Essential difference between the repetitive threshold and asymptotic repetitive threshold

Ľubomíra Dvořáková
joint work with Edita Pelantová

FNSPE Czech Technical University in Prague

One World CoW Seminar
January, 23, 2023

Program

1. Critical exponent
   - Repetitive threshold
   - Repetitive threshold of balanced sequences
2. Asymptotic critical exponent
   - Asymptotic repetitive threshold
   - Asymptotic repetitive threshold of balanced sequences

Program

1. Critical exponent
   - Repetitive threshold
   - Repetitive threshold of balanced sequences
2. Asymptotic critical exponent
   - Asymptotic repetitive threshold
   - Asymptotic repetitive threshold of balanced sequences

Rational powers

Definition

If a word z is a prefix of $uuuuuu\cdots = u^\omega$, then z is a **fractional power** *of u, we write $z = u^e$, where $e = \frac{|z|}{|u|}$.*

Example

$$\texttt{kabelkabel} = (\texttt{kabel})^2$$
$$\texttt{kabelkabelkabel} = (\texttt{kabel})^3$$
$$\texttt{kabelka} = (\texttt{kabel})^{\frac{7}{5}}$$

Critical exponent

Definition

The **critical exponent** *of a sequence* $\mathbf{u}$ *is*

$$E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e \text{ is a non-empty factor of } \mathbf{u}\}.$$

The **repetitive threshold** *for* $d \geq 2$*:*

$$RT(d) = \inf\{E(\mathbf{u}) : \mathbf{u} \text{ is a } d\text{-ary sequence}\}.$$

Example

The Thue–Morse sequence $\mathbf{u}_{TM} = \text{abbabaabbaababbabaab}\ldots$

$\mathbf{u}_{TM} = \varphi(\mathbf{u}_{TM})$, where $\varphi : \text{a} \to \text{ab}, \quad \text{b} \to \text{ba}$

$\mathbf{u}_{TM}$ does not contain overlaps: $xwxwx$, where w is a factor and x is a letter. Hence $E(\mathbf{u}_{TM}) = 2$ and $RT(2) = 2$.

Critical exponent

Definition

The **critical exponent** *of a sequence* $\mathbf{u}$ *is*

$$E(\mathbf{u}) = \sup\{e \in \mathbb{Q} : u^e \textit{ is a non-empty factor of } \mathbf{u}\}.$$

The **repetitive threshold** *for* $d \geq 2$*:*

$$RT(d) = \inf\{E(\mathbf{u}) : \mathbf{u} \textit{ is a } d\textit{-ary sequence}\}.$$

Example

The Thue–Morse sequence $\mathbf{u}_{TM} = \mathtt{abbabaabbaababbabaab}\ldots$
$\mathbf{u}_{TM} = \varphi(\mathbf{u}_{TM})$, where $\varphi : \mathtt{a} \to \mathtt{ab}, \quad \mathtt{b} \to \mathtt{ba}$
$\mathbf{u}_{TM}$ does not contain overlaps: $xwxwx$, where w is a factor and x is a letter. Hence $E(\mathbf{u}_{TM}) = 2$ and $RT(2) = 2$.

Repetitive threshold

Theorem (Dejean)

- $RT(2) = 2$;
- $RT(3) = 7/4$;
- $RT(4) = 7/5$;
- $RT(d) = 1 + \frac{1}{d-1}$ for $d \geq 5$.

Repetitive threshold of balanced sequences

Definition

A sequence is **balanced** if, for any two of its factors u and v, holds $|u| = |v| \Rightarrow ||u|_a - |v|_a| \leq 1$.
The **repetitive threshold of balanced sequences** for $d \geq 2$ $RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v}\ d\text{-ary balanced sequence}\}$.

Theorem (Carpi, de Luca, 2000)

$RTB(2) = 2 + \frac{1+\sqrt{5}}{2}$.

Repetitive threshold of balanced sequences

Definition

A sequence is **balanced** if, for any two of its factors u and v, holds $|u| = |v| \Rightarrow ||u|_a - |v|_a| \leq 1$.
The **repetitive threshold of balanced sequences** for $d \geq 2$
$RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v}\ d\text{-ary balanced sequence}\}$.

Theorem (Carpi, de Luca, 2000)

$RTB(2) = 2 + \frac{1+\sqrt{5}}{2}$.

Repetitive threshold of balanced sequences

Definition

A sequence is **balanced** if, for any two of its factors u and v, holds $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1$.
The **repetitive threshold of balanced sequences** for $d \geq 2$
$RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v}\ d\text{-ary balanced sequence}\}$.

Theorem (Rampersad, Shallit, Vandomme, 2019)

$RTB(3) = 2 + \frac{1}{\sqrt{2}}$ *and* $RTB(4) = 1 + \frac{1+\sqrt{5}}{4}$.

