Taking and merging games as rewrite games

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Joint work with Victor Marsault and Michel Rigo

One World Combinatorics on Words Seminar, February 27th 2023

First part
Combinatorial games

Combinatorial games: definition

Berlekamp, Conway and Guy, *Winning Ways*, 1981

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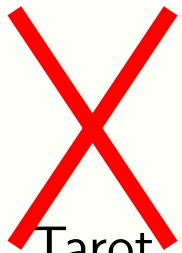
2 players

Chess



Tarot

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- 2 players
- Total information, no chance
- Finite number of turns, no draw

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Total information, no chance

Finite number of turns, no draw

Winner given by the last move.
Normal Convention: the player who cannot play loses.


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“Taking and breaking” games

Board: Heaps of tokens

Rules: A player takes tokens in a single heap, with some constraints on the number, and possibly splits the heap.

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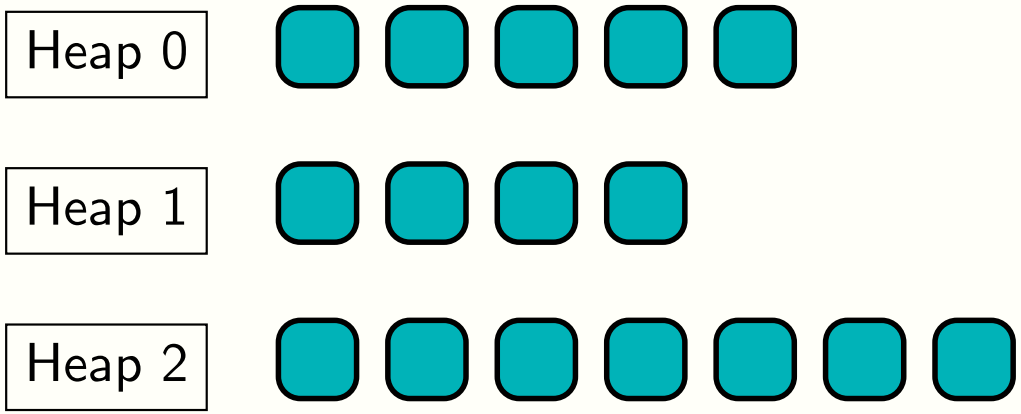

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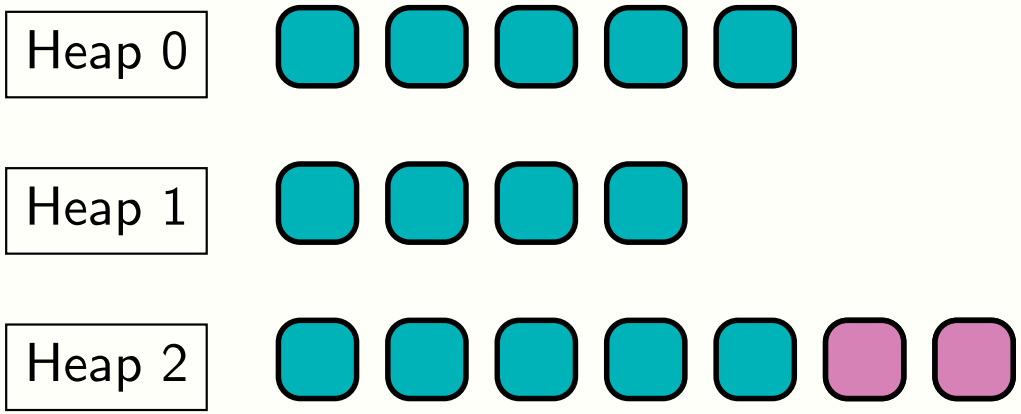

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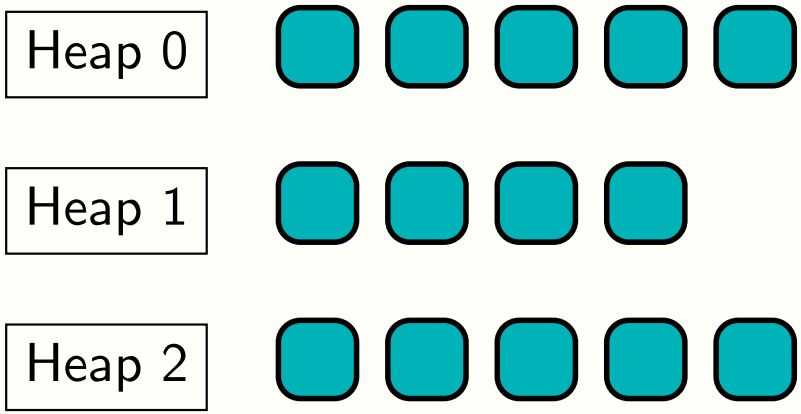

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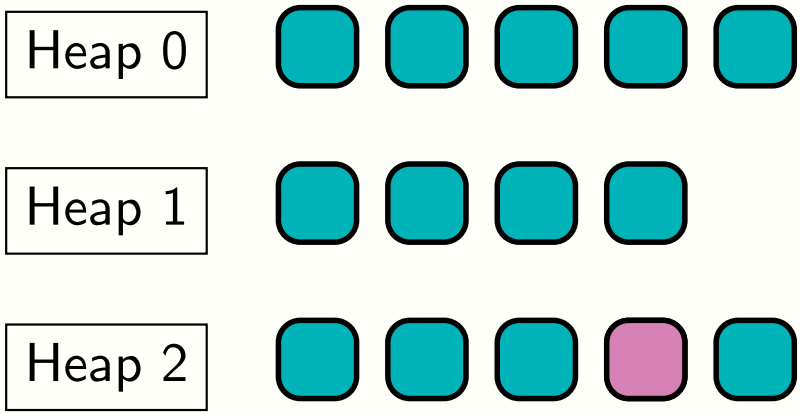

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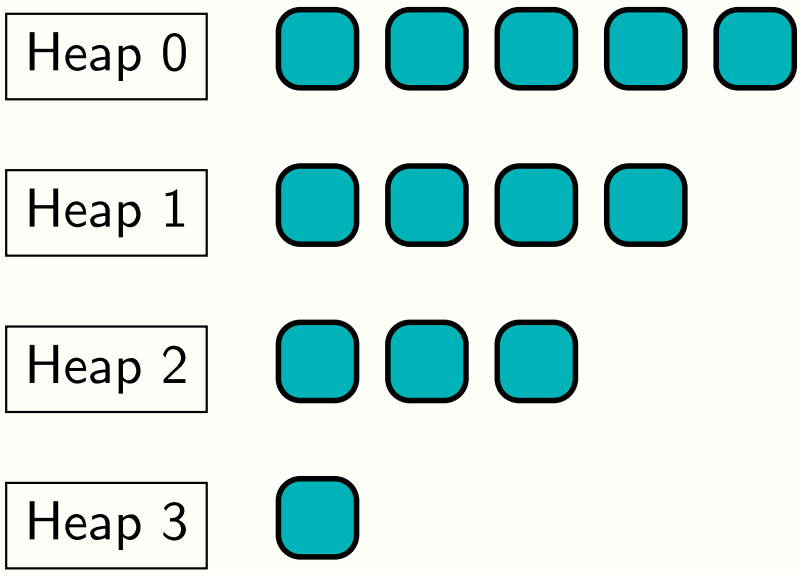

