k-recursive Sequences and their Asymptotic Analysis

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(Un-)bordered Factors

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- word w bordered:
  - exists non-empty word $v \neq w$
  - v is prefix and suffix of w
- otherwise unbordered

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bordered factor	border	length
00	0	2
11	1	2
010	0	3
101	1	3
1010	10	4
0110100110	0110	10

unbordered factor	length
ε	0
0	1
1	1
01	2
10	2
011	3
110	3
100	3
001	3

(Un-)bordered Factors

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  - exists non-empty word $v \neq w$
  - v is prefix and suffix of w
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bordered factor	border	length
$t[5..6] = 00$	0	2
$t[1..2] = 11$	1	2
$t[3..5] = 010$	0	3
$t[2..4] = 101$	1	3
$t[2..5] = 1010$	10	4
$t[0..9] = 0110100110$	0110	10

unbordered factor	length
ε	0
$t[0..0] = 0$	1
$t[1..1] = 1$	1
$t[0..1] = 01$	2
$t[2..3] = 10$	2
$t[0..2] = 011$	3
$t[1..3] = 110$	3
$t[4..6] = 100$	3
$t[5..7] = 001$	3

Thue–Morse Sequence

$$t = 01101001\,10010110\,10010110\,01101001\ldots$$

Number of Unbordered Factors

Theorem (Goč–Henshall–Shallit 2013)

exists unbordered factor of length n in Thue–Morse sequence $\iff$ $(n)_2 \notin 1(01^*0)^*10^*1$

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- number $f(n)$ of unbordered factors of length n in the Thue–Morse sequence

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(n)$	1	2	2	4	2	4	6	0	4	4	4	4	12	0	4	4

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Theorem (Goč–Mousavi–Shallit 2013)

- *inequality $f(n) \leq n$ holds for all $n \geq 4$*
- *$f(n) = n$ infinitely often*
- $\limsup_{n\geq 1} \frac{f(n)}{n} = 1$

Recurrence Relations

- number $f(n)$ of unbordered factors of length n in Thue–Morse sequence
- recurrence relations

$$
\begin{aligned}
f(4n) &= 2f(2n) && (n \geq 2)\\
f(4n+1) &= f(2n+1) && (n \geq 0)\\
f(8n+2) &= f(2n+1) + f(4n+3) && (n \geq 1)\\
f(8n+3) &= -f(2n+1) + f(4n+2) && (n \geq 2)\\
f(8n+6) &= -f(2n+1) + f(4n+2) + f(4n+3) && (n \geq 2)\\
f(8n+7) &= 2f(2n+1) + f(4n+3) && (n \geq 3)
\end{aligned}
$$

Theorem (Goč–Mousavi–Shallit 2013)

$f(n)$ satisfies recurrence relations above

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$$

- $f(n)$ is a 2-recursive sequence

k-recursive Sequence

- integer $k \geq 2$

k-recursive Sequence $x(n)$

there exist

- integers $M > m \geq 0$, $\ell \leq u$, $n_0 \geq \max\{-\ell/k^m, 0\}$
- constants $c_{s,j} \in \mathbb{C}$

such that

$$x(k^M n + s) = \sum_{\ell \leq j \leq u} c_{s,j}\, x(k^m n + j)$$

holds for all $n \geq n_0$ and $0 \leq s < k^M$

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- $h(n)$... largest power of 2 less than or equal to n
  - $h(2n) = 2h(n)$, $h(2n+1) = 2h(n)$ for $n \geq 1$, $h(1) = 1$
  - $k = 2$, $M = 1$, $m = 0$, $\ell = 0$, $u = 1$, $n_0 = 1$

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  - $k = 2$, $M = 1$, $m = 0$, $\ell = 0$, $u = 1$, $n_0 = 1$
- binary sum of digits
  - $s(2n) = s(n)$, $s(2n+1) = s(n) + 1$
  - no direct fit because of constant sequence
  - deal with inhomogeneities by increasing the exponents