Repetitive threshold of balanced sequences

Definition

A sequence is **balanced** if, for any two of its factors u and v, holds $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1\,.$
The **repetitive threshold of balanced sequences** for $d \geq 2$
$RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v}\ d\text{-ary balanced sequence}\}\,.$

Theorem (Baranwal, Shallit, 2019, 2020)

$RTB(d) = 1 + \frac{1}{d-3}$ *for* $5 \leq d \leq 8$.

Repetitive threshold of balanced sequences

Definition

A sequence is **balanced** if, for any two of its factors u and v, holds $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1\,.$
The **repetitive threshold of balanced sequences** for $d \geq 2$
$RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v}\ d\text{-ary balanced sequence}\}\,.$

Theorem (Dolce, D., Pelantová, 2021)

$RTB(d) = 1 + \frac{1}{d-3}$ *for* $9 \leq d \leq 10$.

Repetitive threshold of balanced sequences

Definition

A sequence is **balanced** if, for any two of its factors u and v, holds $|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1$.
The **repetitive threshold of balanced sequences** for $d \geq 2$
$RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v}\ d\text{-ary balanced sequence}\}$.

Theorem (D., Opočenská, Pelantová, Shur, 2022)

$RTB(d) = 1 + \frac{1}{d-2}$ for $d = 11$ and all even numbers $d \geq 12$.

Repetitive threshold of balanced sequences

Definition

A sequence is **balanced** if, for any two of its factors u and v, holds
$|u| = |v| \Rightarrow |u|_a - |v|_a \leq 1\,.$
The **repetitive threshold of balanced sequences** for $d \geq 2$
$RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v}\ d\text{-ary balanced sequence}\}\,.$

Theorem (D., Opočenská, Pelantová, Shur, 2022)

$RTB(d) \geq 1 + \frac{1}{d-2}$ for $d \geq 11$.

Program

1. Critical exponent
   - Repetitive threshold
   - Repetitive threshold of balanced sequences
2. Asymptotic critical exponent
   - Asymptotic repetitive threshold
   - Asymptotic repetitive threshold of balanced sequences

Asymptotic critical exponent

Definition

The **asymptotic critical exponent** *of a sequence* $\mathbf{u}$ *is* $E^*(\mathbf{u}) = +\infty$ *if* $E(\mathbf{u}) = +\infty$ *and otherwise*

$$E^*(\mathbf{u}) = \limsup_{n\to\infty}\{e \in \mathbb{Q} : \ u^e \text{ is a factor of } \mathbf{u} \text{ for some } u \text{ of length } n\}.$$

The **asymptotic repetitive threshold** *for* $d \geq 2$*:*

$$RT^*(d) = \inf\{E^*(\mathbf{u}) : \mathbf{u} \text{ is a } d\text{-ary sequence}\}.$$

- $E^*(\mathbf{u}) \leq E(\mathbf{u})$

Asymptotic critical exponent

Definition

The **asymptotic critical exponent** *of a sequence* $\mathbf{u}$ *is* $E^*(\mathbf{u}) = +\infty$ *if* $E(\mathbf{u}) = +\infty$ *and otherwise*

$$E^*(\mathbf{u}) = \limsup_{n\to\infty}\{e \in \mathbb{Q} : \ u^e \text{ is a factor of } \mathbf{u} \text{ for some } u \text{ of length } n\}.$$

The **asymptotic repetitive threshold** *for* $d \geq 2$*:*

$$RT^*(d) = \inf\{E^*(\mathbf{u}) : \mathbf{u} \text{ is a } d\text{-ary sequence}\}.$$

- $E^*(\mathbf{u}) \leq E(\mathbf{u})$
- if $E(\mathbf{u})$ is irrational, then $E^*(\mathbf{u}) = E(\mathbf{u})$ (Fibonacci $2 + \frac{1+\sqrt{5}}{2}$)

Asymptotic critical exponent

Definition

The **asymptotic critical exponent** *of a sequence* $\mathbf{u}$ *is* $E^*(\mathbf{u}) = +\infty$ *if* $E(\mathbf{u}) = +\infty$ *and otherwise*

$$E^*(\mathbf{u}) = \limsup_{n\to\infty}\{e \in \mathbb{Q} : \ u^e \text{ is a factor of } \mathbf{u} \text{ for some } u \text{ of length } n\} .$$

The **asymptotic repetitive threshold** *for* $d \geq 2$*:*

$$RT^*(d) = \inf\{E^*(\mathbf{u}) : \mathbf{u} \text{ is a } d\text{-ary sequence}\} .$$