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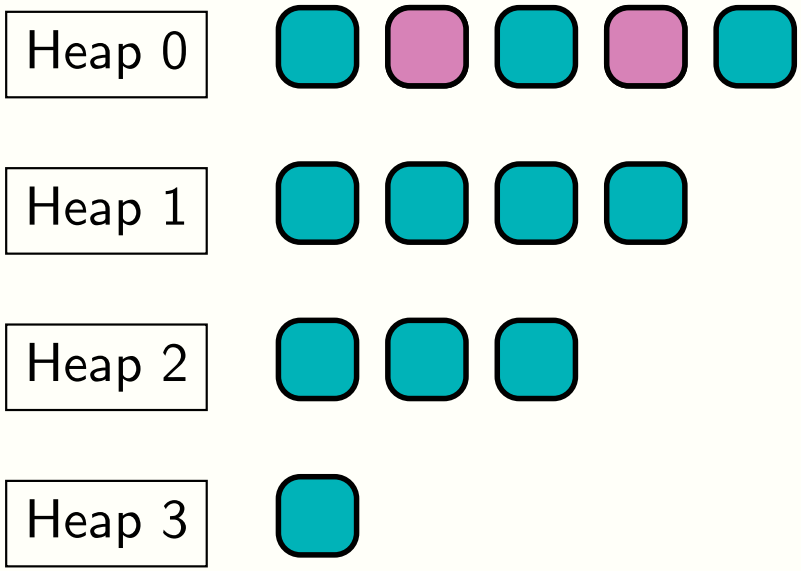

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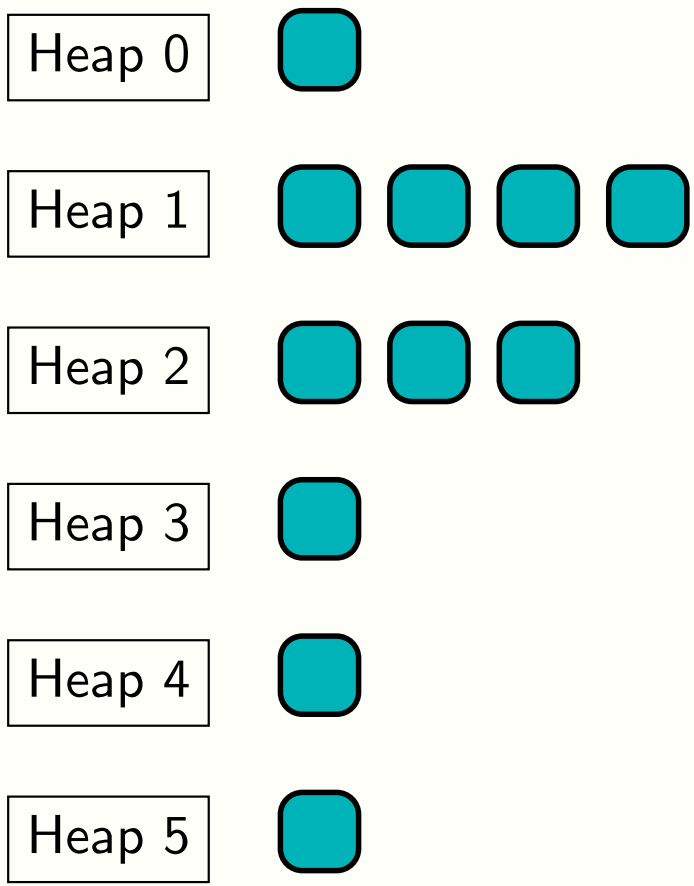

Subtraction games

One cannot split a heap ! Subtraction game
Defined by a set S N:

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Example : $S = f1; 2; 4g$

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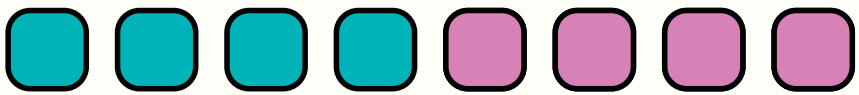

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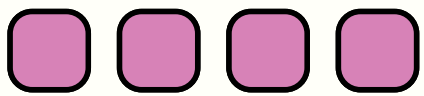

Existence of a winning strategy

Any combinatorial game can be represented by a nite DAG.

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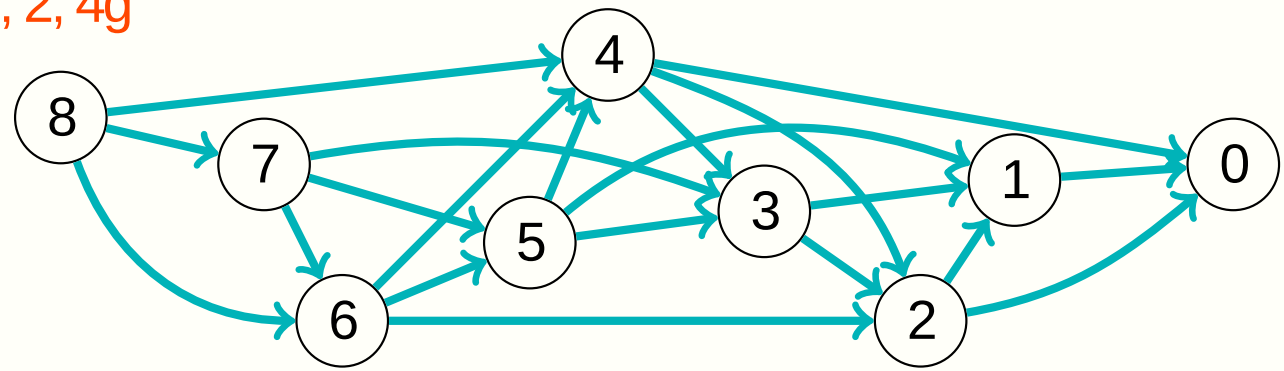

Playing in the game Moving a token along the arcs

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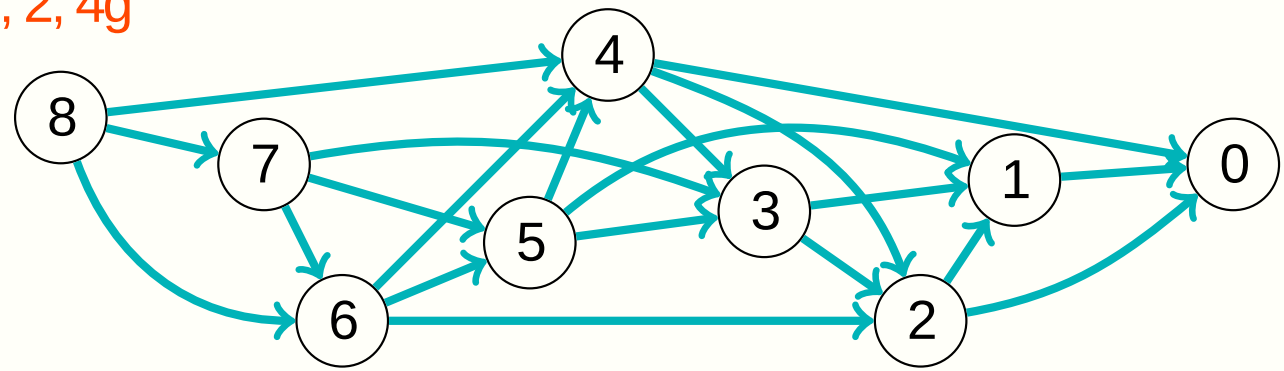

