Synchronizing Times for k-sets in Automata

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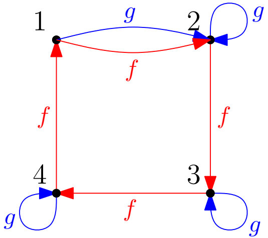
Joint work with Robert Johnson, QMUL

What Are Synchronizing Automata?

An automaton is a set of states $[n]$ and a set of mappings from the set of states to itself.



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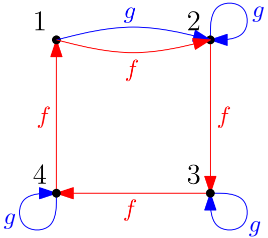

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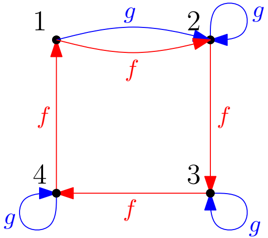

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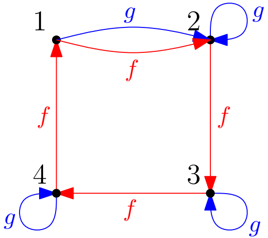
If true would be best possible.



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Best known: $\leq 0.1654n^3 + o\left(n^3\right)$ [Shitov 19].

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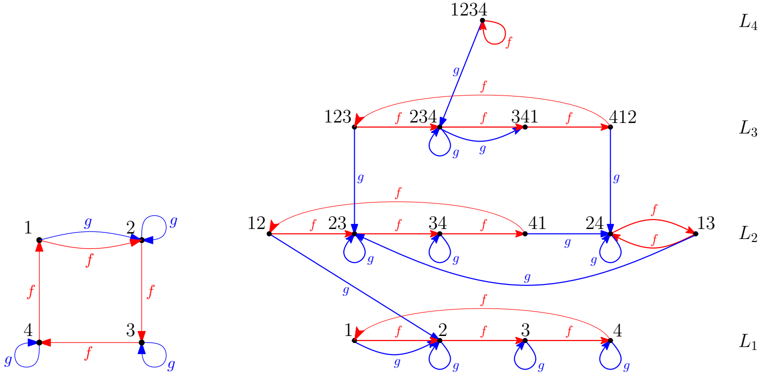
Conjecture holds if:

- automaton is orientable [Eppstein 90]
- the underlying digraph is Eulerian [Kari 03]
- one transition function is a cyclic permutation of all the states [Dubuc 98]
- ...

The Power Automaton



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Upper bounds

Applying Frankl–Pin directly gives

$$\mathrm{rdv}(k, n) \leq 1 + \sum_{i=2}^{k-1} \binom{n-i+2}{2} = (k-2)\frac{n^2}{2} + O(n).$$

Gonze and Jungers showed $\mathrm{rdv}(3, n) \leq 0.1545n^2$.

Our Results

Theorem 1 [B., Johnson '22]

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Theorem 2 [B., Johnson '22]

$$\begin{aligned} \mathrm{rdv}(3,n) &\leq \frac{3-\sqrt{5}}{4}n^2 + \frac{3}{2}n \approxeq 0.191n^2 + O(n) \\ \mathrm{rdv}(4,n) &\leq 0.459n^2 + O(n) \\ \mathrm{rdv}(5,n) &\leq 0.798n^2 + O(n) \end{aligned}$$

Proof of Theorem 1

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In general,

$$\sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \binom{i+1}{2} + \sum_{i=1}^{\lceil \frac{k}{2} \rceil - 1} \binom{n-i+1}{2}.$$

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Exists word sending some triple to a singleton of length $\leq n + \binom{r+2}{2}$

Claim 2

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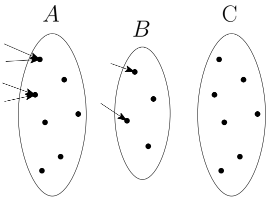

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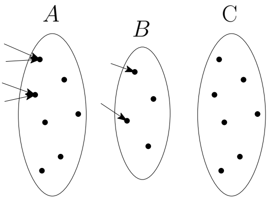

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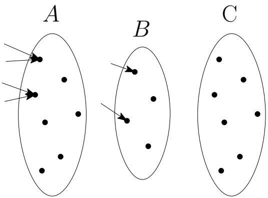

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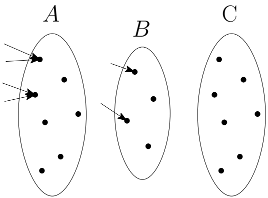

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Apply Frankl-Pin to A.

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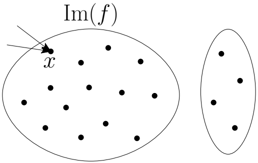

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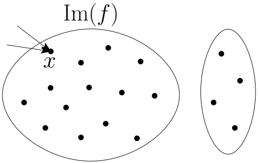

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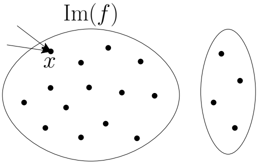

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Put two claims together.

Exists word sending some triple to a singleton of length

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$$\mathrm{rdv}(3, n) \leq \frac{3-\sqrt{5}}{4}n^2 + \frac{3}{2}n$$

$\square$

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A 'yes' would give an improvement on the Frankl–Pin bound for Černý's conjecture.

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If $k \geq 3$ and n suff. large there exists a (non-sync.) automaton where the shortest path from L_k to L_1 is of length $\geq \frac{4}{3}\left(\frac{n}{4k}\right)^{k-1}$.

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Construction for $k = 3$