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- number $f(n)$ of unbordered factors of length n in Thue–Morse sequence

k-regular Sequences

explicitly:

- there exist sequences $f_1(n)$, ... $f_s(n)$ such that
- for all $j \geq 0$, $0 \leq r < k^j$
- there exist c_1, ..., c_s
- with

$$f(k^j n + r) = c_1 f_1(n) + \cdots + c_s f_s(n)$$

k-regular Sequences

k-regular Sequence $f(n)$

k-kernel $\left\{ f(k^j n + r) \mid j \geq 0,\, 0 \leq r < k^j \right\}$
is contained in
finitely generated module

$\langle \dots \rangle$

explicitly:

- there exist sequences $f_1(n), \dots f_s(n)$ such that
- for all $j \geq 0$, $0 \leq r < k^j$
- there exist $c_1, \dots, c_s$
- with

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k-linear Representation

binary sum of digits $s(n)$:

- recurrence relations

even numbers: $s(2n) = s(n)$

odd numbers: $s(2n+1) = s(n) + 1$

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- vector-valued sequence

  set $v(n) = (s(n), 1)^T$

  even $v(2n) = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(n)$

  odd $v(2n+1) = \begin{pmatrix} s(n)+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v(n)$

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k-regular Sequence $f(n)$

- square matrices $M_0, \ldots, M_{k-1}$
- vectors u and w
- k-linear representation

  $$f(n) = u^T M_{n_0} M_{n_1} \ldots M_{n_{\ell-1}} w$$

  with standard k-ary expansion

  $$n = (n_{\ell-1} \ldots n_1 n_0)_k$$

Some k-regular Sequences

- $h(n)\dots$ largest power of 2 less than or equal to n
  - $h(2^j n + r) = 2^j h(n)$ for $n \geq 1$, $j \geq 0$, $0 \leq r < 2^j$
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- $h(n)\dots$ largest power of 2 less than or equal to n
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- binary sum of digits
- k-recursive sequences:

Theorem (Heuberger–K–Lipnik 2022)

- *k-recursive sequence $x(n)$*

Then

- *$x(n)$ is k-regular sequence*
- *k-linear representation of $x(n)$*
  - *vector-valued sequence $v(n)$ in block form*
  - *block matrices $M_0, \dots, M_{k-1}$*
  - *computed by coefficients of k-recursive sequence*
  - *explicit formulæ for the rows available*

Unboarded Factors: Coefficient Matrices

- number $f(n)$ of unbordered factors of length n in Thue–Morse sequence

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f(8n+7) &= 2f(4n+1) + f(4n+3)
\end{aligned}
\implies
B_0 = \begin{pmatrix} 2 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & -1 & 1 & 0 \end{pmatrix}
\quad
B_1 = \begin{pmatrix} 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\\ 0 & -1 & 1 & 1\\ 0 & 2 & 0 & 1 \end{pmatrix}
$$

Unboardered Factors: Linear Representation

- coefficient matrices B_0, B_1
- 2-linear representation of $f(n)$:

$$v = \begin{pmatrix} f \\ f \circ (n \mapsto 2n) \\ f \circ (n \mapsto 2n+1) \\ f \circ (n \mapsto 4n) \\ f \circ (n \mapsto 4n+1) \\ f \circ (n \mapsto 4n+2) \\ f \circ (n \mapsto 4n+3) \end{pmatrix}$$

$$M_0 = \begin{pmatrix} J_{00} & J_{01} \\ 0 & B_0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} J_{10} & J_{11} \\ 0 & B_1 \end{pmatrix}$$