- $E^*(\mathbf{u}) \leq E(\mathbf{u})$
- if $E(\mathbf{u})$ is irrational, then $E^*(\mathbf{u}) = E(\mathbf{u})$ (Fibonacci $2 + \frac{1+\sqrt{5}}{2}$)
- $E^*(\mathbf{u}_{TM}) = E(\mathbf{u}_{TM}) = 2$

Asymptotic critical exponent

Definition

The **asymptotic critical exponent** *of a sequence* $\mathbf{u}$ *is* $E^*(\mathbf{u}) = +\infty$ *if* $E(\mathbf{u}) = +\infty$ *and otherwise*

$$E^*(\mathbf{u}) = \limsup_{n\to\infty}\{e \in \mathbb{Q} : \ u^e \text{ is a factor of } \mathbf{u} \text{ for some } u \text{ of length } n\}.$$

The **asymptotic repetitive threshold** *for* $d \geq 2$*:*

$$RT^*(d) = \inf\{E^*(\mathbf{u}) : \mathbf{u} \text{ is a } d\text{-ary sequence}\}.$$

- $E^*(\mathbf{u}) \leq E(\mathbf{u})$
- if $E(\mathbf{u})$ is irrational, then $E^*(\mathbf{u}) = E(\mathbf{u})$ (Fibonacci $2 + \frac{1+\sqrt{5}}{2}$)
- $E^*(\mathbf{u}_{TM}) = E(\mathbf{u}_{TM}) = 2$

Asymptotic repetitive threshold

Theorem (D., Opočenská, Pelantová, 2022)

$RT^*(d) = 1$ for every $d \geq 2$.

Proof : Construction of a uniformly recurrent binary sequence $\mathbf{u}^{(k)}$ with $E^*\left(\mathbf{u}^{(k)}\right) \leq 1 + \frac{2}{F_k - 3}$ for infinitely many k $\Rightarrow$ $RT^*(2) = 1$.

Asymptotic repetitive threshold

Theorem (D., Opočenská, Pelantová, 2022)

$RT^*(d) = 1$ *for every* $d \geq 2$.

Proof : Construction of a uniformly recurrent binary sequence $\mathbf{u}^{(k)}$ with $E^*\left(\mathbf{u}^{(k)}\right) \leq 1 + \frac{2}{F_k - 3}$ for infinitely many k $\Rightarrow$ $RT^*(2) = 1$.

- Set $d = F_k$ if F_k even, $F_k \geq 12$

Asymptotic repetitive threshold

Theorem (D., Opočenská, Pelantová, 2022)

$RT^*(d) = 1$ *for every* $d \geq 2$.

Proof : Construction of a uniformly recurrent binary sequence $\mathbf{u}^{(k)}$ with $E^*\left(\mathbf{u}^{(k)}\right) \leq 1 + \frac{2}{F_k - 3}$ for infinitely many $k \ \Rightarrow \ RT^*(2) = 1$.

- Set $d = F_k$ if F_k even, $F_k \geq 12$
- Find a uniformly recurrent d-ary balanced sequence $\mathbf{v}$ with $E(\mathbf{v}) = 1 + \frac{1}{d-2}$

Asymptotic repetitive threshold

Theorem (D., Opočenská, Pelantová, 2022)

$RT^*(d) = 1$ *for every* $d \geq 2$.

Proof : Construction of a uniformly recurrent binary sequence $\mathbf{u}^{(k)}$ with $E^*\left(\mathbf{u}^{(k)}\right) \leq 1 + \frac{2}{F_k - 3}$ for infinitely many $k \Rightarrow RT^*(2) = 1$.

- Set $d = F_k$ if F_k even, $F_k \geq 12$
- Find a uniformly recurrent d-ary balanced sequence $\mathbf{v}$ with $E(\mathbf{v}) = 1 + \frac{1}{d-2}$
- Define a morphism $\psi : \{0, 1, \ldots, d-1\}^* \to \{0, 1\}^*$ $\psi(i) = 110(i)_{\text{Fib}}$

Asymptotic repetitive threshold

Theorem (D., Opočenská, Pelantová, 2022)

$RT^*(d) = 1$ *for every* $d \geq 2$.

Proof : Construction of a uniformly recurrent binary sequence $\mathbf{u}^{(k)}$ with $E^*\left(\mathbf{u}^{(k)}\right) \leq 1 + \frac{2}{F_k - 3}$ for infinitely many k $\Rightarrow$ $RT^*(2) = 1$.

- Set $d = F_k$ if F_k even, $F_k \geq 12$
- Find a uniformly recurrent d-ary balanced sequence $\mathbf{v}$ with $E(\mathbf{v}) = 1 + \frac{1}{d-2}$
- Define a morphism $\psi : \{0, 1, \ldots, d-1\}^* \to \{0, 1\}^*$ $\psi(i) = 110(i)_{\text{Fib}}$
- Define $\mathbf{u}^{(k)} = \psi(\mathbf{v})$

Asymptotic repetitive threshold

Theorem (D., Opočenská, Pelantová, 2022)

$RT^*(d) = 1$ for every $d \geq 2$.

