Pattern complexity of 2D substitutive shifts

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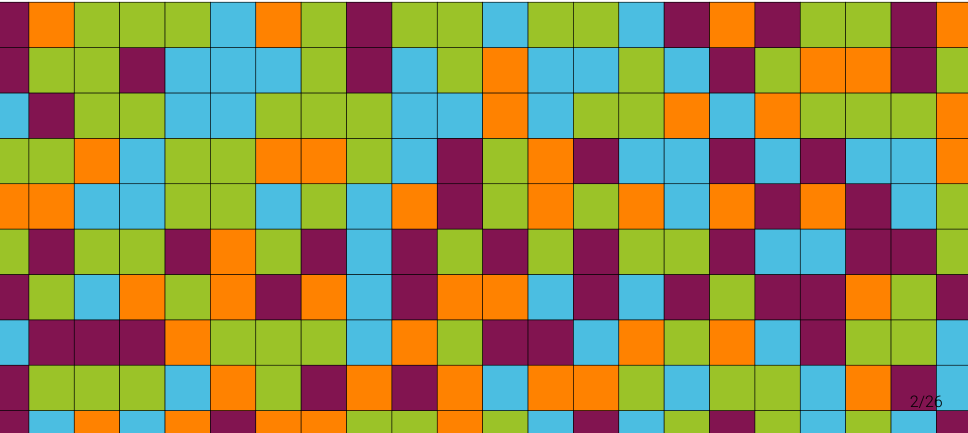
Shift spaces and pattern complexity

Shifts Spaces – Configurations

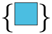
Finite alphabet: $\mathcal{A} = \{$  $\}$

Configuration: $c \in \mathcal{A}^{\mathbb{Z}^2}$



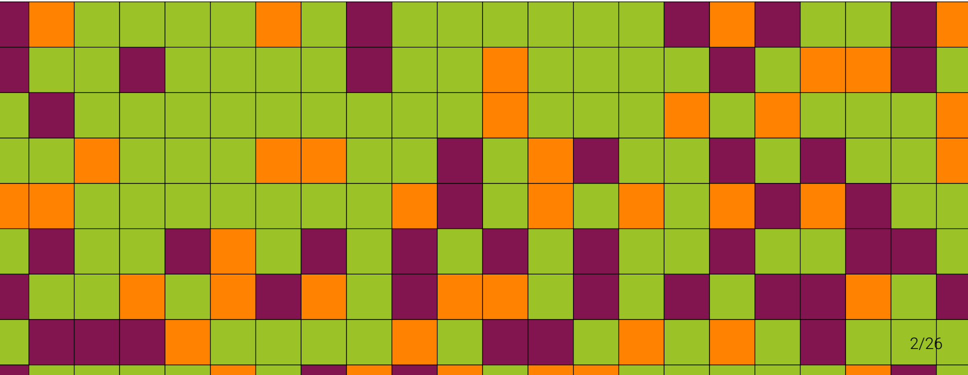
Shifts Spaces

Finite alphabet: $\mathcal{A} =$ 

Set of forbidden patterns: $F =$ 

(sub)Shift :

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^2} \mid \forall m \in F, m \text{ does not appear in } c\}$$



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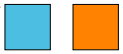
Set of forbidden patterns: $F = \{$  $\}$

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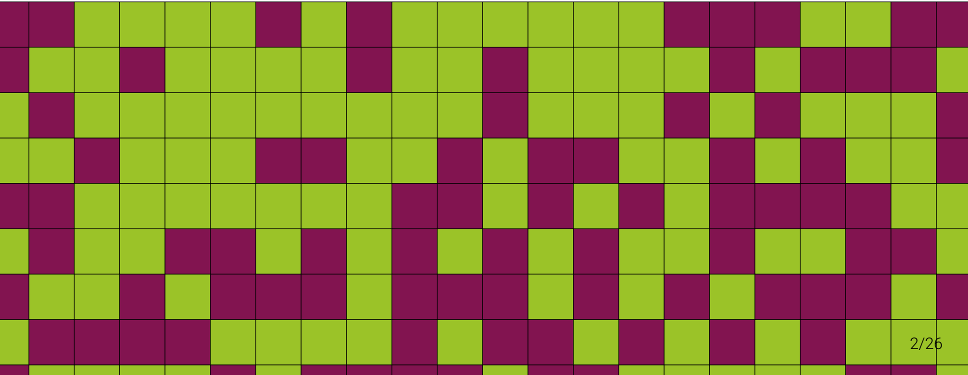

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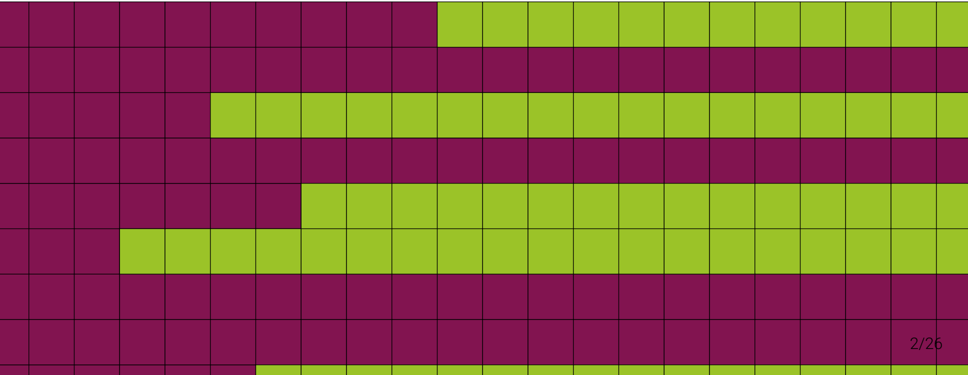