  - J_{r0}, J_{r1} entries 0, 1

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  - J_{r0}, J_{r1} entries 0, 1
- initial value compensation $\rightsquigarrow$ 2-linear representation of $f(n)$:

$$\widetilde{M}_r = \begin{pmatrix} M_0 & W_0 \\ 0 & J_0 \end{pmatrix} \quad \text{and} \quad \widetilde{M}_1 = \begin{pmatrix} M_1 & W_1 \\ 0 & J_1 \end{pmatrix}$$

  - J_r entries 0, 1
  - W_r entries from initial values

Unboardered Factors: Linear Representation

- coefficient matrices B_0, B_1
- 2-linear representation of $f(n)$:

$$v = \begin{pmatrix} f \\ f \circ (n \mapsto 2n) \\ f \circ (n \mapsto 2n+1) \\ f \circ (n \mapsto 4n) \\ f \circ (n \mapsto 4n+1) \\ f \circ (n \mapsto 4n+2) \\ f \circ (n \mapsto 4n+3) \end{pmatrix} \qquad M_0 = \begin{pmatrix} J_{00} & J_{01} \\ 0 & B_0 \end{pmatrix} \qquad M_1 = \begin{pmatrix} J_{10} & J_{11} \\ 0 & B_1 \end{pmatrix}$$

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  - J_r entries 0, 1
  - W_r entries from initial values
- minimization algorithm: dimension 10 $\rightsquigarrow$ dimension 8

Asymptotics of Partial Sums

- k-regular sequence $f(n)$
- partial sums $F(N) = \sum_{n<N} f(n)$

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Theorem (Heuberger–K–Prodinger 2018, Heuberger–K 2020)

$$F(N) = \sum_{\substack{\lambda\in\sigma(M_0+\cdots+M_{k-1})\\ |\lambda|>\rho}} N^{\log_k \lambda} \sum_{0\le\ell<m(\lambda)} (\log_k N)^\ell \Phi_{\lambda\ell}(\{\log_k N\}) + O(N^{\log_k R}(\log N)^{\widehat{m}})$$

- *1-periodic (Hölder) continuous functions* $\Phi_{\lambda\ell}$
- *functional equation*

$$\Big(I - \frac{1}{k^s}(M_0+\cdots+M_{k-1})\Big)\mathcal{V}(s) = \sum_{n=1}^{k-1}\frac{v(n)}{n^s} + \frac{1}{k^s}\sum_{r=0}^{k-1} M_r \sum_{\ell\ge1}\binom{-s}{\ell}\Big(\frac{r}{k}\Big)^\ell \mathcal{V}(s+\ell)$$

- *meromorphic continuation on the half plane* $\Re s > \log_k R$
- *Fourier series* $\Phi_{\lambda\ell}(u) = \sum_{h\in\mathbb{Z}} \varphi_{\lambda\ell h} \exp(2\ell\pi i u)$

$$\varphi_{\lambda\ell h} = \frac{(\log k)^\ell}{\ell!} \operatorname{Res}\Bigg(\frac{\big(f(0)+\mathcal{F}(s)\big)\big(s-\log_k\lambda-\frac{2h\pi i}{\log k}\big)^\ell}{s}, s = \log_k \lambda + \frac{2h\pi i}{\log k}\Bigg)$$

Binary Sum-of-Digits Function: Analysis

- eigenvalues etc.
  - $C = M_0 + M_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$
  - C has eigenvalue $\lambda = 2$ with multiplicity 2
  - joint spectral radius 1
  - $\|M_{r_1} \cdots M_{r_\ell}\| = O\bigl(R^\ell\bigr)$ for any $R > 1$
- $\rightsquigarrow$ analysis of summatory function:
  - $S(N) = N(\log_2 N)\,\Phi_{21}(\{\log_2 N\}) + N\,\Phi_{20}(\{\log_2 N\})$
  - 1-periodic continuous functions Φ_{21} and Φ_{20}
  - $\Phi_{21}(u) = \frac{1}{2}$ via functional equation
  - no error term
- recovering:

