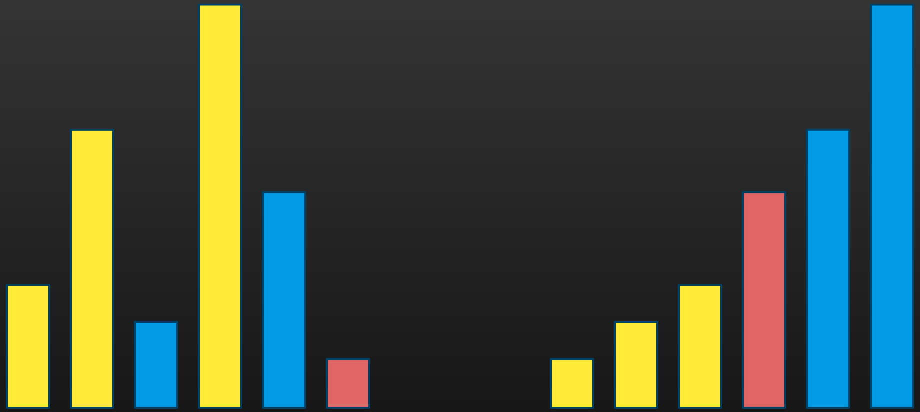

What is the *Ulam number* $U(\alpha,\beta)$ for a pair of *random* permutations of length n?

Thank You!