Proof : Construction of a uniformly recurrent binary sequence $\mathbf{u}^{(k)}$ with $E^*\left(\mathbf{u}^{(k)}\right) \leq 1 + \frac{2}{F_k - 3}$ for infinitely many $k \Rightarrow RT^*(2) = 1$.

- Set $d = F_k$ if F_k even, $F_k \geq 12$
- Find a uniformly recurrent d-ary balanced sequence $\mathbf{v}$ with $E(\mathbf{v}) = 1 + \frac{1}{d-2}$
- Define a morphism $\psi : \{0, 1, \ldots, d-1\}^* \to \{0, 1\}^*$ $\psi(i) = 110(i)_{\text{Fib}}$
- Define $\mathbf{u}^{(k)} = \psi(\mathbf{v})$

Corollary (RT versus RT^*)

Let $d \geq 2$, then $RT^*(d) = 1 < 1 + \frac{1}{d-1} \leq RT(d)$.

Asymptotic repetitive threshold

Theorem (D., Opočenská, Pelantová, 2022)

$RT^*(d) = 1$ for every $d \geq 2$.

Proof : Construction of a uniformly recurrent binary sequence $\mathbf{u}^{(k)}$ with $E^*\left(\mathbf{u}^{(k)}\right) \leq 1 + \frac{2}{F_k - 3}$ for infinitely many $k \;\Rightarrow\; RT^*(2) = 1$.

- Set $d = F_k$ if F_k even, $F_k \geq 12$
- Find a uniformly recurrent d-ary balanced sequence $\mathbf{v}$ with $E(\mathbf{v}) = 1 + \frac{1}{d-2}$
- Define a morphism $\psi : \{0, 1, \ldots, d-1\}^* \to \{0, 1\}^*$ $\psi(i) = 110(i)_{\text{Fib}}$
- Define $\mathbf{u}^{(k)} = \psi(\mathbf{v})$

Corollary (RT versus RT^*)

Let $d \geq 2$, then $RT^*(d) = 1 < 1 + \frac{1}{d-1} \leq RT(d)$.

Asymptotic repetitive threshold of balanced sequences

Definition

The **asymptotic repetitive threshold of balanced sequences** for $d \geq 2$

$$RTB^*(d) = \inf\{E^*(\mathbf{v}) : \mathbf{v} \text{ balanced } d\text{-ary sequence}\}\,.$$

Theorem (D., Opočenská, Pelantová, 2022)

$RTB^*(d) \geq 1 + \frac{1}{2^{d-2}}$ for $d \geq 2$.

Asymptotic repetitive threshold of balanced sequences

Definition

The **asymptotic repetitive threshold of balanced sequences** for $d \geq 2$

$$RTB^*(d) = \inf\{E^*(\mathbf{v}) : \mathbf{v} \text{ balanced } d\text{-ary sequence}\}.$$

Theorem (D., Opočenská, Pelantová, 2022)

$RTB^*(d) \geq 1 + \frac{1}{2^{d-2}}$ for $d \geq 2$.

Theorem (D., Opočenská, Pelantová, 2022)

$RTB^*(d)$ for $2 \leq d \leq 10$.

Asymptotic repetitive threshold of balanced sequences

Definition

The **asymptotic repetitive threshold of balanced sequences** for $d \geq 2$

$$RTB^*(d) = \inf\{E^*(\mathbf{v}) : \mathbf{v} \text{ balanced } d\text{-ary sequence}\}\,.$$

Theorem (D., Opočenská, Pelantová, 2022)

$RTB^*(d) \geq 1 + \frac{1}{2^{d-2}}$ *for* $d \geq 2$.

Theorem (D., Opočenská, Pelantová, 2022)

$RTB^*(d)$ *for* $2 \leq d \leq 10$.

$RTB^*(d)$ for $d \leq 10$

d	$RTB^*(d)$	$RTB(d)$
2	$2+\frac{1+\sqrt{5}}{2} \doteq 3.618034$	$2+\frac{1+\sqrt{5}}{2}$
3	$2+\frac{\sqrt{2}}{2} \doteq 2.707107$	$2+\frac{\sqrt{2}}{2}$
4	$1+\frac{\sqrt{5}+1}{4} \doteq 1.809017$	$1+\frac{\sqrt{5}+1}{4}$
5	$\frac{3}{2} = 1.5$	$\frac{3}{2}$
6	$\frac{75+3\sqrt{65}}{80} \doteq 1.239835$	$\frac{4}{3} \doteq 1.333333$
7	$\frac{49+\sqrt{577}}{64} \doteq 1.140950$	$\frac{5}{4} = 1.25$
8	$1+\frac{3-\sqrt{5}}{16} \doteq 1.047746$	$\frac{6}{5} = 1.2$
9	$\frac{21-\sqrt{20}}{16} \doteq 1.032992$	$\frac{7}{6} \doteq 1.166667$
10	$\frac{364-21\sqrt{7}}{304} \doteq 1.014603$	$\frac{8}{7} \doteq 1.142857$