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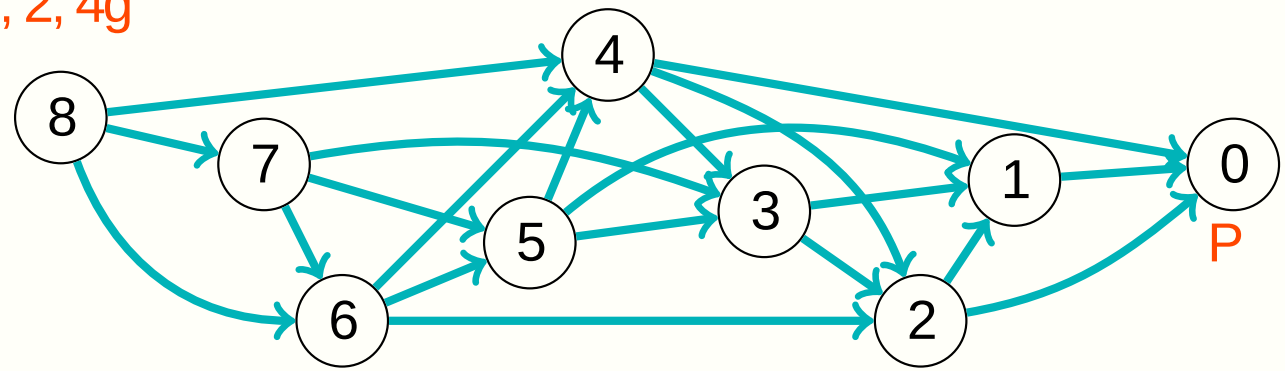

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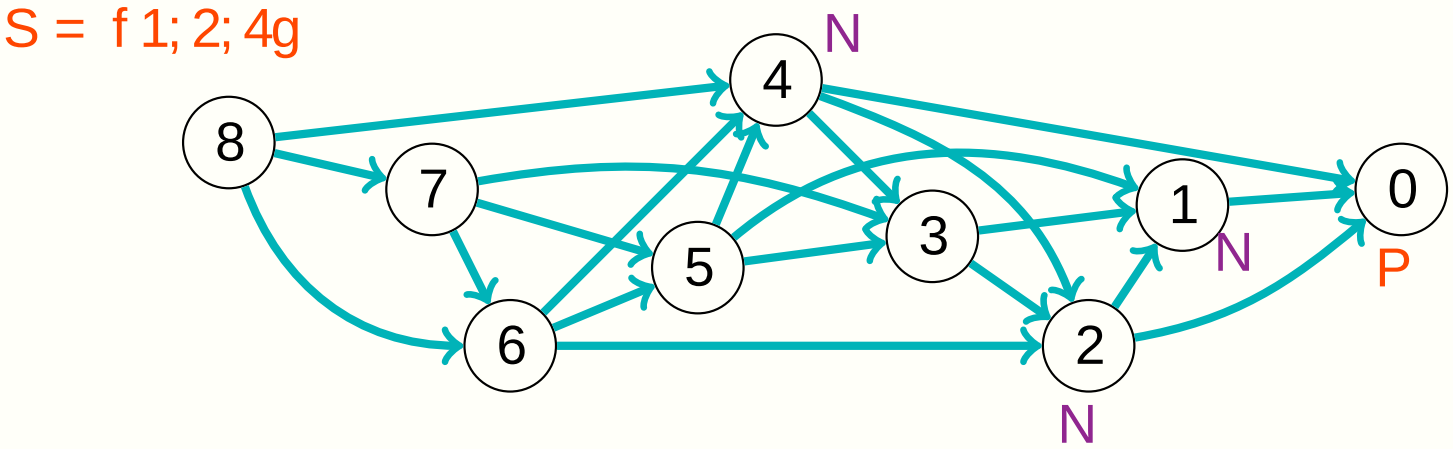

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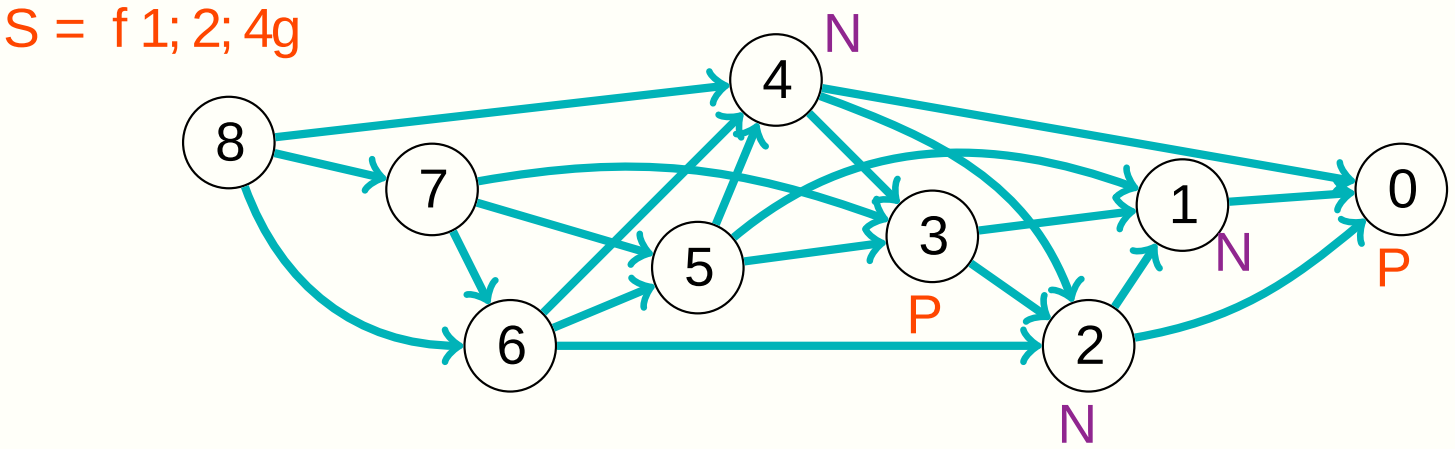

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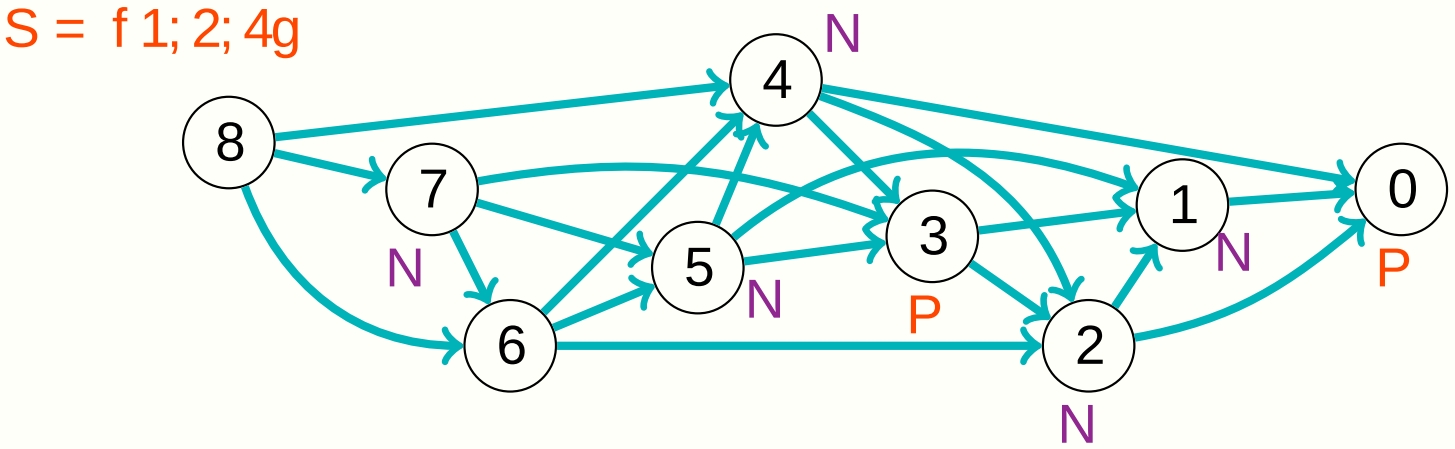

