Scattered Factor Universality - Investigating Simon's Congruence

Pamela Fleischmann

One World Combinatorics on Words Seminar 2023

m-Nearly k-Universality

What is a Scattered Factor?

Example

`palindrome`

What is a Scattered Factor?

Example

palindrome

palm

What is a Scattered Factor?

Example

palindrome

palm

and

What is a Scattered Factor?

palindrome

palm

&

and

lime

What is a Scattered Factor?

Example

palindrome

palm and lime dome

Scattered Factors

Definition.
A word $u = u[1] \cdots u[n] \in \Sigma^*$ scattered factor of $v \in \Sigma^*$ if

$$\exists x_1, \ldots x_{n+1} \in \Sigma^* : \; v = x_1 u[1] x_2 u[2] \cdots x_n u[n] x_{n+1}$$

Simon's Congruence

$$\sim_k$$

Simon's Congruence

abaa $\sim_3$ abaaa?

$$\sim_k$$

Simon's Congruence

$$\mathtt{abaa} \sim_3 \mathtt{abaaa}?$$

$$\mathrm{ScatFact}_3(\mathtt{abaa}) = \{\mathtt{aaa}, \mathtt{aba}, \mathtt{baa}\}$$

$\sim_k$

Simon's Congruence

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✓

$\sim_k$

$\mathtt{abba} \sim_3 \mathtt{baab}?$

Simon's Congruence

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abba $\sim_3$ baab?

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Simon's Congruence

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Simon's Congruence

Definition.
The words u and v are Simon congruent modulo $k \in \mathbb{N}_0$ if

$$\mathrm{ScatFact}_\ell(u) = \mathrm{ScatFact}_\ell(v) \quad \text{for all } \ell \leq k$$

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- what is the index $|\Sigma^* / \sim_k|$ for a fixed $k \in \mathbb{N}$?

Simon's Congruence

Motivated by the classical universality problem for languages:

Motivation for Universality

Motivated by the classical universality problem for languages:

Problem.
Given: $w \in \Sigma^*$, $k \in \mathbb{N}_0$
Goal: Decide whether $\mathrm{ScatFact}_k(w) = \Sigma^k$?

Scattered Factor Universality

Definition.
A word $w \in \Sigma^*$ is called *k-universal* if

$$\mathrm{ScatFact}_k(w) = \Sigma^k.$$

- $\iota(w)$ largest number k such that $\mathrm{ScatFact}_k(w) = \Sigma^k$

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- this is only a small part of the way towards the index, thus...

Scattered Factor Universality

Definition.
A word $w \in \Sigma^*$ is called *m-nearly k-universal* if

$$|\operatorname{ScatFact}_k(w)| = |\Sigma|^k - m.$$

m-nearly *k*-Universality

- abacdbaacdbadbacba is 3-universal

m-nearly *k*-Universality

Examples

- abacdbaacdbadbacba is 3-universal
  - let's write it more conveniently: abacd.baacd.badbac.ba

m-nearly *k*-Universality

- abacdbaacdbadbacba is 3-universal
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- abacdbaacdbacbaba is nearly 3-universal

m-nearly *k*-Universality

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  - let’s write it more conveniently, too: abacd.baacd.bacbaba

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  - ddd is indeed absent!

m-nearly *k*-Universality

Examples

- abacdbaacdbadbacba is 3-universal
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- abacdbaacdbacbaba is nearly 3-universal
  - let's write it more conveniently, too: abacd.baacd.bacbaba
  - ddd is indeed absent!
  - why is it the only one?

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

Arch Factorisation (Hebrard)

aabacbcbcabcabc

a b c

✓

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a	b	c
✓	✓	

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Arch Factorisation (Hebrard)

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a	b	c
✓	✓	✓

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a b c

✓ ✓ ✓

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

✓

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

✓ ✓

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

✓ ✓ ✓

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

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"Formalisation"

aabacbcbcabcabc

a b c

✓

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"Formalisation"

aabacbcbcabcabc

a b c

✓ ✓

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

✓ ✓

Rest

$r(w) = bc$

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

✓ ✓

# of Arches $\iota(w) = 3$

Rest $\mathrm{r}(w) = \mathrm{bc}$

Arch Factorisation (Hebrard)

"Formalisation"

aabacbcbcabcabc

a b c

✓ ✓

# of Arches	Rest	Modus
$\iota(w) = 3$	$\mathrm{r}(w) = \mathrm{bc}$	m(aabacbcbcabcabc) =

Arch Factorisation (Hebrard)

"Formalisation"

$$\underbrace{\text{aaba}}\underbrace{\text{cbcbca}}\underbrace{\text{bca}}\text{bc}$$

a b c

✓ ✓

# of Arches	Rest	Modus
$\iota(w) = 3$	$\mathrm{r}(w) = \mathrm{bc}$	$\mathrm{m}(\text{aabacbcbcabcabc}) = \text{caa}$

k-universal Words

Theorem.
A word $w \in \Sigma^*$ is k-universal iff $\iota(w) \geq k$.

k-universal Words

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A word $w \in \Sigma^*$ is k-universal iff $\iota(w) \geq k$.