Summatory Binary Sum-of-Digits (Delange 1975)

$$S(N) = \sum_{n<N} s(n) = \tfrac{1}{2} N \log_2 N + N\,\Phi_{20}(\{\log_2 N\})$$

- explicit Fourier coefficients of $\Phi_{20}(u)$

Joint Spectral Radius

- finite set S of $n \times n$ matrices
- $\|\cdot\|$ any matrix norm

Joint Spectral Radius

$$\rho(S) = \lim_{\ell \to \infty} \max\{\|F_1 \cdots F_\ell\|^{1/\ell} \mid F_i \in S\}$$

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- example $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$
  - $\rho(S) \leq 1$ (maximal spectral norm)
  - $\rho(S) \geq 1$ (joint eigenvalue 1)
  - $\rightsquigarrow \rho(S) = 1$

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  - $\rho(S) \geq 1$ (joint eigenvalue 1)
  - $\rightsquigarrow \rho(S) = 1$
- approximation algorithms available

Unboardered Factors: towards Asymptotics

Asymptotics of k-recursive Sequences

Only properties of coefficient matrices needed from k-linear representation!

- coefficient matrices

$$B_0 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

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- spectrum

$$\sigma(B_0 + B_1) = \left\{1 - \sqrt{3}, 1, 2, 1 + \sqrt{3}\right\}$$

Unboardered Factors: towards Asymptotics

Asymptotics of k-recursive Sequences

Only properties of coefficient matrices needed from k-linear representation!

- coefficient matrices

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- spectrum

$$\sigma(B_0 + B_1) = \left\{1 - \sqrt{3}, 1, 2, 1 + \sqrt{3}\right\}$$

- joint spectral radius of $\{B_0, B_1\}$ is 2
- has simple growth property

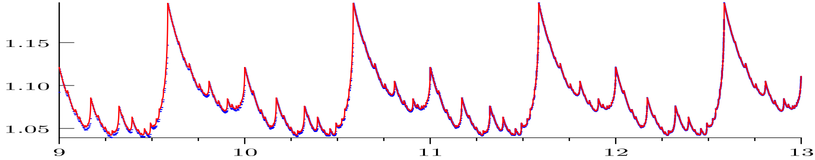
Unboardered Factors: Asymptotics

- number $f(n)$ of unbordered factors of length n in the Thue–Morse sequence

Theorem (Heuberger–K–Lipnik 2022)

$$F(N) = \sum_{0 \le n < N} f(n) = N^{\kappa} \cdot \Phi_F(\{\log_2 N\}) + O(N \log N) \quad \textit{as } N \to \infty$$

- $\kappa = \log_2(1+\sqrt{3}) = 1.44998431347650\ldots$
- *1-periodic continuous function* Φ_F*,*
  Hölder continuous with any exponent smaller than $\kappa - 1$
- *explicit functional equation for Dirichlet series* *+ analyticity properties, poles*
- *efficiently computable Fourier coefficients of* Φ_F



Stern's Diatomic Sequence

Stern's Diatomic Sequence

$$d(2n) = d(n)$$
$$d(2n+1) = d(n) + d(n+1)$$

for all $n \geq 0$ and

$$d(0) = 0, \; d(1) = 1$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d(n)$	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4

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$$d(2n) = d(n)$$
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for all $n \geq 0$ and
$$d(0) = 0,\ d(1) = 1$$

- number of different hyperbinary representations *(Northshield 2010)*
- number of integers $r \in \mathbb{N}_0$ such that Stirling partition numbers $\left\{ {n \atop 2r} \right\}$ are even and non-zero *(Carlitz 1964)*
- number of different representations as a sum of distinct Fibonacci numbers F_{2k} *(Bicknell-Johnson 2003)*
- number of different alternating bit sets *(Finch 2003)*
- relation to the Towers of Hanoi *(Hinz–Klavžar–Milutinović–Parisse–Petr 2005)*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d(n)$	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4