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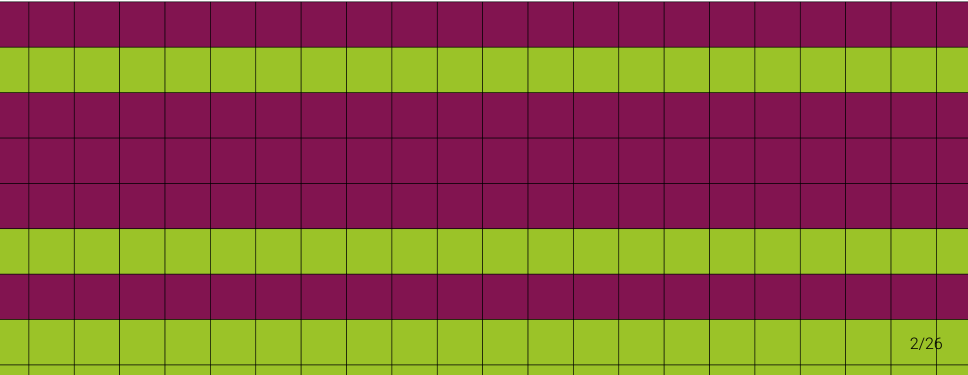

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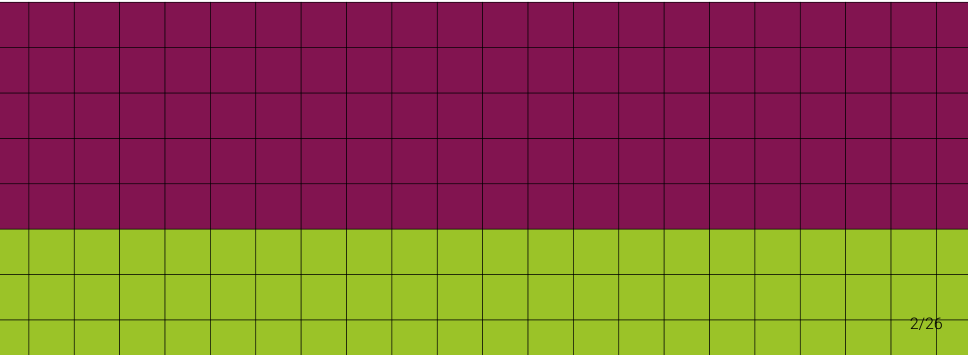

Shifts Spaces

Finite alphabet: $\mathcal{A} = \{$  $\}$

Finite Set of forbidden patterns: $F = \{$  $\}$

(sub)Shift of Finite Type (SFT):

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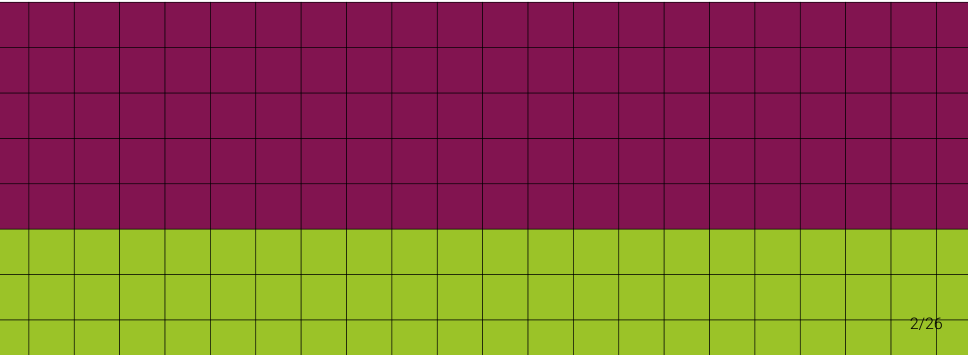
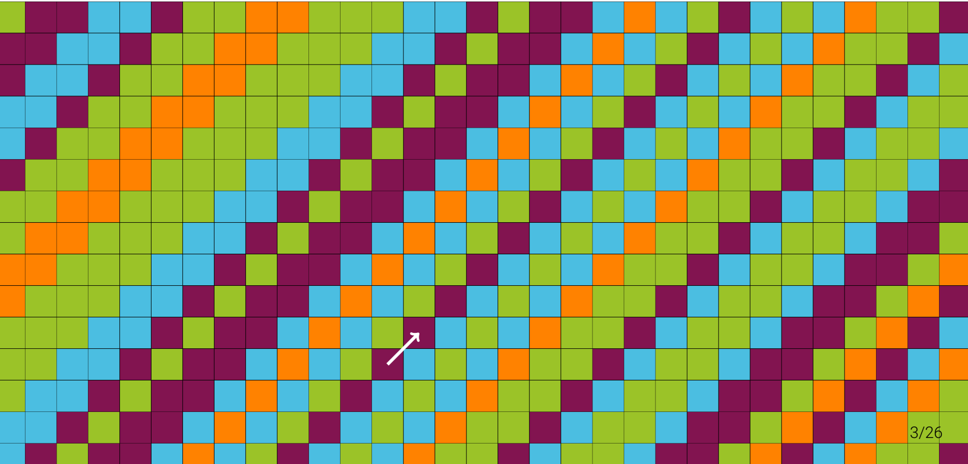