Corollary.
All words with $\geq k$ arches are in one congruence class w.r.t. $\sim_k$.

Nearly k-universal Words

Is it possible to have only $k-2$ arches?

Nearly k-universal Words

Considerations

Is it possible to have only $k-2$ arches?

Nearly k-universal Words

Is it possible to have only $k-2$ arches?

- $a_1a_2a_3a_4a_5ba_1$ and $a_1a_2a_3a_4a_5ba_2$ not scattered factors of length 7!

Nearly k-universal Words

Considerations

Is it possible to have only $k-2$ arches?

- $a_1 a_2 a_3 a_4 a_5 b a_1$ and $a_1 a_2 a_3 a_4 a_5 b a_2$ not scattered factors of length 7!

Nearly k-universal words have $k-1$ arches.

Nearly *k*-universal Words

Is it possible to have two or more missing letters in the rest?

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Nearly k-universal Words

Is it possible to have two or more missing letters in the rest?

- $a_1a_2a_3a_4a_5a$ and $a_1a_2a_3a_4a_5b$ not scattered factors of length 6!

Nearly k-universal Words

Considerations

Is it possible to have two or more missing letters in the rest?

- $a_1a_2a_3a_4a_5a$ and $a_1a_2a_3a_4a_5b$ not scattered factors of length 6!

Nearly k-universal words have $|\Sigma| - 1$ letters in the rest.

Nearly k-universal Words

Do we have w nearly k-universal iff $\iota(w) = k - 1$ and $|\mathrm{alph}(r(w))| = |\Sigma| - 1$?

Nearly k-universal Words

Characterisation?

Do we have w nearly k-universal iff $\iota(w) = k - 1$ and $|\mathrm{alph}(r(w))| = |\Sigma| - 1$?

cbbacabbccaab

- 3 arches

Nearly k-universal Words

Characterisation?

Do we have w nearly k-universal iff $\iota(w) = k - 1$ and $|\mathrm{alph}(r(w))| = |\Sigma| - 1$?

$$\underbrace{\mathrm{cbba}}\underbrace{\mathrm{cabb}}\underbrace{\mathrm{cca}}\mathrm{ab}$$

- 3 arches
- $r(w) = \{\mathtt{ab}\} = \Sigma \setminus \{\mathtt{c}\}$

Nearly k-universal Words

Characterisation?

Do we have w nearly k-universal iff $\iota(w) = k - 1$ and $|\mathrm{alph}(r(w))| = |\Sigma| - 1$?

cbbacabbccaab

- abac absent

Nearly k-universal Words

Characterisation?

Do we have w nearly k-universal iff $\iota(w) = k - 1$ and $|\mathrm{alph}(r(w))| = |\Sigma| - 1$?

cbbacabbccaab

- abac absent
- aaac absent

Nearly *k*-Universal

bcbaaccbabcabacbcbaac

Nearly *k*-Universal

Example

bcbaaccbabcabacbcbaac

Nearly *k*-Universal

Example

bcbaaccbabcabacbcbaac

Nearly *k*-Universal

Example

bcbaaccb ab cabacbcbaac

Nearly k-Universal

bcbaaccbabc ab acbcbaac

Nearly k-Universal

bcbaaccbabcabac bc baac

Nearly k-Universal

bcbaaccbabcabacbcba ac

Nearly k-Universal

Theorem.
A word $w \in \Sigma^*$ is nearly k-universal iff

1. $\iota(w) = k - 1$
2. for all $u \in \mathrm{PerfUniv}_{k_1}$ and all $v \in \mathrm{PerfUniv}_{k_2}$ with $k = k_1 + k_2 + 1$ and $x \in \Sigma^*$ with $w = uxv^R$ we have $|\mathrm{alph}(x)| = |\Sigma| - 1$

Each word in Σ^k determines a congruence class w.r.t. $\sim_k$.