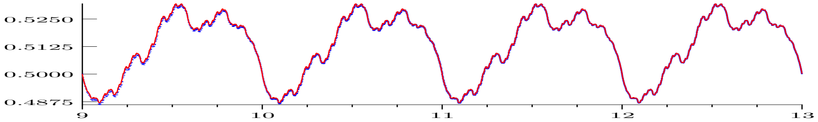
Stern's Diatomic Sequence: Asymptotics

- Stern's diatomic sequence $d(n)$
- 2-recursive, 2-regular

Asymptotics

$$D(N) = \sum_{0 \le n < N} d(n) = N^{\kappa} \cdot \Phi_D(\{\log_2 N\}) + O(N^{\log_2 \varphi}) \quad \text{as } N \to \infty$$

-
  - $\kappa = \log_2 3 = 1.5849625007211\ldots$
  - $\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887498\ldots$, $\log_2 \varphi = 0.69424191363061\ldots$
  - 1-periodic continuous function Φ_D,
    Hölder continuous with any exponent smaller than $\kappa - \log_2 \varphi$
-
  - explicit functional equation for Dirichlet series
    $+$ analyticity properties, poles
  - efficiently computable Fourier coefficients of Φ_D



Generalized Pascal's Triangle

Binomial Coefficients of Words

binomial coefficient $\binom{u}{v}$ equals
number of different occurrences of v
as a scattered subword of u

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	k	0	1	2	3	4	5	6	7	8	
n	$(n)_2$ \ $(k)_2$	ε	1	10	11	100	101	110	111	1000	$z(n)$
0	ε	**1**	0	0	0	0	0	0	0	0	1
1	1	**1**	**1**	0	0	0	0	0	0	0	2
2	10	1	1	1	0	0	0	0	0	0	3
3	11	**1**	**2**	0	**1**	0	0	0	0	0	3
4	100	1	1	2	0	1	0	0	0	0	4
5	101	1	2	1	1	0	1	0	0	0	5
6	110	1	2	2	1	0	0	1	0	0	5
7	111	**1**	**3**	0	**3**	0	0	0	**1**	0	4
8	1000	1	1	3	0	3	0	0	0	1	5

Generalized Pascal's Triangle

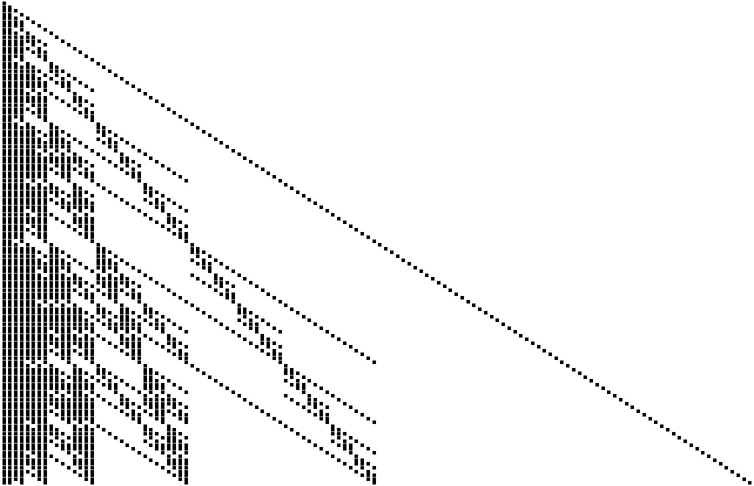
Binomial Coefficients of Words

binomial coefficient $\binom{u}{v}$ equals
number of different occurrences of v
as a scattered subword of u