Periodic Configuration, Aperiodic Shift

$c \in \mathcal{A}^{\mathbb{Z}^2}$ is:

- **1-periodic** / **periodic**: $\exists \mathbf{u}, \forall \mathbf{v}, \quad c_{\mathbf{v}-\mathbf{u}} = c_{\mathbf{v}}$



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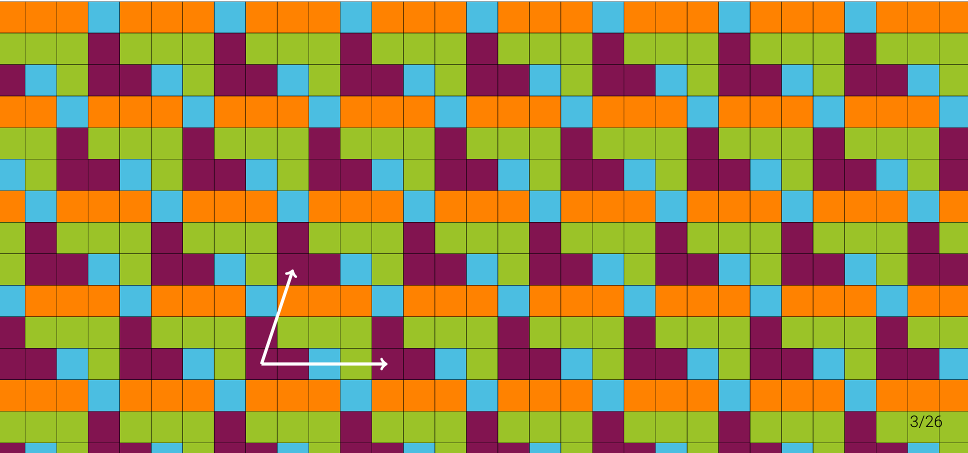

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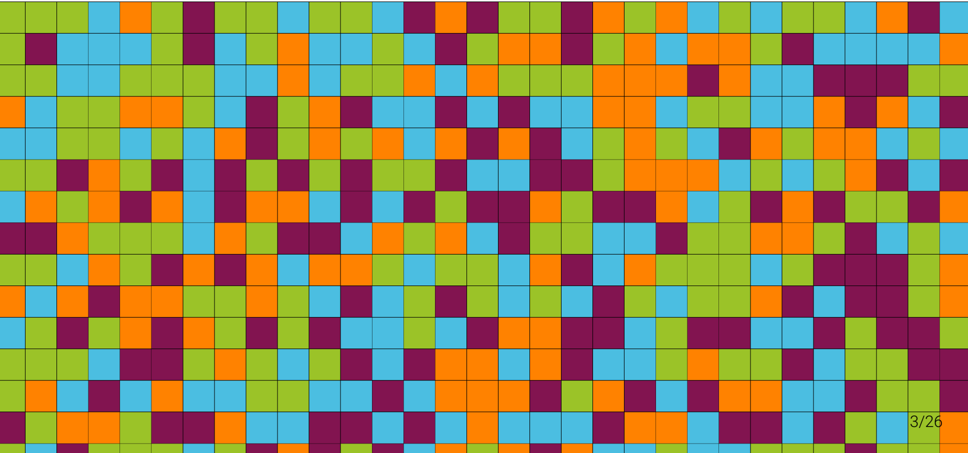

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Property

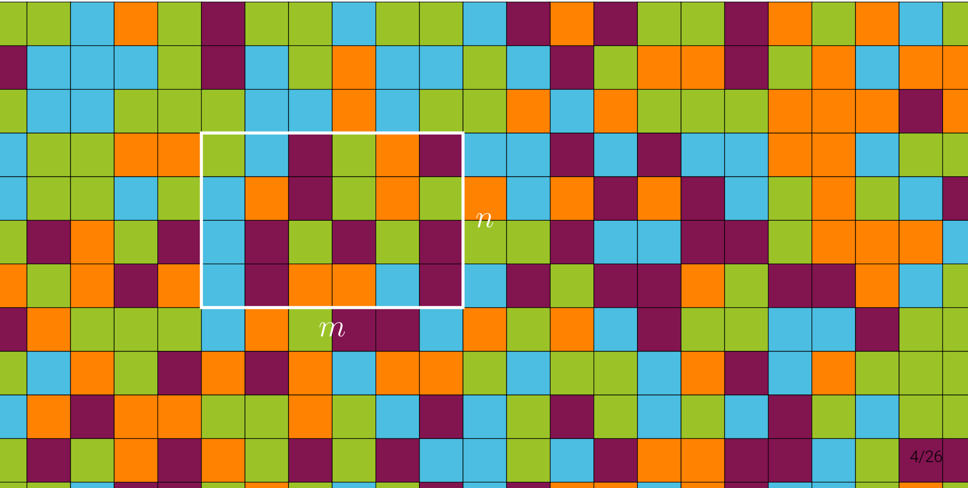
In dimension 2, X SFT:

$$X \text{ contains a 1-periodic configuration} \Leftrightarrow X \text{ contains a 2-periodic configuration}$$