Nearly k-Universal Words

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$$|u| = |\mathtt{abccab}| = 6$$

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Each word in Σ^k determines a congruence class w.r.t. $\sim_k$.

$|u| = |\mathtt{abccab}| = 6$ → 5 arches and rest

Nearly k-Universal Words

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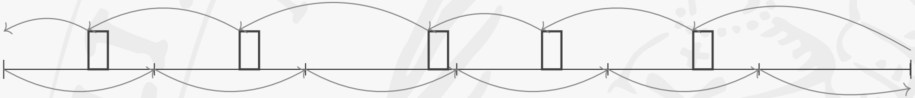
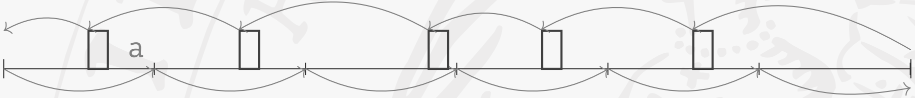


$|u| = |\mathtt{abccab}| = 6$ ⟹ 5 arches and rest

Shortlex Normal Form

$u = \text{abccab}$



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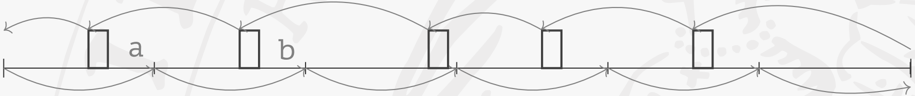

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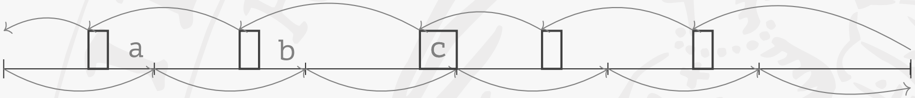

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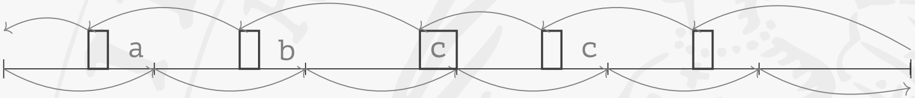

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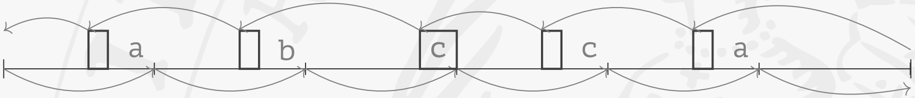

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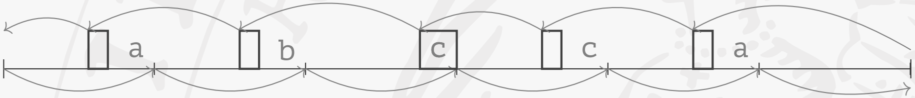

$\mathtt{b} \notin \mathrm{r}(w)$

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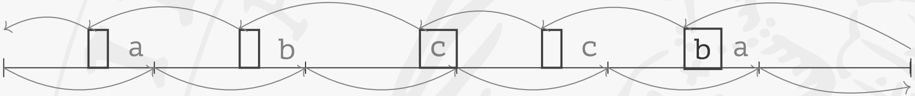

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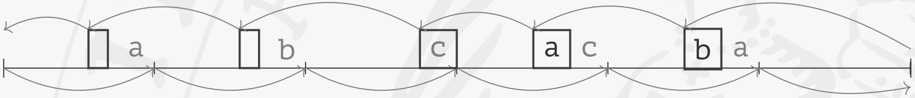

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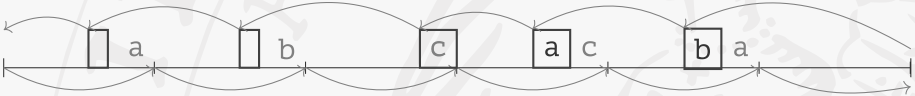

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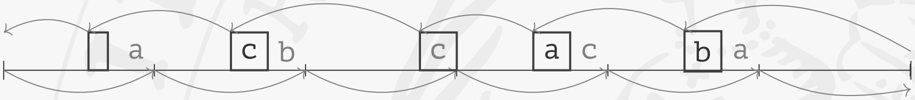

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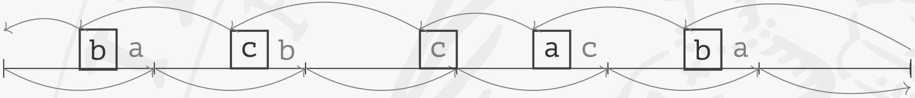