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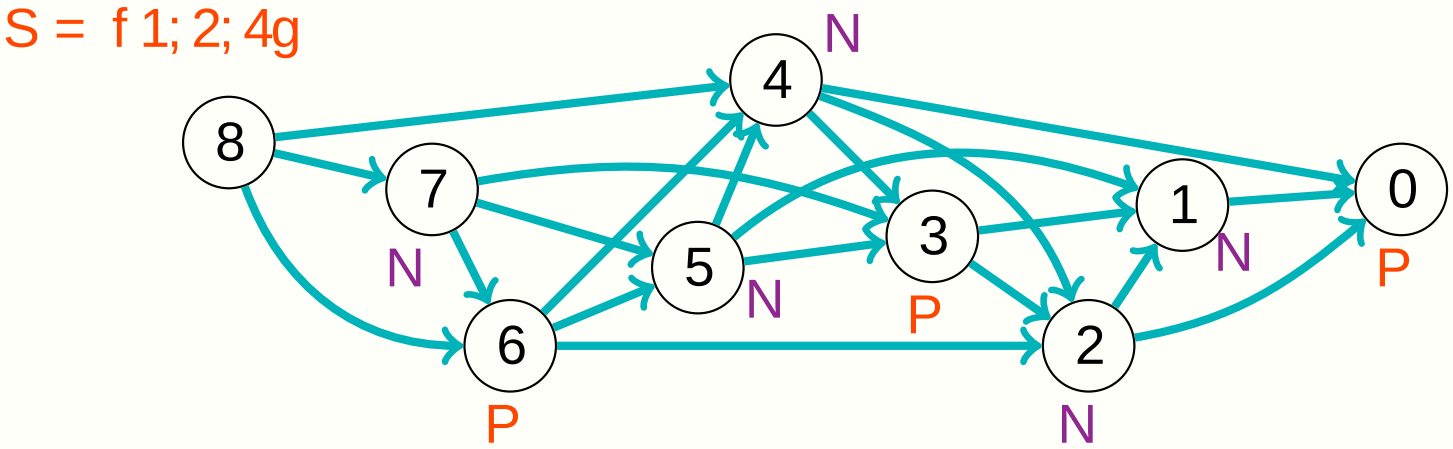

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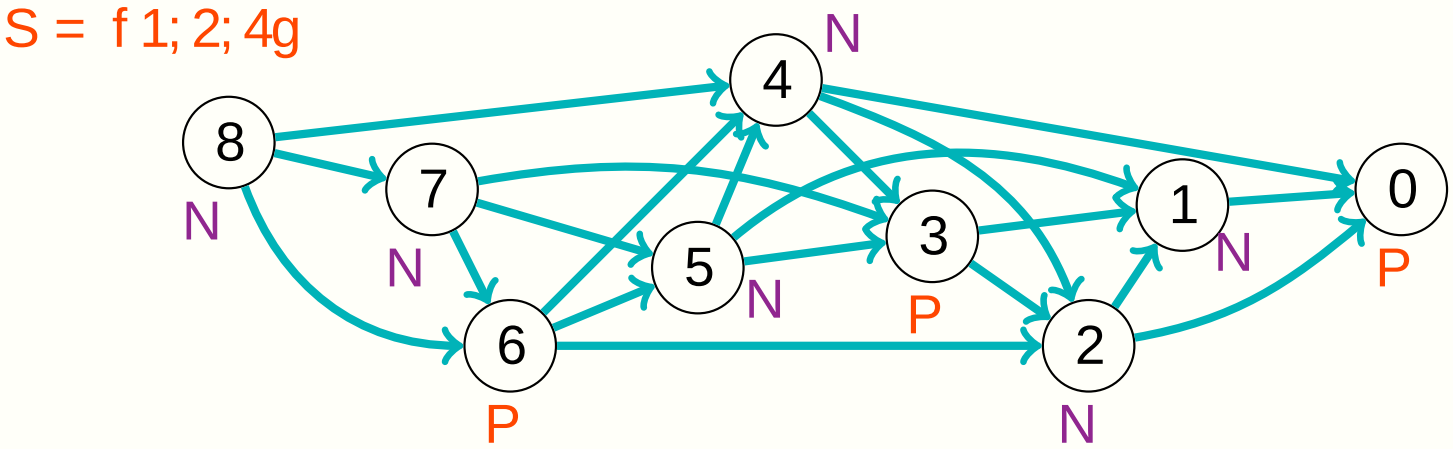

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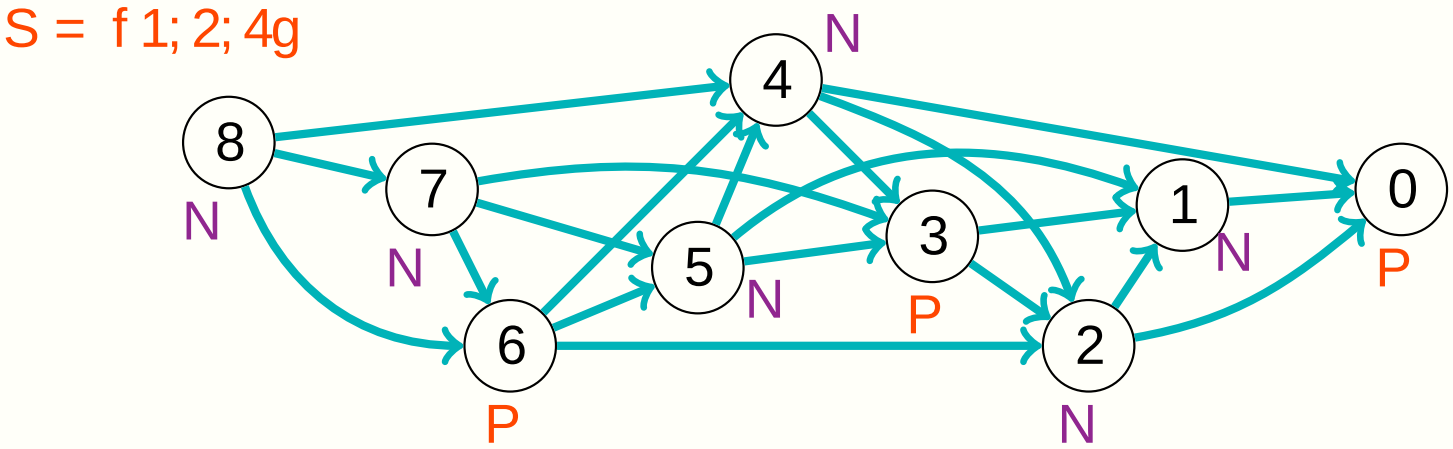

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Theorem

One of the players has a winning strategy.

Main issue

Outcome of the game

Input : Game position
Output : First ($\mathcal{N}$) or second ($\mathcal{P}$) player wins?

Winning strategy

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These two problems can be solved using the DAG but its size is often exponential !

They are generally in Pspace

A standard Pspace problem

Quanti ed Boolean Formula (QBF)

Input : $Q_1 x_1 Q_2 x_2 : : : Q_n x_n \; (\; x_1 ; : : : ; x_n) \; : \; Q_i \; 2 \; f 8 \; ; 9 g$

Output : Is the formula true?

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QBF-game :

Board: logic formula $(x_1;::::;x_n)$

Players assign boolean values x_1,...,x_n, following this order.

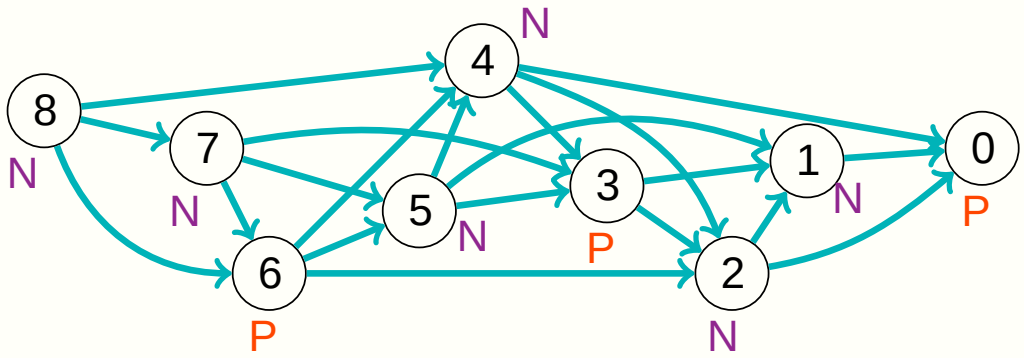
First player wins if at the end the formula is true.