Pattern complexity & Nivat's conjecture

A finite alphabet
$c \in A^{\mathbb{Z}^2}$ a configuration

$$P_c(m,n) = \text{ number of } m \times n \text{ patterns in } c$$



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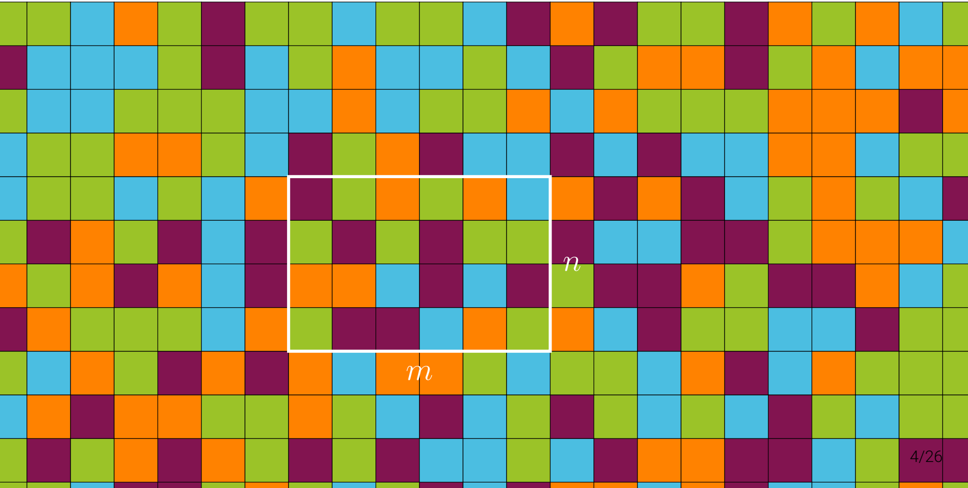

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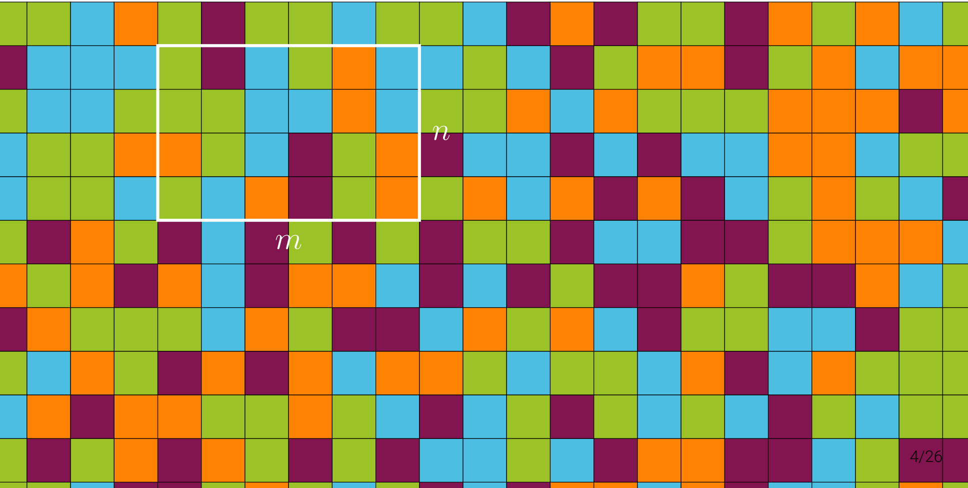

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Theorem (Morse & Hedlund, 1938)

$$\forall w \in \mathcal{A}^{\mathbb{Z}},$$
$$\exists n > 0,\ P_w(n) \leq n \quad \Rightarrow \quad w \textit{ periodic}$$

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Conjecture (Nivat, 1997)

$$\forall c \in \mathcal{A}^{\mathbb{Z}^2},$$
$$\exists m, n > 0, \; P_c(m, n) \leq mn \quad \Rightarrow \quad c \textit{ periodic}$$

Aperiodic shift complexity

Theorem (Kari, M., 2020)

If $\exists c \in X, m, n \in \mathbb{N}$ s.t. $P_c(m, n) \leq mn$, then $\exists d \in X$ which is *periodic*.

Corollary

Let X be an aperiodic subshift. Then $\forall m, n \in \mathbb{N}, c \in X$,

$$P_c(m, n) \geq mn + 1$$

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Can we do **better** ?

How small can be the complexity of aperiodic shifts?

- **Cassaigne** 1999: Characterization of configurations with complexity $mn + 1$ for all m, n (their orbit closure is *not* aperiodic)

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What about aperiodic **tilings** (SFTs) ?

- **Jeandel, Rao** 2015/2021: You need 11 Wang tiles to have aperiodicity ($m = n = 1$ for Wang tiles)

How small can be the complexity of aperiodic shifts?

→ Build aperiodic shifts with complexity as small as possible

→ **Lower bound on the complexity of aperiodic shifts**

Substitutive shifts

1D substitutions

$\sigma : \mathcal{A} \to \mathcal{A}^*$
$w \in \mathcal{A}^* : \sigma(w) = \sigma(w_0)\sigma(w_1)\sigma(w_2)\cdots$

The **substitutive shift** associated with σ:

$$X^\sigma = \{w \in \mathcal{A}^\mathbb{Z} \mid \forall p \sqsubset c, \exists a \in \mathcal{A}, \exists k \in \mathbb{N}, p \sqsubseteq \sigma^k(a)\}$$

$$\sigma : 0 \mapsto 01, \quad 1 \mapsto 0$$

$$p = \qquad 1010$$

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$$0100101001001$$

$$=\sigma^5(0)$$

1D substitutions: complexity

Well understood !