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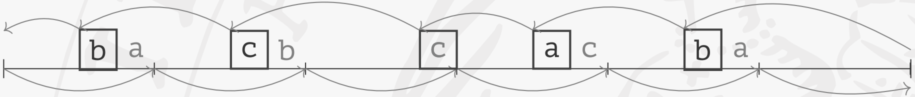

$\text{a} \notin \text{r}(w^R)$

$\text{b} \notin \text{r}(w)$

Shortlex Normal Form

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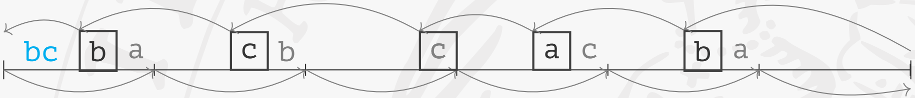

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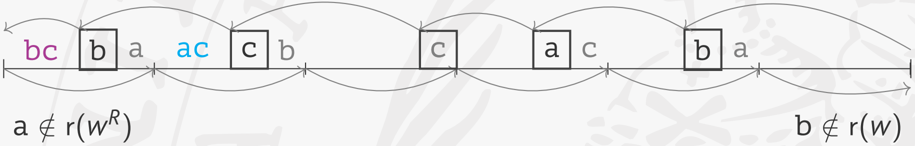

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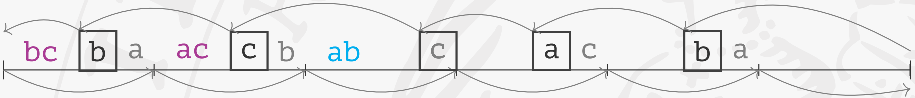

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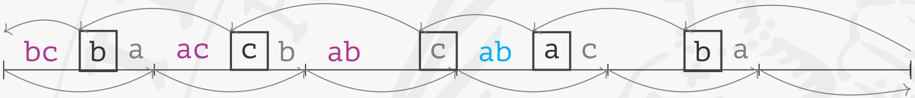

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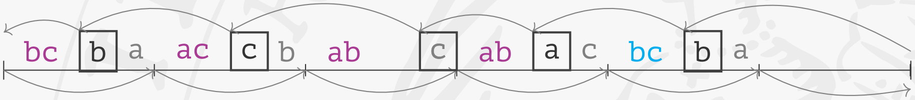

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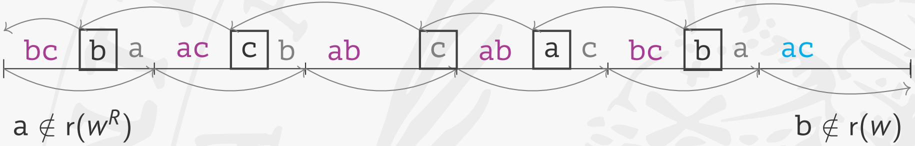

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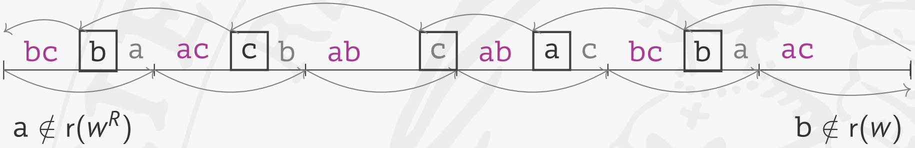
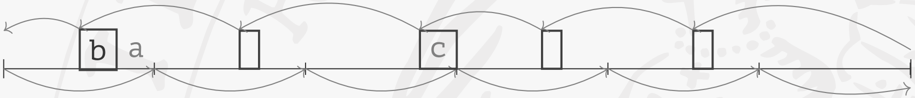


bcba.accb.abc.abac.bcba.ac is the shortlex normal form for abccab.

Shortlex Normal Form (faster)

$u = \text{abccab}$



$a \notin r(w^R)$

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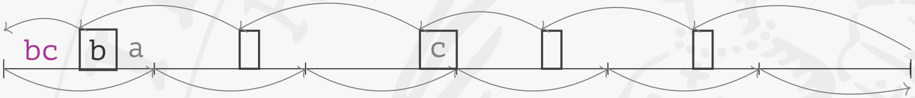

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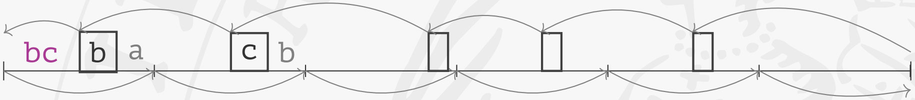

