A few words on games

Robbert Fokkink (TU Delft)

CoW seminar 2022

Motivating Question

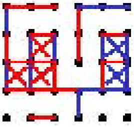
Éric Duchêne and Michel Rigo asked (2008): is there an impartial game with losing positions coded by the 4-bonacci word:

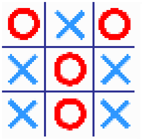
$$abacabadabacabaabacabadabacabab\cdots$$

Led to joint work with Dan Rust and Cisco Ortega.

Our results are related to Dekking, Shallit, Sloane (2019): Queens in exile.

Impartial Games





What is a Game?

- Two players take turns.
- A game has positions (vertices) and moves (edges).
- The game is over after a finite number of moves.
- Last player to move wins.

The archetype Impartial Game is a take-away-game.

Tennis

Lenstra's Wimbledon Theorem (1978)

<u>Definition.</u>- A game is a set.

A few words of explanation may be in order.

Hypergame

Sets and games are the same, we run into the same existential issues.

Hypergame: Which game? You choose, I start!

Is hypergame a game?

Russian dolls

How can a set be a game? In axiomatic set theory, all sets are made of sets.

$$\emptyset,\ \{\emptyset\},\ \{\emptyset,\{\emptyset\}\},\ \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\ \ldots$$

A move is: select an element of the set. First player to select the empty set wins. A bit like Russian dolls. But more interesting.

Win-lose

Theorem (Zermelo, 1912)

A game is either won or lost for Alice.

Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels.

Quick Question: is $\mathbb{N}$ won or lost? How about $\mathbb{Z}$?

So who wins?

How does Alice win a game?

She has a perfect move. No matter what Bob does, she has a perfect reply. And again, no matter what Bob does, she has a perfect reply.... Etc.

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots \text{ Alice wins}$$

So now the only problem is: how do we find the perfect move?

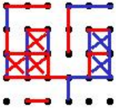
Complexity

An NP complete problem:

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \cdots \text{ Boolean Expression True}$$

This is a game. We have a Boolean Expression. Alice chooses the x's. Bob chooses the y's. Alice wins if the Expression is True.

This is an NP-complete game. But it is a bit boring. How about dots-and-boxes?



Complexity

Erik Demaine (2001): It remains open whether Dots-and-Boxes is in NP or PSPACE-complete from an arbitrary configuration.

Solved in 2021 by a group from TU Eindhoven: PSPACE-complete.

Winning Ways, p225:
Some people consider a class of problems "finished" when it has been shown to be NP-complete. Philosophically, this is a viewpoint we strongly oppose:

Some games which are NP-hard are very interesting

Periodic words

Winning Ways considers small games, like the Game of Officers:

Rule: remove one stone from a pile. You can split the remaining pile into two non-empty piles, or keep one pile.

PNNPNNNNNNNPNNNNNNNPNNNNNNNNPNNNN

See A046695 in the OEIS. Is it eventually periodic?

Periodic words

Octal games. Remove some stones, split into some piles.

A002186-87,071074,046695,071433-34,071461,071515,071426,125940

Conjecture: all octal games are eventually periodic. The blue colored OEIS entries are of octal games for which the conjecture has been settled.

So how about a word like

NPNNPNPNNPNPNNPNPNPNNP

Do we have a game for that?

Wythoff's Game

We sure do! It is a two-pile game.

Wythoff's game (1907): Take any number from a single pile. Or take any equal number from both piles. Last player to move wins.

The losing positions are:

$$(1,2),\ (3,5),\ (4,7),\ (6,10),\ (8,13),\ \ldots$$

Wythoff's Game

Let's assemble these losing positions in a table

1	3	4	6	8	9	11	12	14
2	5	7	10	13	15	18	20	23

Wythoff proved that the *n*-th losing position is given by

$$(\lfloor n\phi \rfloor, \lfloor n(\phi + 1) \rfloor)$$

where $\phi^2 = \phi + 1$ is the golden ratio.

Morphic Games

Duchêne and Rigo (2008) observed that the positions can be coded by the Fibonacci word

$$abaababaabaababaababaab\cdots$$

The n-th losing position is given by the location of the n-th a and the n-th b.

Morphic Game: any game in which the n-th losing position is given by the n-th a, b, c, ... in a morphic word.