Asymptotic behavior of $RTB^*(d)$

Observing the table

$$RTB^*(7) < 1+\tfrac{1}{7}, \quad RTB^*(8) < 1+\tfrac{1}{20},$$
$$RTB^*(9) < 1+\tfrac{1}{30}, \quad RTB^*(10) < 1+\tfrac{1}{68}.$$

Conjecture

There exists $q>1$

$$RTB^*(d) < 1+\frac{1}{q^d} \text{ for sufficiently large } d.$$

Corollary (RTB versus RTB^*)

$$RTB^*(d) < 1+\frac{1}{q^d} < 1+\frac{1}{d-2} \le RTB(d).$$

Balanced sequences

Theorem (Graham 1973, Hubert 2000)

$\mathbf{v}$ recurrent aperiodic is balanced iff $\mathbf{v}$ obtained from a Sturmian sequence $\mathbf{u}$ over $\{\mathtt{a}, \mathtt{b}\}$ by replacing

- *$\mathtt{a}$'s with a constant gap sequence $\mathbf{y}$ over $\mathcal{A}$,*
- *$\mathtt{b}$'s with a constant gap sequence $\hat{\mathbf{y}}$ over $\mathcal{B}$,*

where $\mathcal{A}$ and $\mathcal{B}$ disjoint. We write $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \hat{\mathbf{y}})$.

Example

$\mathbf{v} = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}, \hat{\mathbf{y}})$, where $\mathbf{y} = (1323)^\omega$ and $\hat{\mathbf{y}} = (\hat{1}\hat{3}\hat{2}\hat{3})^\omega$

$$\mathbf{u}_F = \mathtt{abaababaabaababaababaabaababaab}\cdots$$
$$\mathbf{v} = 1\hat{1}323\hat{3}3\hat{2}13\hat{3}23\hat{1}1\hat{3}32\hat{2}3\hat{3}13\hat{1}23\hat{3}1\hat{2}32\hat{3}\cdots$$

"discolouration map" π replaces all letters from $\mathcal{A}$ by $\mathtt{a}$ and all letters from $\mathcal{B}$ by $\mathtt{b}$, i.e., $\pi(\mathbf{v}) = \mathbf{u}_F$ and $\pi(1\hat{1}32\hat{3}3\hat{2}) = \mathtt{abaabab}$.

Balanced sequences

Theorem (Graham 1973, Hubert 2000)

$\mathbf{v}$ recurrent aperiodic is balanced iff $\mathbf{v}$ obtained from a Sturmian sequence $\mathbf{u}$ over $\{\mathtt{a}, \mathtt{b}\}$ by replacing

- *$\mathtt{a}$'s with a constant gap sequence $\mathbf{y}$ over $\mathcal{A}$,*
- *$\mathtt{b}$'s with a constant gap sequence $\hat{\mathbf{y}}$ over $\mathcal{B}$,*

where $\mathcal{A}$ and $\mathcal{B}$ disjoint. We write $\mathbf{v} = \mathrm{colour}(\mathbf{u}, \mathbf{y}, \hat{\mathbf{y}})$.

Example

$\mathbf{v} = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}, \hat{\mathbf{y}})$, where $\mathbf{y} = (1323)^\omega$ and $\hat{\mathbf{y}} = (\hat{1}\hat{3}\hat{2}\hat{3})^\omega$

$$\mathbf{u}_F = \mathtt{abaababaabaababaababaabaababaab}\cdots$$

$$\mathbf{v} = 1\hat{1}32\hat{3}3\hat{2}13\hat{3}23\hat{1}1\hat{3}32\hat{2}3\hat{3}13\hat{1}23\hat{3}1\hat{2}32\hat{3}\cdots$$

"discolouration map" π replaces all letters from $\mathcal{A}$ by $\mathtt{a}$ and all letters from $\mathcal{B}$ by $\mathtt{b}$, i.e., $\pi(\mathbf{v}) = \mathbf{u}_F$ and $\pi(1\hat{1}32\hat{3}3\hat{2}) = \mathtt{abaabab}$.