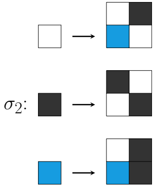
Theorem (Pansiot 1984)

Let σ be a substitution, $w \in X^\sigma$. Then $P_w(n) = \Theta(c(n))$

with $c(n) = \begin{cases} 1 \\ n \\ n \log\log n \\ n \log n \\ n^2 \end{cases}$ depending only on σ.

2D substitutions

$\sigma : \mathcal{A} \to \mathcal{A}^{[0,M)}$ (uniform square case)



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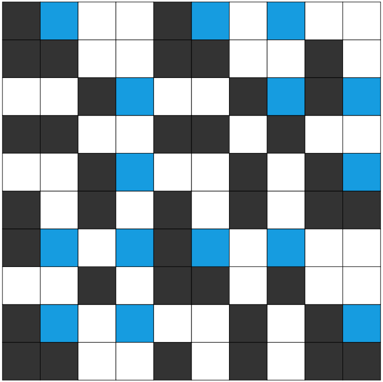
Substitutive shift (2D)



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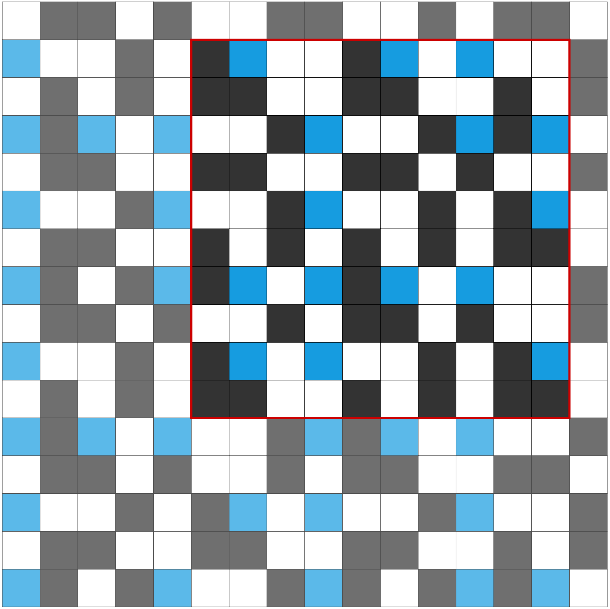

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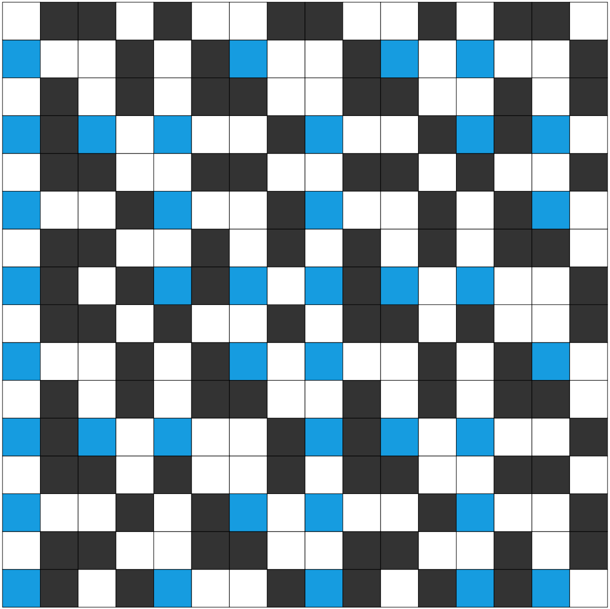

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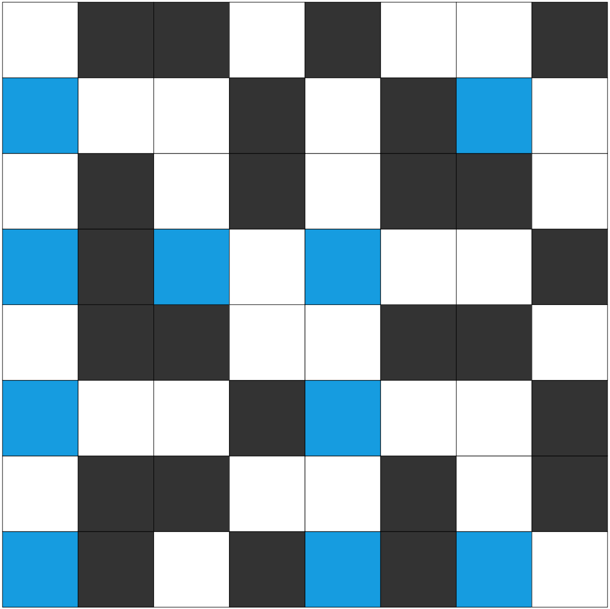

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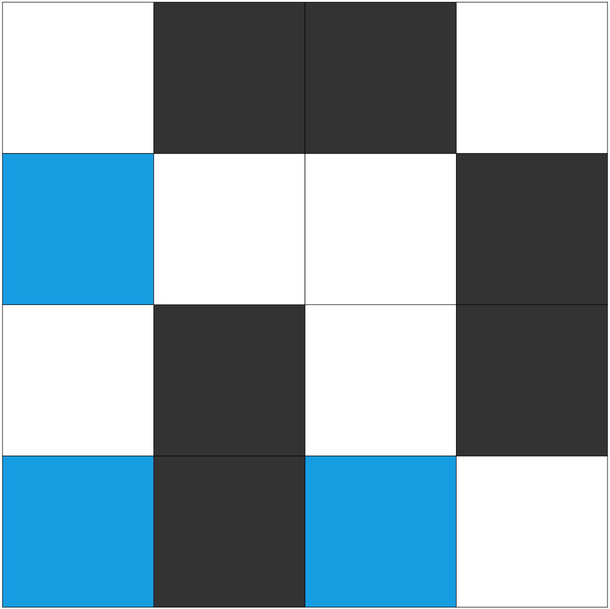