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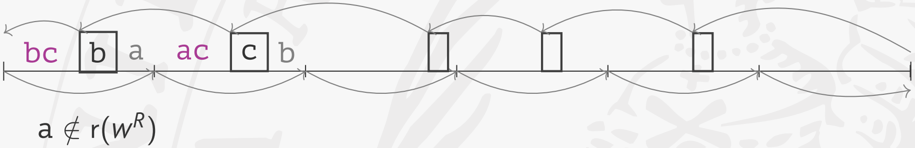

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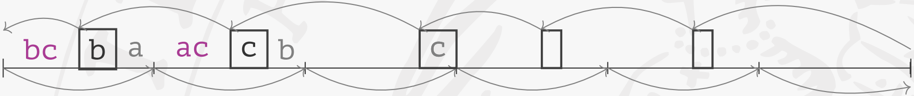

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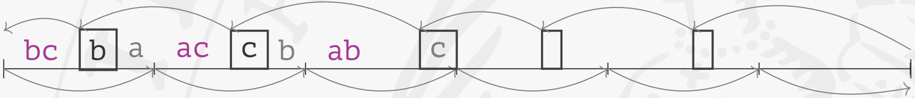

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Shortlex Normal Form (faster)

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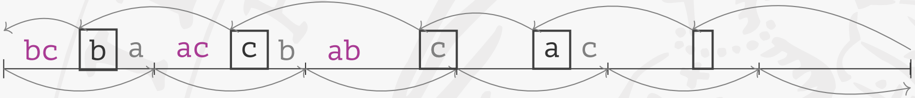

$\mathsf{a} \notin \mathsf{r}(w^R)$

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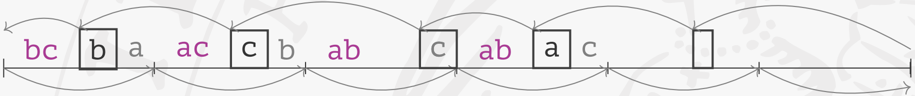

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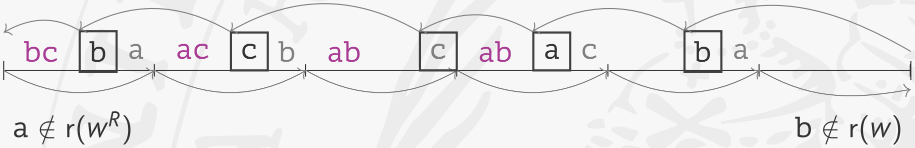

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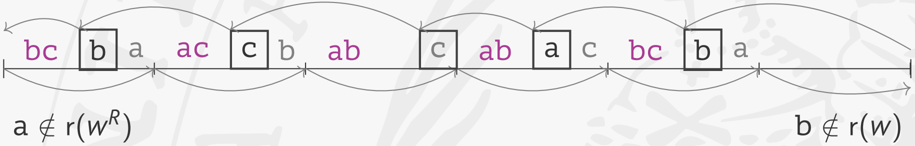

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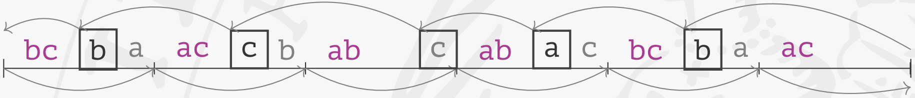

$a \notin r(w^R)$

$b \notin r(w)$

Shortlex Normal Form (faster)

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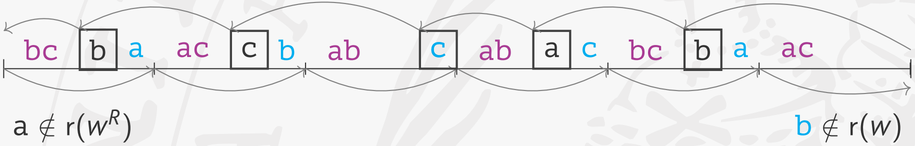

$\alpha\beta$-Factorisation

$\alpha\beta$-factorisation

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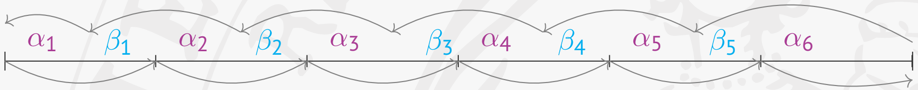

- first and last letter in β_i unique

- α_i to fill up arches with β_{i-1} and β_i
- letters in α_i arbitrarily often and permuted

2-Nearly k-Universal

Still nice

Theorem.
A word $w \in \Sigma^*$ is 2-nearly k-universal iff

1. $\iota(w) = k - 1$,
2. there exists $i \in [k]$ such that for all $j \in [k] \setminus \{i\}$ we have
   - $|\text{alph}(\alpha_i(w))| = |\Sigma| - 2$ and
   - $|\text{alph}(\alpha_j(w))| = |\Sigma| - 1$.