Computation of asymptotic critical exponent

Proposition (D., Pelantová)

Let $\mathbf{u}$ be a uniformly recurrent aperiodic sequence. Let w_n be the n-th bispecial of $\mathbf{u}$ and v_n a shortest return word to w_n. Then

$$E(\mathbf{u}) = 1 + \sup\{\tfrac{|w_n|}{|v_n|} : n \in \mathbb{N}\} \quad \text{and} \quad E^*(\mathbf{u}) = 1 + \limsup_{n\to\infty} \tfrac{|w_n|}{|v_n|}.$$

Colouring of the Fibonacci sequence

Definition

Let $\delta \geq 1$ and $d = 2\delta$. Define a d-ary sequence $\mathbf{v}_\delta = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}_\delta, \hat{\mathbf{y}}_\delta)$, where the period of $\mathbf{y}_\delta$ and $\hat{\mathbf{y}}_\delta$ is $H = 2^{\delta-1}$.

The sequence $\mathbf{v}_\delta$ is aperiodic, balanced and uniformly recurrent.

Example

$\mathbf{y}_1 = 1^\omega, \quad \mathbf{y}_2 = (12)^\omega, \quad \mathbf{y}_3 = (1323)^\omega, \quad \mathbf{y}_4 = (14342434)^\omega$, etc.

Return words to bispecials in $\mathbf{v}_\delta$

- Consider a sufficiently long factor w in $\mathbf{v}_\delta$, then w is bispecial in $\mathbf{v}_\delta$ iff $\pi(w) = b$ is bispecial in $\mathbf{u}_F$.
- If v is a return word to w, then $\pi(v)$ is a concatenation of return words to b.
- $|\pi(v)|_{\mathtt{a}}$ and $|\pi(v)|_{\mathtt{b}}$ are divisible by H.

Example

$\mathbf{v}_3 = \mathrm{colour}(\mathbf{u}_F, \mathbf{y}_3, \hat{\mathbf{y}}_3)$, $\mathbf{y}_3 = (1213)^\omega$ and $\hat{\mathbf{y}}_3 = (\hat{1}\hat{2}\hat{1}\hat{3})^\omega$, $H = 4$
$w = 1\hat{1}32\hat{3}3$, $b = \pi(w) = \mathtt{abaaba}$, $|\pi(v)|_{\mathtt{a}} = 32$, $|\pi(v)|_{\mathtt{b}} = 20$.

$\mathtt{abaababaabaababaabaabaababaabaabaababaababaaba}\cdots$

$1\hat{1}32\hat{3}3\hat{2}13\hat{3}23\hat{1}1\hat{3}32\hat{2}3\hat{3}13\hat{1}23\hat{3}1\hat{2}32\hat{3}31\hat{1}2\hat{2}13\hat{1}1\hat{3}21\hat{1}31\hat{2}2\hat{1}13\hat{3}1\hat{1}21\hat{2}3\cdots$

Return words to bispecials in $\mathbf{v}_\delta$

- Consider a sufficiently long factor w in $\mathbf{v}_\delta$, then w is bispecial in $\mathbf{v}_\delta$ iff $\pi(w) = b$ is bispecial in $\mathbf{u}_F$.
- If v is a return word to w, then $\pi(v)$ is a concatenation of return words to b.
- $|\pi(v)|_{\mathtt{a}}$ and $|\pi(v)|_{\mathtt{b}}$ are divisible by H.

Lemma

Consider a sufficiently long bispecial w in $\mathbf{v}_\delta$ and its return word v. There exist $N, \kappa, \lambda \in \mathbb{N}, \kappa + \lambda \geq 1$, such that

1. $|w| = F_{N+3} - 2$;
2. $|v| = H(\kappa F_{N+2} + \lambda F_{N+1})$;
3. $|\kappa - \tau\lambda| < \frac{\tau^2}{H}$.

Theorem (coloured Fibonacci)

If $\tau^{N_0+1} < H = 2^{\delta-1} < \tau^{N_0+2}$ *for some* $N_0 \geq 1$*, then*

$$E^*(\mathbf{v}_\delta) \leq 1 + \frac{1}{H\tau^{N_0-1}}.$$

Proof.

Consider a long bispecial w and its shortest return word v.

- By Lemma $\exists N, \kappa, \lambda \in \mathbb{N}, \kappa + \lambda \geq 1$
  $|w| = F_{N+3} - 2, \quad |v| = H(\kappa F_{n+2} + \lambda F_{n+1}), \quad |\kappa - \tau\lambda| < \frac{\tau^2}{H}$

Theorem (coloured Fibonacci)

If $\tau^{N_0+1} < H = 2^{\delta-1} < \tau^{N_0+2}$ *for some* $N_0 \geq 1$*, then*

$$E^*(\mathbf{v}_\delta) \leq 1 + \frac{1}{H\tau^{N_0-1}}.$$

Proof.

Consider a long bispecial w and its shortest return word v.