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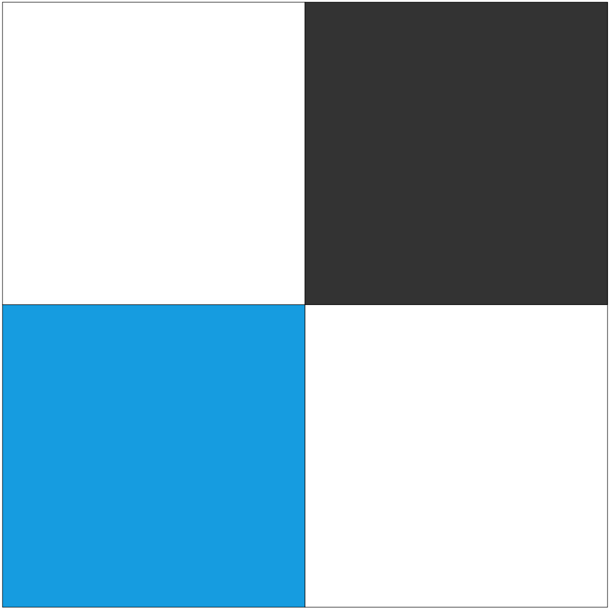

Substitutive shift (2D)

2D substitutions: complexity

Theorem (Folklore / Robinson 2004)

Let σ be a uniform square substitution, primitive and invertible, $c \in X^\sigma$. Then

$$P_c(n, n) = O(n^2)$$

$\rightarrow$ If X^σ is aperiodic, $P_c(n, n) = \Theta(n^2)$ for all $c \in X^\sigma$

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But what is the constant ?

Our lower bound

Theorem (M., Petit-Jean 2021)

Let σ be a uniform square substitution, primitive, recognizable and marked, $c \in X^\sigma$. Then for n large enough,

$$P_c(n,n) \geq Cn^2$$

with $C > 1$ depending only on σ.

Strictly higher than $mn + 1$!

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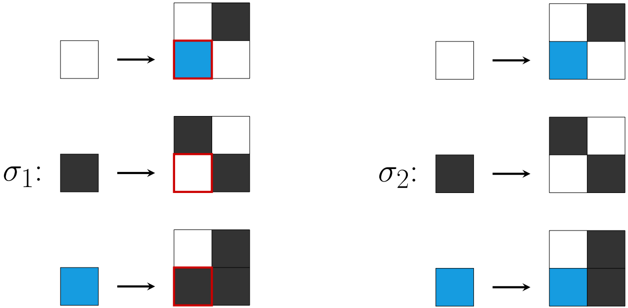
Marked substitutions

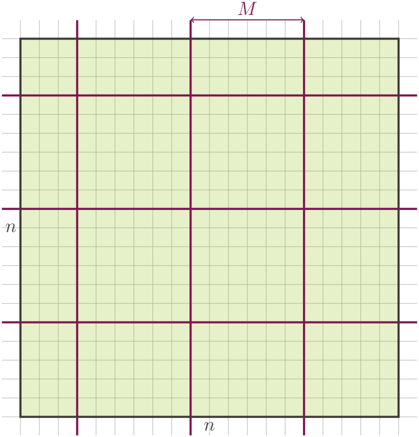
σ marked:
$\rightarrow$ when a *single position* determines the antecedent



Recognizable substitutions

σ recognizable:

$n \geq \rho \Rightarrow p \in \mathcal{L}_c(n, n)$ have a unique "position of de-substitution"



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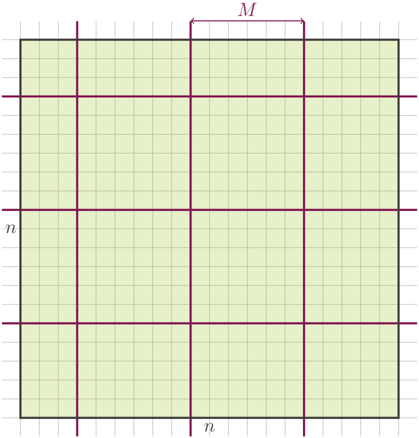

Theorem (Solomiak 1998 / many others in 1D)

X^σ is aperiodic $\Leftrightarrow$ σ is recognizable

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Let σ be a uniform square substitution, primitive, recognizable and marked, $c \in X^\sigma$. Then for n large enough,

$$P_c(n,n) \geq Cn^2$$

with $C > 1$ depending only on σ.