Theorem Schae er, 1989 and Arora,Barak, 2009

Deciding if there is a winning strategy for the rst player at QBF-game is Pspace -complete.

Polynomiality of subtraction games

Consider the subtraction game $S = \{1, 2, 4\}$ on n token.



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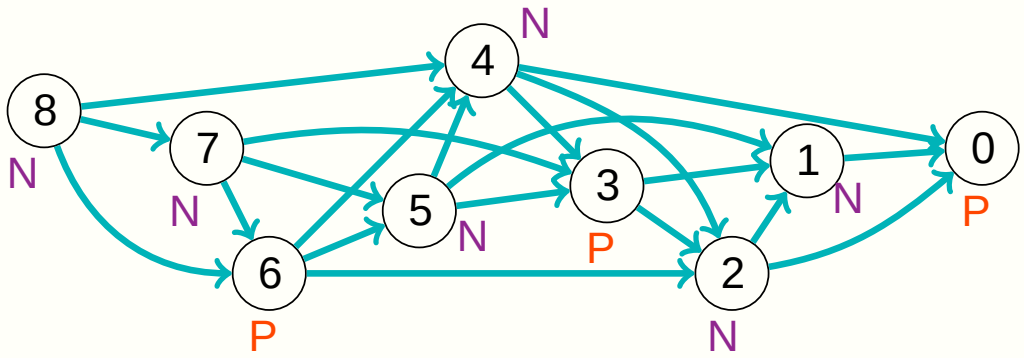

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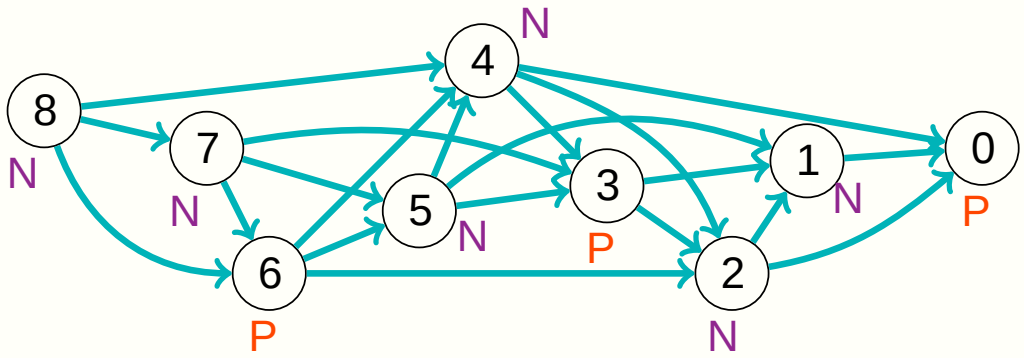

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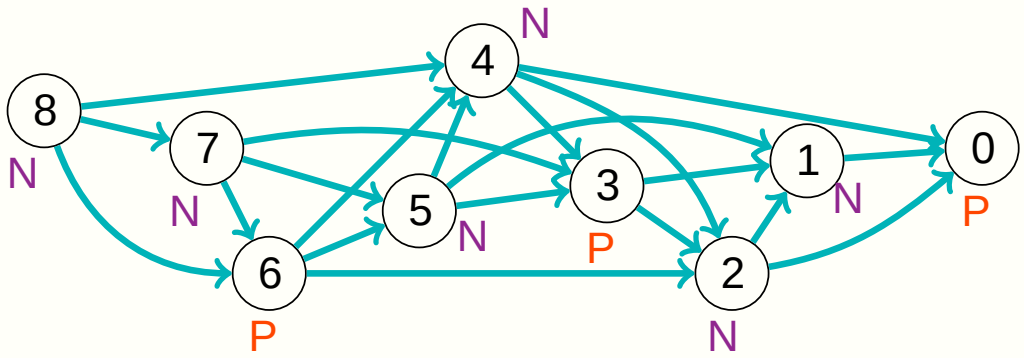

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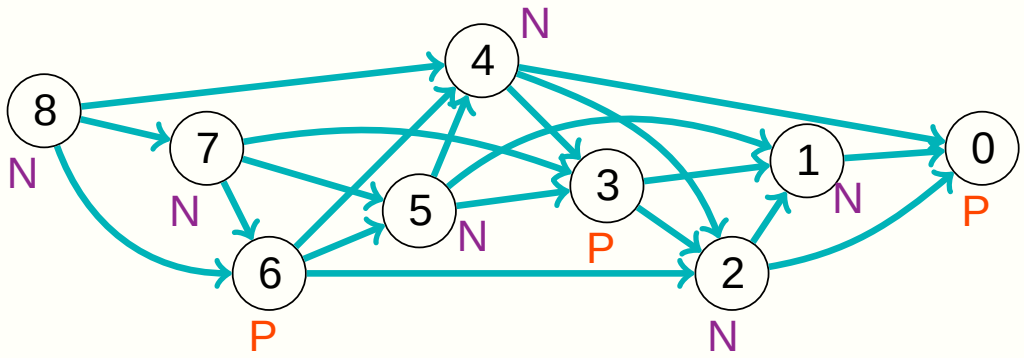

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Open

Size of the preperiod and the period in function of S?

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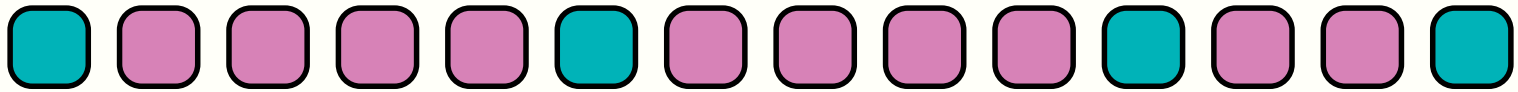

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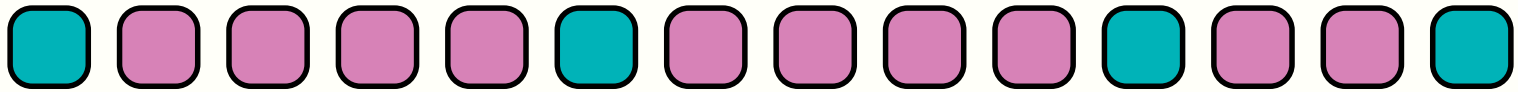

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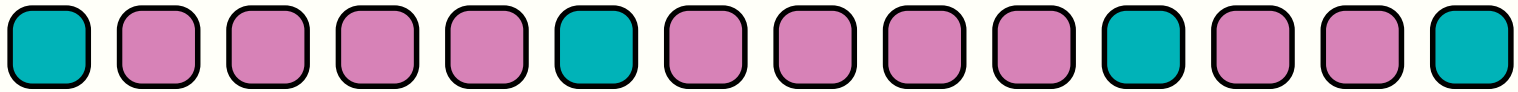