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Corollary.
The 2-nearly k-universal words contribute $k\binom{|\Sigma|}{2}|\Sigma|^{k-1}$ congruence classes to $\Sigma^*/\sim_k$.

3-Nearly k-Universal

$\Sigma = \{a, b\}, k = 2$

aaaa . . .

3-Nearly k-Universal

$\Sigma = \{a, b\}, k = 2$

aaaa . . .

- ab, ba, and bb absent

3-Nearly k-Universal

$\Sigma = \{a, b\}, k = 2$

$$\mathtt{aaaa}\ldots$$

- ab, ba, and bb absent
- $\iota(w) = 0$ (**not** $k - 1$!)

3-Nearly k-Universal

Theorem.
A word $w \in \Sigma^*$ is 3-nearly k-universal iff

1. either $w \in x^2x^*$ for $k = 2$, $|\Sigma| = 2$, and $x \in \Sigma$ or

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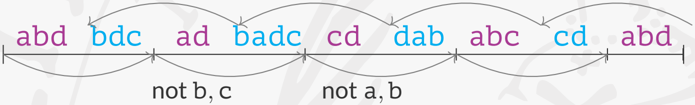
   - there exists $i \in [k-1]$ with $|\mathrm{alph}(\alpha_i(w))| = |\alpha_{i+1}(w)| = |\Sigma| - 2$ and the concatenation of one pair of letters is a scattered factor of $\beta_i(w)$; all other α_j miss exactly one letter.

3-Nearly *k*-Universal

Example



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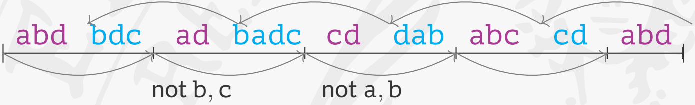
Example

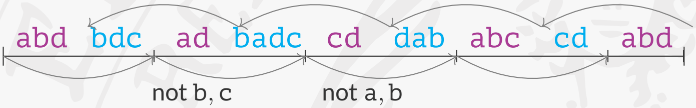


absent:

- ccbdc

3-Nearly *k*-Universal

Example



absent:

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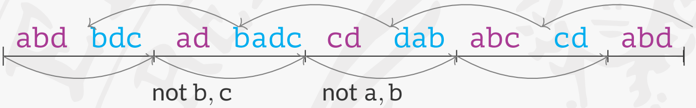
- ccadc

3-Nearly *k*-Universal

Example



absent:

- ccbdc
- ccadc
- cbbdc

Repetition

k-Universality

Theorem.
A word $w \in \Sigma^*$ is k-universal iff

$$\mathrm{ScatFact}_k(w) = \mathrm{ScatFact}_k(w^2).$$

k-Universality

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A word $w \in \Sigma^*$ is k-universal iff

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- $\Rightarrow \surd$
- $\Leftarrow w \geq \frac{k}{2}$ arches $\Rightarrow \surd$
- $\Leftarrow w < \frac{k}{2}$ arches $\Rightarrow m(w)\bar{r}m(w)$ or $m(w)m(w) \notin \mathrm{ScatFact}_{\leq k}(w)$ ↯

Universality of Repetitions

- $\iota(w) = k \Rightarrow w^n$ has at least kn arches

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aabb

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$$\text{aabb}\,|\,\text{aabb}$$

Universality of Repetitions

- $\iota(w) = k \Rightarrow w^n$ has at least kn arches

$$\texttt{aabb}|\texttt{aabb}$$

when do we have the additional arch?

Circular k-Universality

Definition.
A word $w \in \Sigma^*$ is circular k-universal ($\zeta(w) = k$) if a conjugate of w is k-universal.