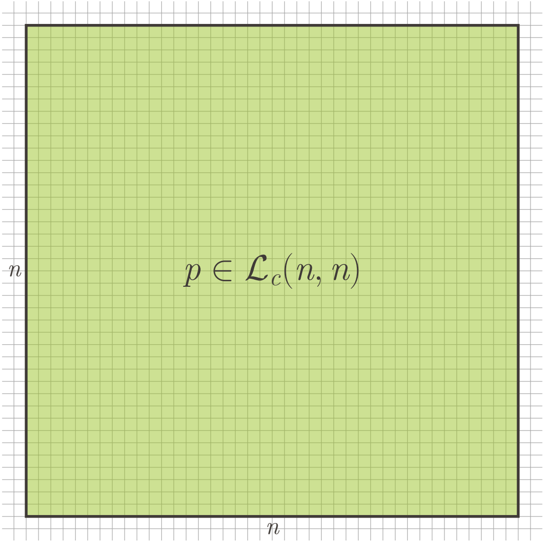
Proof idea: de-substitute as much as possible and count

$$\sigma \to \sigma^k$$

k maximal to have squares (n,n) **recognizable**

Proof: ”just” de-substitute

σ^k: size $M^k \times M^k$



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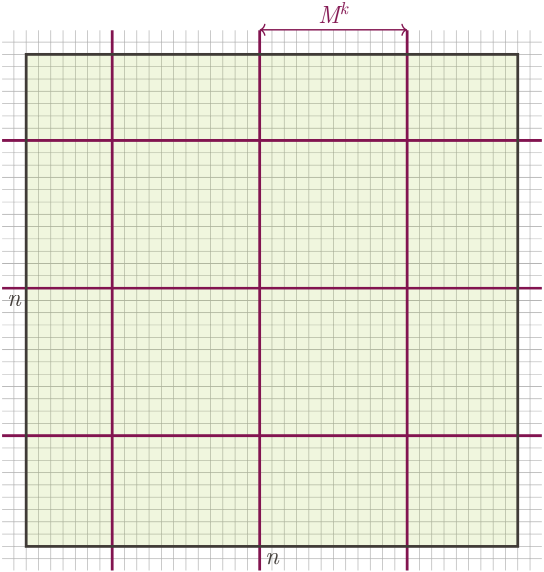

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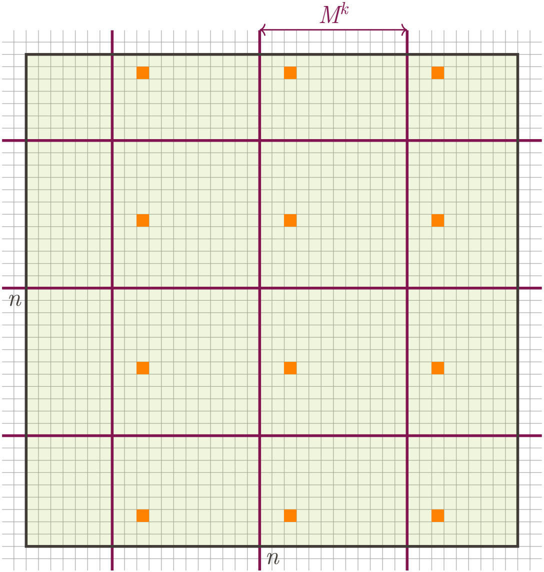

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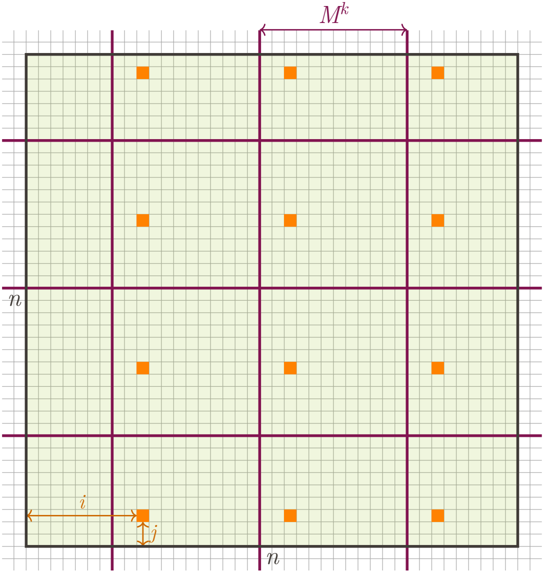

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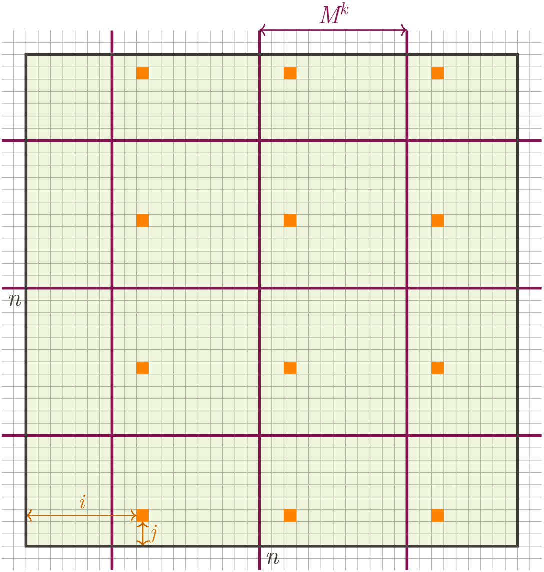

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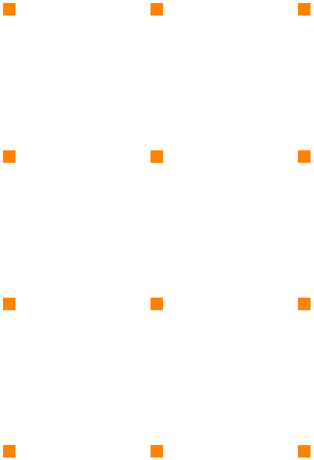