- By Lemma $\exists N, \kappa, \lambda \in \mathbb{N}, \kappa + \lambda \geq 1$
  $|w| = F_{N+3} - 2, \quad |v| = H(\kappa F_{n+2} + \lambda F_{n+1}), \quad |\kappa - \tau\lambda| < \frac{\tau^2}{H}$
- By properties of Fibonacci numbers
  $|F_{N_0+1} - \tau F_{N_0}| = \frac{1}{\tau^{N_0}} < \frac{\tau^2}{H} < \frac{1}{\tau^{N_0-1}} = |F_{N_0} - \tau F_{N_0-1}|$

Theorem (coloured Fibonacci)

If $\tau^{N_0+1} < H = 2^{\delta-1} < \tau^{N_0+2}$ for some $N_0 \geq 1$, then

$$E^*(\mathbf{v}_\delta) \leq 1 + \frac{1}{H\tau^{N_0-1}}.$$

Proof.

Consider a long bispecial w and its shortest return word v.

- By Lemma $\exists N, \kappa, \lambda \in \mathbb{N}, \kappa + \lambda \geq 1$
  $|w| = F_{N+3} - 2, \quad |v| = H(\kappa F_{n+2} + \lambda F_{n+1}), \quad |\kappa - \tau\lambda| < \frac{\tau^2}{H}$
- By properties of Fibonacci numbers
  $|F_{N_0+1} - \tau F_{N_0}| = \frac{1}{\tau^{N_0}} < \frac{\tau^2}{H} < \frac{1}{\tau^{N_0-1}} = |F_{N_0} - \tau F_{N_0-1}|$
- By theory of Diophantine approximations $\kappa \geq F_{N_0+1}, \ \lambda \geq F_{N_0}$

Theorem (coloured Fibonacci)

If $\tau^{N_0+1} < H = 2^{\delta-1} < \tau^{N_0+2}$ *for some* $N_0 \geq 1$, *then*

$$E^*(\mathbf{v}_\delta) \leq 1 + \frac{1}{H\tau^{N_0-1}}.$$

Proof.

Consider a long bispecial w and its shortest return word v.

- By Lemma $\exists N, \kappa, \lambda \in \mathbb{N}, \kappa + \lambda \geq 1$
  $|w| = F_{N+3} - 2, \quad |v| = H(\kappa F_{n+2} + \lambda F_{n+1}), \quad |\kappa - \tau\lambda| < \frac{\tau^2}{H}$
- By properties of Fibonacci numbers
  $|F_{N_0+1} - \tau F_{N_0}| = \frac{1}{\tau^{N_0}} < \frac{\tau^2}{H} < \frac{1}{\tau^{N_0-1}} = |F_{N_0} - \tau F_{N_0-1}|$
- By theory of Diophantine approximations $\kappa \geq F_{N_0+1}, \ \lambda \geq F_{N_0}$
- $\frac{|w|}{|v|} = \frac{F_{N+3}-2}{H(\kappa F_{N+2}+\lambda F_{N+1})} \leq \frac{F_{N+3}-2}{H(F_{N_0+1}F_{N+2}+F_{N_0}F_{N+1})} \leq \frac{F_{N+3}}{HF_{N_0+N+2}}$

Theorem (coloured Fibonacci)

If $\tau^{N_0+1} < H = 2^{\delta-1} < \tau^{N_0+2}$ *for some* $N_0 \geq 1$*, then*

$$E^*(\mathbf{v}_\delta) \leq 1 + \frac{1}{H\tau^{N_0-1}}.$$

Proof.

Consider a long bispecial w and its shortest return word v.

- By Lemma $\exists N, \kappa, \lambda \in \mathbb{N}, \kappa + \lambda \geq 1$
  $|w| = F_{N+3} - 2, \quad |v| = H(\kappa F_{n+2} + \lambda F_{n+1}), \quad |\kappa - \tau\lambda| < \frac{\tau^2}{H}$
- By properties of Fibonacci numbers
  $|F_{N_0+1} - \tau F_{N_0}| = \frac{1}{\tau^{N_0}} < \frac{\tau^2}{H} < \frac{1}{\tau^{N_0-1}} = |F_{N_0} - \tau F_{N_0-1}|$
- By theory of Diophantine approximations $\kappa \geq F_{N_0+1}, \ \lambda \geq F_{N_0}$
- $\frac{|w|}{|v|} = \frac{F_{N+3}-2}{H(\kappa F_{N+2}+\lambda F_{N+1})} \leq \frac{F_{N+3}-2}{H(F_{N_0+1}F_{N+2}+F_{N_0}F_{N+1})} \leq \frac{F_{N+3}}{HF_{N_0+N+2}}$
- $E^*(\mathbf{v}_\delta) = 1 + \limsup_{N\to\infty} \frac{|w|}{|v|} \leq 1 + \frac{1}{H\tau^{N_0-1}}$

□

Theorem (coloured Fibonacci)

If $\tau^{N_0+1} < H = 2^{\delta-1} < \tau^{N_0+2}$ *for some* $N_0 \geq 1$, *then*

$$E^*(\mathbf{v}_\delta) \leq 1 + \frac{1}{H\tau^{N_0-1}}.$$

Proof.