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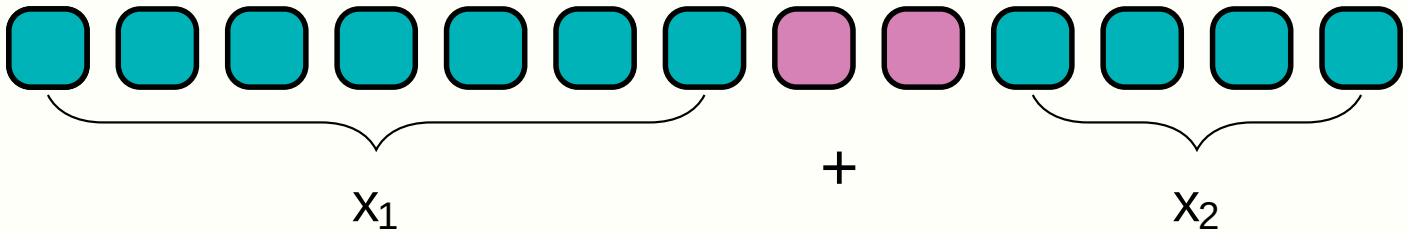
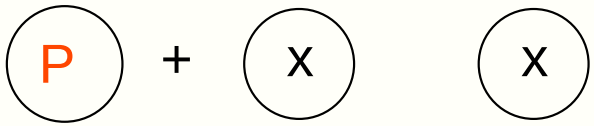
! with sum of games



Computing the outcome of a sum

With a P -position:



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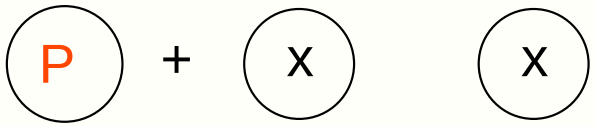
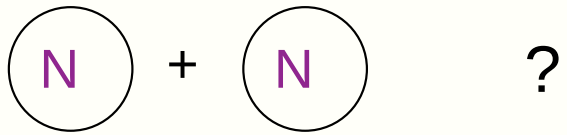


With only N -positions:



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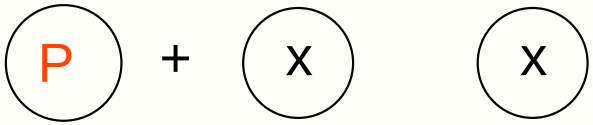
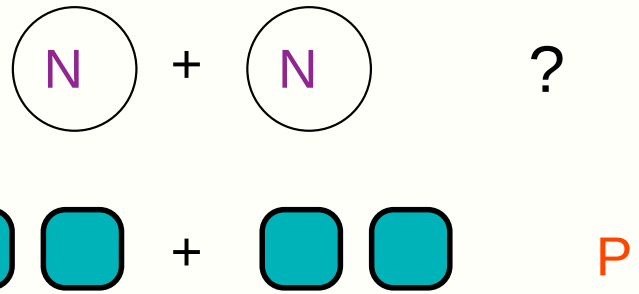


With only N -positions:



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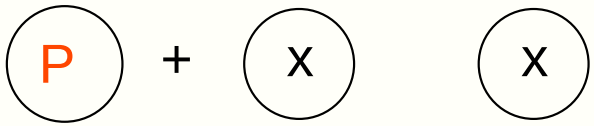

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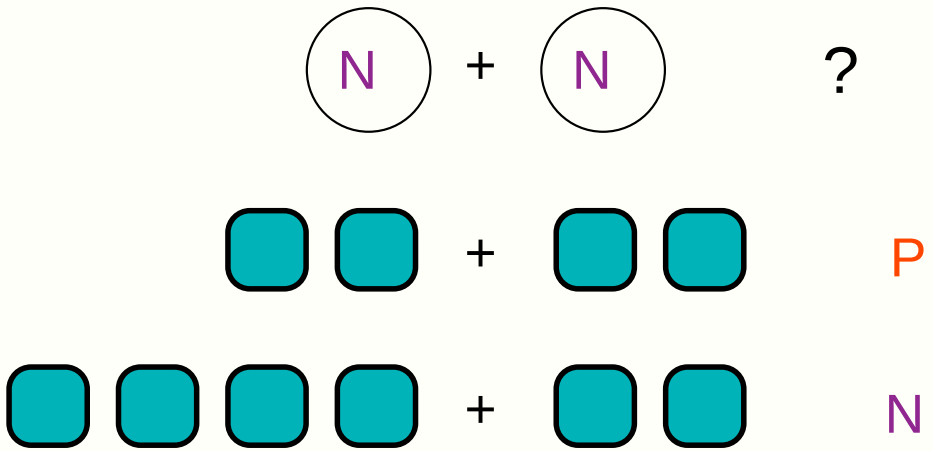

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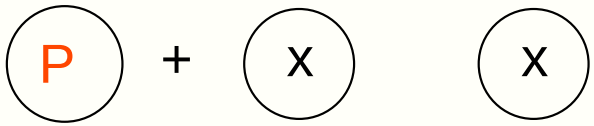

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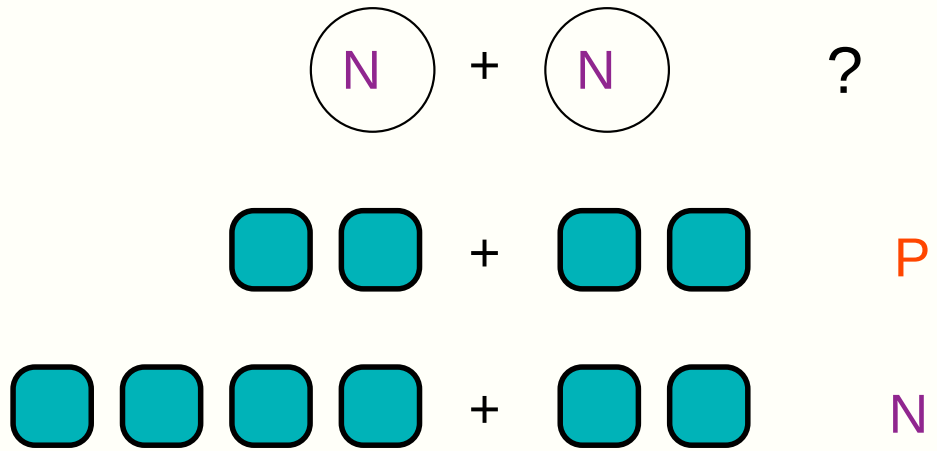

+	P	N
P	P	N
N	N	P or N

Grundy values

Let $I \subseteq \mathbb{N}$. MeX (minimum excluded value) $:= \min \mathbb{N} \setminus I$.

$$\mathrm{MeX}(\{0,1,3,5\}) = 2\,; \quad \mathrm{MeX}(\{2,3,6\}) = 0\,; \quad \mathrm{MeX}(\emptyset) = 0\,.$$

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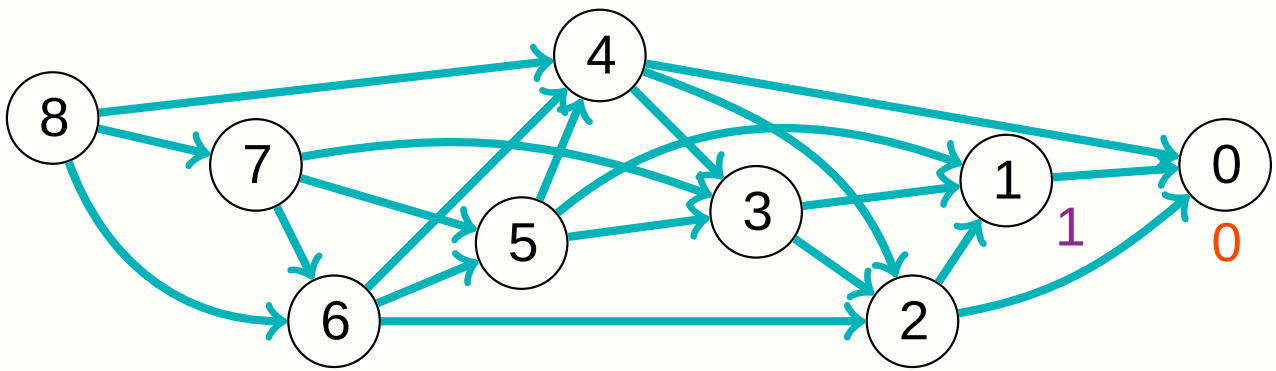

Proposition

$G(x) = 0$ if and only if x is P.