"classical"
binomial coefficient
$\binom{n}{k} = \binom{1^n}{1^k}$

n	k / $(k)_2$ \ $(n)_2$	0: ε	1: 1	2: 10	3: 11	4: 100	5: 101	6: 110	7: 111	8: 1000	$z(n)$
0	ε	**1**	0	0	0	0	0	0	0	0	1
1	1	**1**	**1**	0	0	0	0	0	0	0	2
2	10	1	1	1	0	0	0	0	0	0	3
3	11	**1**	**2**	0	**1**	0	0	0	0	0	3
4	100	1	1	2	0	1	0	0	0	0	4
5	101	1	2	1	1	0	1	0	0	0	5
6	110	1	2	2	1	0	0	1	0	0	5
7	111	**1**	**3**	0	**3**	0	0	0	**1**	0	4
8	1000	1	1	3	0	3	0	0	0	1	5

Non-zeros in Generalized Pascal's Triangle



Stern's Diatomic Sequence & Generalized Pascal's Triangle

- Stern's diatomic sequence $d(n)$
- number $z(n)$ of non-zero elements in nth row of generalized Pascal's triangle $\binom{(n)_2}{(k)_2}$

Theorem (Leroy–Rigo–Stipulanti 2017)

$z(n) = d(2n+1)$ *for all* $n \geq 0$

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Theorem (Leroy–Rigo–Stipulanti 2017)

recurrence relations

$$\begin{aligned} z(2n+1) &= 3z(n) - z(2n) \\ z(4n) &= -z(n) + 2z(2n) \\ z(4n+2) &= 4z(n) - z(2n) \end{aligned}$$

for all $n \geq 0$

Generalized Pascal's Triangle: Asymptotic Analysis

- reformulate as 2-recursive sequence:

$$\begin{aligned} z(4n) &= \tfrac{5}{3}z(2n) - \tfrac{1}{3}z(2n+1) \\ z(4n+1) &= \tfrac{4}{3}z(2n) + \tfrac{1}{3}z(2n+1) \\ z(4n+2) &= \tfrac{1}{3}z(2n) + \tfrac{4}{3}z(2n+1) \\ z(4n+3) &= -\tfrac{1}{3}z(2n) + \tfrac{5}{3}z(2n+1) \end{aligned}$$

Generalized Pascal's Triangle: Asymptotic Analysis

- reformulate as 2-recursive sequence & read off coefficient matrices:

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- 2-linear representation of dimension 3

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- towards asymptotics: spectrum & joint spectral radius
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Generalized Pascal's Triangle: Asymptotic Analysis

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- 2-linear representation of dimension 3
- towards asymptotics: spectrum & joint spectral radius
- investigating eigenstructure ⇝ no error term
- reconsider connection to Stern's diatomic sequence
- (compute Fourier coefficients of fluctuation)

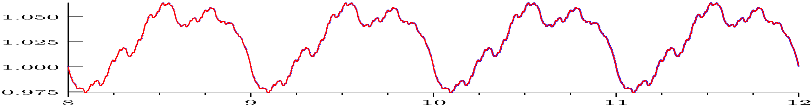
Generalized Pascal's Triangle: Asymptotics

- number $z(n)$ of non-zero elements in nth row of generalized Pascal's triangle $\binom{(n)_2}{(k)_2}$

Asymptotics

$$Z(N) = \sum_{0 \le n < N} z(n) = N^{\kappa} \cdot \Phi_Z(\{\log_2 N\}) \quad \text{for } N \ge 1$$

- $\kappa = \log_2 3$
- $\Phi_Z = 2\Phi_D$ (with Φ_D from Stern's diatomic sequence)



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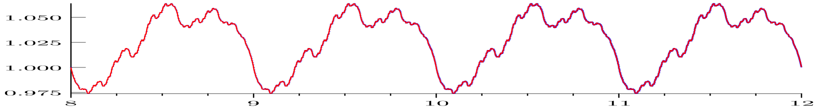

Corollary (Heuberger–K–Lipnik 2022)

- *Stern's diatomic sequence $d(n)$*

$$\sum_{0 \le n < N} d(n) + \frac{1}{2} d(N) = N^{\kappa} \cdot \Phi_D(\{\log_2 N\})$$