Consider a long bispecial w and its shortest return word v.

- By Lemma $\exists N, \kappa, \lambda \in \mathbb{N}, \kappa + \lambda \geq 1$
  $|w| = F_{N+3} - 2, \quad |v| = H(\kappa F_{n+2} + \lambda F_{n+1}), \quad |\kappa - \tau\lambda| < \frac{\tau^2}{H}$
- By properties of Fibonacci numbers
  $|F_{N_0+1} - \tau F_{N_0}| = \frac{1}{\tau^{N_0}} < \frac{\tau^2}{H} < \frac{1}{\tau^{N_0-1}} = |F_{N_0} - \tau F_{N_0-1}|$
- By theory of Diophantine approximations $\kappa \geq F_{N_0+1}, \ \lambda \geq F_{N_0}$
- $\frac{|w|}{|v|} = \frac{F_{N+3}-2}{H(\kappa F_{N+2}+\lambda F_{N+1})} \leq \frac{F_{N+3}-2}{H(F_{N_0+1}F_{N+2}+F_{N_0}F_{N+1})} \leq \frac{F_{N+3}}{HF_{N_0+N+2}}$
- $E^*(\mathbf{v}_\delta) = 1 + \limsup_{N\to\infty} \frac{|w|}{|v|} \leq 1 + \frac{1}{H\tau^{N_0-1}}$

□

Theorem

Let $d \in \mathbb{N}, d > 1$, d even. Then

$$1 + \frac{1}{2^{d-2}} \leq RTB^*(d) < 1 + \frac{\tau^3}{2^{d-2}}\,.$$

Corollary

Let $d \in \mathbb{N}, d > 1$, d even. Then

$$RTB^*(d+1) \leq RTB^*(d) < 1 + \frac{\tau^3}{2^{d-2}}\,.$$

Theorem

Let $d \in \mathbb{N}, d > 1$, d even. Then

$$1 + \frac{1}{2^{d-2}} \leq RTB^*(d) < 1 + \frac{\tau^3}{2^{d-2}}\,.$$

Corollary

Let $d \in \mathbb{N}, d > 1$, d even. Then

$$RTB^*(d+1) \leq RTB^*(d) < 1 + \frac{\tau^3}{2^{d-2}}\,.$$

Conjecture (proven)

There exists $q > 1$

$$RTB^*(d) < 1 + \tfrac{1}{q^d} \text{ for sufficiently large } d.$$

Theorem

Let $d \in \mathbb{N}, d > 1$, d even. Then

$$1 + \frac{1}{2^{d-2}} \leq RTB^*(d) < 1 + \frac{\tau^3}{2^{d-2}}\,.$$

Corollary

Let $d \in \mathbb{N}, d > 1$, d even. Then

$$RTB^*(d+1) \leq RTB^*(d) < 1 + \frac{\tau^3}{2^{d-2}}\,.$$

Conjecture (proven)

There exists $q > 1$

$$RTB^*(d) < 1 + \tfrac{1}{q^d} \text{ for sufficiently large } d.$$

- The upper bound from Theorem (coloured Fibonacci) is optimal for $d = 2, 4, 8$ and it is close to the value of $RTB^*(d)$ for every even $d \leq 10$. We conjecture that the bound is attained for infinitely many even d's.
- What is the value of $RTB^*(d)$ for $d \geq 11$?

- The upper bound from Theorem (coloured Fibonacci) is optimal for $d = 2, 4, 8$ and it is close to the value of $RTB^*(d)$ for every even $d \leq 10$. We conjecture that the bound is attained for infinitely many even d's.
- What is the value of $RTB^*(d)$ for $d \geq 11$?
- Is $RTB(d) = 1 + \frac{1}{d-2}$ for $d \geq 13$, d odd?
  We know $RTB(d) \geq 1 + \frac{1}{d-2}$.

- The upper bound from Theorem (coloured Fibonacci) is optimal for $d = 2, 4, 8$ and it is close to the value of $RTB^*(d)$ for every even $d \leq 10$. We conjecture that the bound is attained for infinitely many even d's.
- What is the value of $RTB^*(d)$ for $d \geq 11$?
- Is $RTB(d) = 1 + \frac{1}{d-2}$ for $d \geq 13$, d odd?
  We know $RTB(d) \geq 1 + \frac{1}{d-2}$.

Thank you for attention