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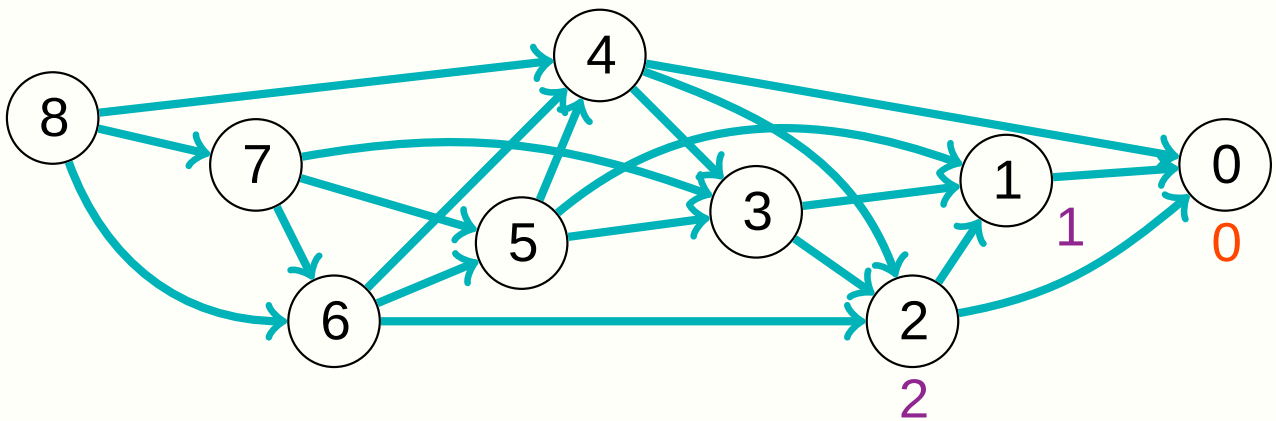

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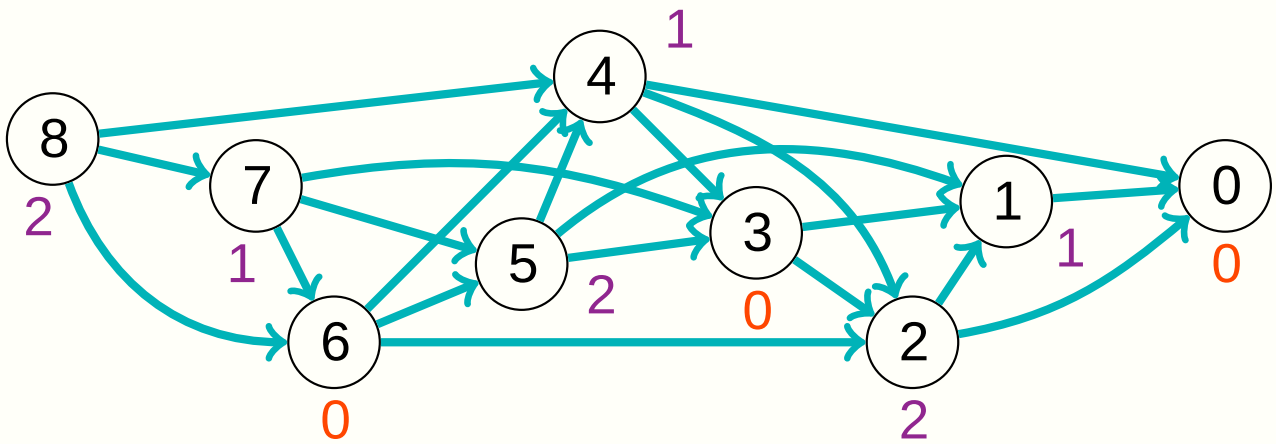

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Grundy value of the sum of games

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Let $x_1; x_2$ be two game positions. Then:

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Corollary

The sum $x_1 + x_2$ is P if and only if $G(x_1) = G(x_2)$.

Grundy sequence

Grundy sequence: sequence of grundy values for 1,2,3,... tokens.

For the subtraction game $\{1, 2, 4\}$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
G(n)	0	1	2	0	1	2	0	1	2	0	1	2	0

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Theorem Guy, Smith, 1956

The Grundy sequence of cram is periodic with period 34 and preperiod 53.

Octal games

Played on several heaps of tokens. A move consists in choosing a heap and, according to the rules:

- remove all the tokens from the heap, and delete this heap,
- remove some tokens from the heap, leaving 1 non-empty heap,
- remove some tokens from the heap, separating the remaining tokens into 2 non-empty heaps.

The number of tokens that can be removed is given by the game rules via an octal code

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Examples:

- The subtraction game$f1; 2; 4g$ is the octal game $0\ 3303$.
- The gamecram corresponds to the octal game 0007.
- The game $0\ 304$ allows you to remove 1 token without splitting the heap, or 3 tokens by necessarily dividing the heap.

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Conjecture Guy, 1956

The Grundy sequence of a finite octal game is ultimately periodic.

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It is open for very simple games like 06!

Which octal games are in P ?

http://wwwhomes.uni-bielefeld.de/achim/octal.html

Second Part
Rewriting games

Rewriting games

Rewriting games (Waldmann, 2002):

Rewriting system (terminal)

Starting from a wordt , players alternate applying rules to the word.

The player who can no longer apply a rule loses.

Example : $R_1 : ab \to \varepsilon$, $R_2 : aaa \to b$ and $t = aabbbaabaaa$

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Allows to model many games, including octal games.

Octal games as rewrite games

Alphabet on two letters: a (for tokens), b (to separate the heaps)

Example: Game CRAM can be modeled with rules $a \to \varepsilon$ and $aa \to b$.

aaaaaa ! aabaa

Octal games as rewrite games

Alphabet on two letters: a (for tokens), b (to separate the heaps)

Example: Game of CRAM can be modeled with rules aa ! " and aa ! b.

aaaaaa ! aabaa

The three rules of octal games can be translated with rewrite rules:

- Emptying a heap of k tokens: ba^kb ! b
- Removing k without emptying: a^{k+1} ! a
- Removing k and splitting in 2: a^{k+2} ! aba

Interpretation of Periodicity

For each word t, there is a corresponding Grundy value $G(t)$.
Grundy class $\mathcal{L}_k$: words with value k.

Theorem Waldmann, 2002

The Grundy sequence of an octal game is ultimately periodic iff, in the associated rewriting game, there is a finite number of non-empty Grundy classes, and each class is rational.

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) Find a DFA that determines if a given word $ba^{x_1}ba^{x_2}\cdots ba^{x_n}b$ satis es $G(x_1)\ :::\ G(x_n) = k$.

There exists a DFA that computes $G(x_i)$ $8i$

Before each new x_i, we keep in memory the previous sum $G(x_1)\ :::\ G(x_{i-1})$: possible because the number of Grundy classes is bounded (by M).

The new sum can be computed by a DFA.

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($\Leftarrow$) The L_k are rational.

$L_k \setminus ba^*b$ is rational.

Rational language with one letter: $\bigcup ba^{kp+\ell}b : k \in \mathbb{N}$

Partition of $\mathbb{N}$ $\Rightarrow$ periods are multiple.

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For any octal rewriting game, there is a finite number of non-empty Grundy classes, and each class is a rational language.

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With di erent rules? Rational classes?

What about other types of rules?

What happens if we can delete edges? ! Taking-and-merging games

What about other types of rules?