Example

$\Sigma = \{a, b, c, d\}$ and

abbccdabacdbdc

Example

$\Sigma = \{a, b, c, d\}$ and

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- not 3-universal: dda is missing

Example

$\Sigma = \{a, b, c, d\}$ and

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- 2-universal

Example

$\Sigma = \{a, b, c, d\}$ and

abbccdabacdbdc·a

- not 3-universal: dda is missing
- 2-universal
- a conjugate is 3-universal

Example

$\Sigma = \{a, b, c, d\}$ and

$$\mathtt{abbccdabacdbdc \cdot a}$$

- not 3-universal: dda is missing
- 2-universal
- a conjugate is 3-universal ($u, v \in \Sigma^*$ are conjugates iff there exist $x, y \in \Sigma^*$ with $u = xy$ and $v = yx$)

Repetitions

Definition.

- $\iota_w(s) = \iota(w^s)$ and $\zeta_w(s) = \zeta(w^s)$

Repetitions

$\iota(w^s), \zeta(w^s)$

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- $\iota_w(s) = \iota(w^s)$ and $\zeta_w(s) = \zeta(w^s)$
- $\nabla \iota_w(s) = \iota_w(s) - \iota_w(s-1)$ (growth of the universality w.r.t. powers)

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Theorem.

$w \in \Sigma^{\geq k}$, $s \in \mathbb{N}$

- if $\zeta(w) = \iota(w) + 1$ then $\iota(w^s) = s \cdot \iota(w) + s - 1$

Repetitions

$\iota(w^s), \zeta(w^s)$

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- if $\zeta(w) = \iota(w) + 1$ then $\iota(w^s) = s \cdot \iota(w) + s - 1$
- if $|\Sigma| = 2$ then $\begin{cases} \iota(w^s) = s \cdot \iota(w) + s - 1, & \text{if } \zeta(w) = \iota(w) + 1, \\ \iota(w^s) = s \cdot \iota(w), & \text{otherwise} \end{cases}$

Example

babccaabc

- $\iota(w) = \zeta(w) = 2$

Example

babccaabc

- $\iota(w) = \zeta(w) = 2$
- $\iota(w^2) = 5 \quad (2k+1)$

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- $\iota(w^3) = 7 \quad (3k+1)$

babccaabc

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- $\iota(w^3) = 7$ (3k+1)

When do we have $\nabla\iota_w(s) = k$ and when $\nabla\iota_w(s) = k + 1$

How to get an Equivalence?

babccaabc ba bccaabc

$\underbrace{\text{ba}}_{p}$

How to get an Equivalence?

$$\underbrace{\text{babccaabc}}\,\underbrace{\text{ba}}_{p}\,\text{bccaabc}$$

- $\iota(w^2) = \iota(w) + \iota(p^{-1}w)$

How to get an Equivalence?

$$\mathtt{babccaabc}\underbrace{\mathtt{ba}}_{p}\mathtt{bccaabc}$$

- $\iota(w^2) = \iota(w) + \iota(p^{-1}w)$
- $p = r(w^R)^R$ is longest prefix of w with $\iota(p^{-1}w) = \iota(w)$

Characterisation of the Growth

$$w^s = w^{s-1}w$$

Proposition.
$\iota(wu) = \iota(w) + \iota(u) + 1$ iff $\mathrm{alph}(r(w)r(u^R)) = \Sigma$

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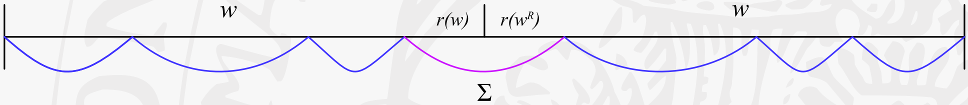
Corollary.
$\nabla\iota_w(s) = \iota(w) + 1$ iff $\text{alph}(w^{s-1}r(w^R)) = \Sigma$



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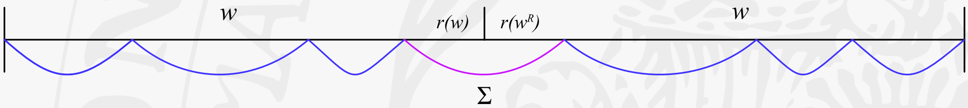

$\rightsquigarrow$ $\nabla\iota_w$ and $s \mapsto r(w^s)$ depend on each other

Remainder Function

$s \mapsto r(w^s)$

- notice $r(w^s) = r(r(w^{s-1}w))$ $\quad$ ($r(w^s) = r(uw)$ for some u)

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  (the converse is not true!)

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Proposition.
The growth of the universality index, $\nabla \iota_w$, is eventually periodic.