A Tribonacci Game

Duchêne and Rigo's 3-pile game

- Take any number from two piles and none from the third.
- If you take from three piles, then twice the max of the three taken cannot exceed the sum of the three taken.
- You can take the same number from two piles and another number from the third, but then you cannot do that in such a way that the largest pile now becomes the middle pile.

Losing positions in this game correspond to the Tribonacci word.

Generalized Wythoff Games

Akiyama observed that in Wythoff's game you remove x, y from the two piles such that

$$xy(x-y)=0$$

In Akiyama's 3-pile game you remove x, y, z from the three piles such that

$$xyz(x-y)(x-z)(y-z)=0$$

Recently, Fraenkel and Klein came up with the same rule and call this Lythoff *K*-pile Wythoff Games, available on Fraenkel's homepage. No clear pattern in the losing positions.

Morphic games and nice rules do not necessarily go together.

Other Wythoff Games

Wythoff also found games with losing positions

$$\lfloor n\phi_a \rfloor, \ \lfloor n(\phi_a + a) \rfloor$$

where $\phi_a^2 = (2 - a)\phi_a + a$.

Wythoff writes: by their aid modified nim-games with suitably chosen game-rules might be constituted.

Beatty

Another way to describe Wythoff's losing positions:

n-th entry in complementary Beatty sequences

$a_n = \lfloor n\alpha \rfloor$ and $b_n = \lfloor n\beta \rfloor$ for $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Cassaigne, Duchêne, Rigo (2016): All complementary Beatty sequences occur as the losing positions in a take-away game.

These are in fact invariant games, which means that every legal move remains legal for bigger piles.

Another sequence

Wythoff gave complementary sequences

$$\lfloor \alpha + n\phi_a \rfloor, \ \lfloor \beta + n(\phi_a + a) \rfloor$$

with ϕ_a as before and special α, β.

Wythoff remarks that these sequences are complementary and have successive differences $b, b + a, b + 2a, \ldots$. He leaves the proof to the reader and does not associate this to a game.

CAT

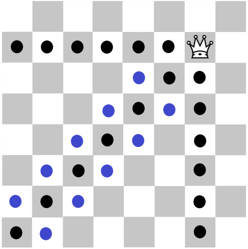
The inhomogeneous Beatty sequences are $a_n = \lfloor n\alpha + \gamma \rfloor$ and $b_n = \lfloor n\beta + \delta \rfloor$ for some further restrictions on γ, δ.

Are there any take-away games on two piles such that the losing positions correspond to inhomogeneous Beatty sequences?

Cassaigne, Duchêne, Rigo (2016): Losing positions correspond to Sturmian words with a CAT condition.

The Queen

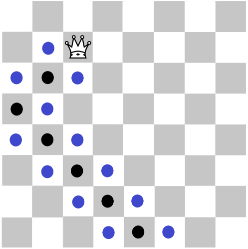
Holladay (1968) rediscovered Wythoff's games. There is a cute interpretation in terms of chess moves.



If you corner the Queen, you win. An ordinary Queen gives ordinary Wythoff. A Queen with broader diagonals produces Wythoff's other games.

The Queen

Now consider a Queen that can bounce against the side.



Here also we can consider Queens with broader diagonals.

The Queen

Tribonacci word:

$$abacabaabacababacabaab\cdots$$

Theorem (with Cisco Ortega and Dan Rust)

Delete the c's from the Tribonacci word to get the losing positions of a bouncing Queen. Delete the b's to get the losing positions of a bouncing Queen with a broadened diagonal.

1	3	4	6	7	9	10	12	14	15	17	18	20
2	5	8	11	13	16	19	22	25	28	31	33	36

The Queen

For more morphic Queens see our paper on arXiv: Corner the Empress.

It's a tribute to the last Queen of Canada.

Problem for you

MATHEMATICAL GAMES

Cornering a queen leads unexpectedly into corners of the theory of numbers

by Martin Gardner

POSITION (n)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A. $[n\phi]$	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24
B. $[n\phi^2]$	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39

The first 15 safe pairs in W. A. Wythoff's nim

Note that each B number is the sum of its A number and its position number. If we add an A number to its B number, the sum is an A number that appears in the A sequence at a position number equal to B. (An example is $8+13=21$. The 13th number of the A sequence is 21.)

The same relation $a_{b_n} = a_n + b_n$ almost holds for the bouncing Queen.

1	3	4	6	7	9	10	12	14	15	17	18	20
2	5	8	11	13	16	19	22	25	28	31	33	36

For some n it holds, for others you need to subtract 1. Can you prove that?

Thank You!