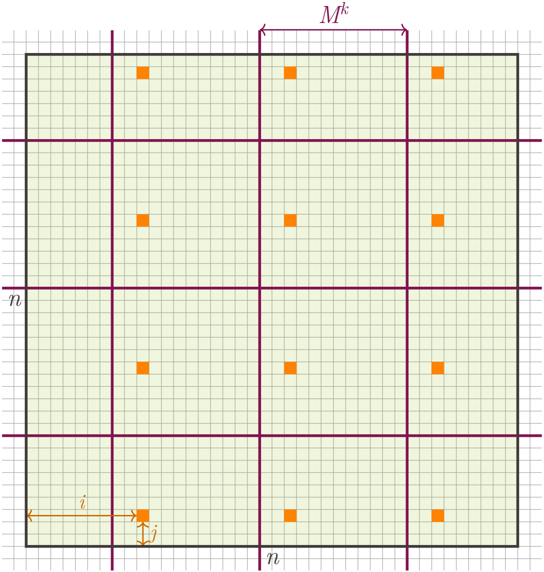
Proof: ”just” de-substitute

σ^k: size $M^k \times M^k$

$$p \in \mathcal{L}_c(n, n)$$

$\uparrow$ injection

$$\bigcup_i \bigcup_j \mathcal{L}_{\sigma(c)}\left(\left\lceil \frac{n-i}{M^k} \right\rceil, \left\lceil \frac{n-j}{M^k} \right\rceil\right)$$



Proof: ”just” de-substitute

$$P_c(n,n) \geq \sum_i \sum_j P_{\sigma(c)}\left(\left\lceil \frac{n-i}{M^k} \right\rceil, \left\lceil \frac{n-j}{M^k} \right\rceil\right)$$

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$$\geq \sum_i \sum_j \left\lceil \frac{n-i}{M^k} \right\rceil \left\lceil \frac{n-j}{M^k} \right\rceil + 1$$

as X^σ is aperiodic !

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$$\geq \ldots \qquad \text{using maximality of } k$$

$$\geq Cn^2$$

with $C = 1 + \frac{1}{(\rho+1)^2}$

□

Generalization ?

Theorem (M., Petit-Jean 2021)

Let σ be a uniform square substitution, primitive, **recognizable** and marked, $c \in X^\sigma$. Then for n large enough,

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with $C > 1$ depending only on σ.

Not recognizable / not aperiodic: **Not true**

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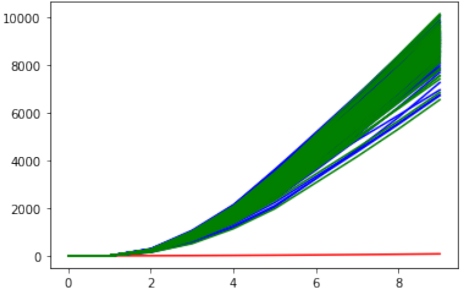
Not marked: *Seems* true...

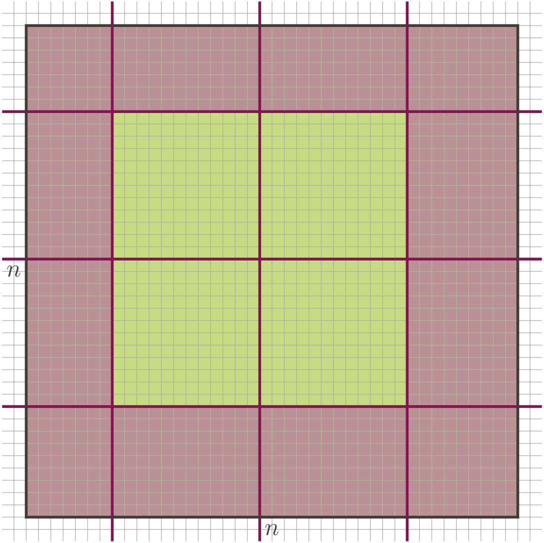


– n^2

– marked (100)

– non marked (100)

Non-marked



Non-uniform case ?

Nice counter-example (thanks to Sebastien Labbé):
cartesian product of two 1D Fibonacci ($0 \mapsto 01, \quad 1 \mapsto 0$)

$$a \mapsto \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} c \\ a \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} a & b \end{pmatrix}$$

$$d \mapsto \begin{pmatrix} a \end{pmatrix}$$

$$P_c(n, n) = (n+1)^2 \qquad \text{(experimentally)}$$

One upper bound

Build aperiodic shifts with complexity as small as possible

For SFTs:

Theorem (Kari, M., 2021)

$f : \mathbb{N} \to \mathbb{N} \notin O(1)$. There exists n and P made of at most $n^2 + f(n)n$ binary square patterns of size $n \times n$ such that $X_{\overline{P}}$ is aperiodic

One upper bound

Theorem (Kari, M., 2021)

T a set of Wang tiles. There exists N, k s.t. $\forall n \geq N, m \geq 2$, and P made of at most $mn + k(n+m)$ binary patterns of size $m \times n$ such that

- T tiles the plane $\Leftrightarrow X_{\overline{P}} \neq \emptyset$
- T tiles the plane aperiodically $\Leftrightarrow X_{\overline{P}}$ aperiodic

$\rightarrow$ re-encode the tileset into an SFT of "pretty low" complexity

What’s next ?

- Get the lower bound for all primitive aperiodic substitutions !
- Get a lower bound for general SFTs ?
- Improve the upper bound for some particular SFTs ?

Thank you !