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Notice: $|\{\text{alph}(r(w^s)) \mid s \in \mathbb{N}_0\}| \leq |\Sigma|$

Periodicity

Theorem.
For all $w \in \Sigma^*$ there exist $s, t \in [|\Sigma|]$ with $s < t$ such that

1. $r(w^{s+i}) = r(w^{t+i})$ for all $i \in \mathbb{N}$,

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- $\rightsquigarrow$ beginning at $s+1$, $\nabla\iota_w$ has period $t-s$

Remainder Function II

$s \mapsto r(w^s)$

- we investigated $w^s = w^{s-1}w$

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- what if we change to $w^s = ww^{s-1}$?

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Lemma.
$r(w^s) \neq r(w^{s+1})$ then

- $\nabla \iota_w(s+1) = \iota(w)$ iff $r(w^s)$ is suffix of $r(w^{s+1})$

Remainder Function II

- we investigated $w^s = w^{s-1}w$
- what if we change to $w^s = ww^{s-1}$?

Lemma.
$r(w^s) \neq r(w^{s+1})$ then

- $\nabla\iota_w(s+1) = \iota(w)$ iff $r(w^s)$ is suffix of $r(w^{s+1})$
- $\nabla\iota_w(s+1) = \iota(w) + 1$ iff $r(w^{s+1})$ is suffix of $r(w^s)$

Remainder Function II

Corollary.
$\text{alph}(r(w^s)) \neq \text{alph}(r(w^{s+1}))$ then

- $\nabla \iota_w(s+1) = \iota(w)$ iff $\text{alph}(r(w^s)) \subset \text{alph}(r(w^{s+1}))$

Remainder Function II

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When is $s \mapsto r(w^s)$ eventually constant? ($\sim$ is the corollary applicable?)

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Lemma.
$s \mapsto r(w^s)$ eventually constant iff $\nabla \iota_w$ eventually constant

Remainder Function II

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Corollary.
$\zeta(w) = \iota(w) + 1$ then $s \mapsto r(w^s)$ eventually constant

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Lemma.
$s \mapsto r(w^s)$ eventually constant iff $\nabla\iota_w$ eventually constant

Corollary.
$\zeta(w) = \iota(w) + 1$ then $s \mapsto r(w^s)$ eventually constant

$\rightsquigarrow$ $\nabla\iota_w(s) = k$ on an interval $[\ell + 1, n]$ then $\mathrm{alph}(r(w^\ell)) \subseteq \ldots \subseteq \mathrm{alph}(r(w^n))$
(equivalence if the chain is strict)

Chains

- $\text{alph}(r(w^{\ell})) \subset \ldots \subset \text{alph}(r(w^{\ell+|\Sigma|+1}))$ implies $|\text{alph}(r(w^{s}))| = s - \ell$ for all $s \in [\ell, \ell + |\Sigma| - 1]$

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- $\rightsquigarrow$ strictly ascending chains of length $|\Sigma| + 1$ cannot exist

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Proposition.

- $\nabla\iota_w(s) = \iota(w)$ for all $s \in [1, |\Sigma|]$ implies $\nabla\iota_w(s) = \iota(w)$ for all $s \in \mathbb{N}$

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- $\nabla\iota_w(s) = \iota(w)$ for all $s \in \mathbb{N}$ iff $\nabla\zeta_w(s) = \iota(w)$ for all $s \in [1, |\Sigma| - 1]$

Generalisation I

Theorem.
$\iota(w) > 0$

- $\zeta(w) = \iota(w) + 1$ implies $\iota(w^s) = s\iota(w) + s - 1$

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- $\zeta(w) = \iota(w) + 1$ implies $\iota(w^s) = s\iota(w) + s - 1$
- $\nabla\zeta_w(t) = \iota(w)$ for all $t \in [1, |\Sigma| - 1]$ implies $\iota(w^s) = s\iota(w)$

Generalisation II

- $\mathrm{alph}(r(w^\ell)) \supset \ldots \supset \mathrm{alph}(r(w^{\ell+|\Sigma|+1}))$ implies $|\mathrm{alph}(r(w^{\ell+s}))| = |\Sigma| - 1 - s$ for all $s \in [0, |\Sigma| - 1]$

Generalisation II

- $\mathrm{alph}(r(w^\ell)) \supset \ldots \supset \mathrm{alph}(r(w^{\ell+|\Sigma|+1}))$ implies $|\mathrm{alph}(r(w^{\ell+s}))| = |\Sigma| - 1 - s$ for all $s \in [0, |\Sigma| - 1]$
- $\rightsquigarrow$ strictly descending chains of length $|\Sigma|$ cannot exist

Generalisation II

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