What happens if we can delete letters? ! Taking-and-merging games

Definition

A rewriting game is said to be \taking-and-merging" if all the rules are of the form $a^k ! "$ or $b^` ! "$

Notation: $a^{k_1}; a^{k_2}; : : : ; a^{k_n}; b^{`_1}; b^{`_2}; : : : b^{`_m}$

What about other types of rules?

What happens if we can delete 0's? ! Taking-and-merging games

Definition

A rewriting game is said to be \taking-and-merging" if all the rules are of the form $a^k \to \varepsilon$ or $b^\ell \to \varepsilon$

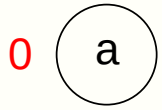
Notation: $a^{k_1}; a^{k_2}; \ldots; a^{k_n}; b^{\ell_1}; b^{\ell_2}; \ldots b^{\ell_m}$

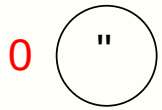
Question: Are the Grundy classes rational?

A first example: the game $a^2; b$

Rules: $aa \to \varepsilon$ and $b \to \varepsilon$

We construct the DAG starting with small words:





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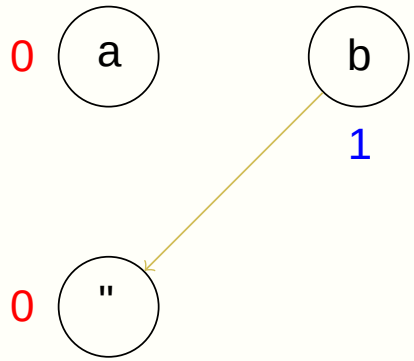

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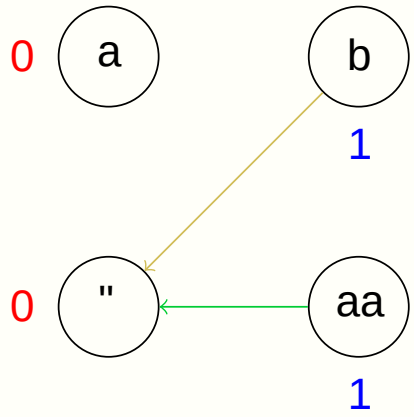

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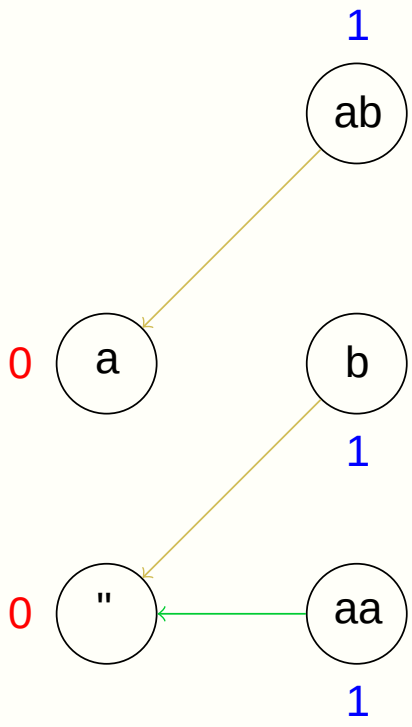

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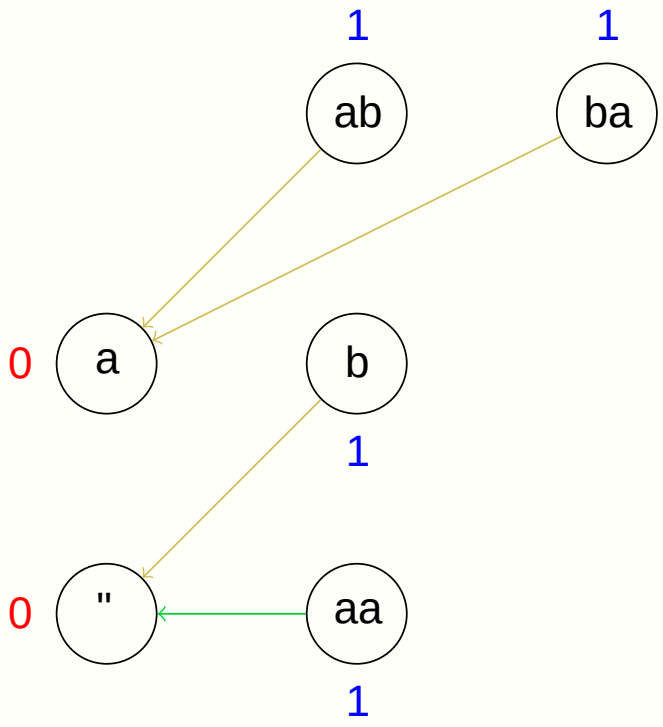

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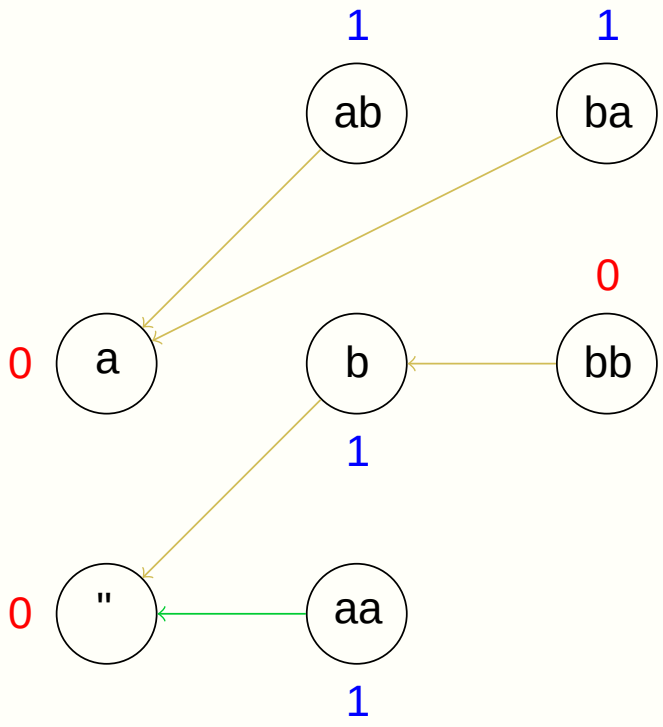

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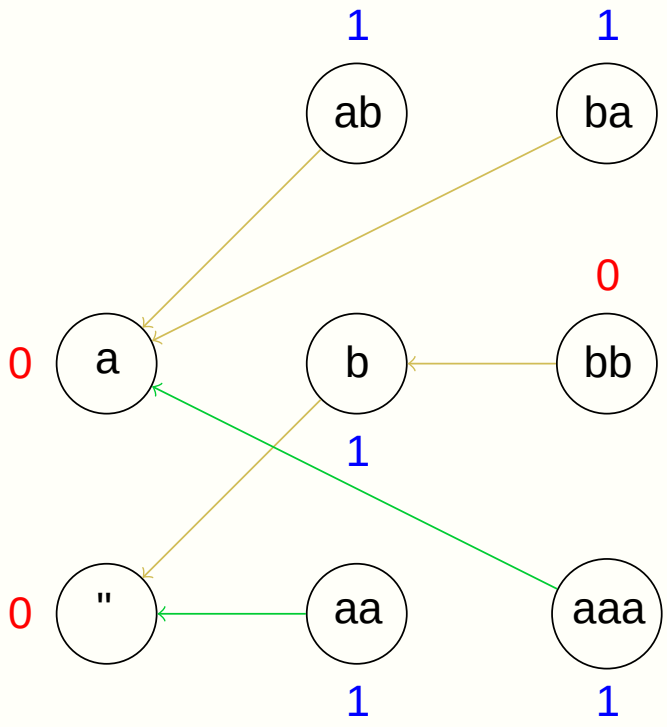

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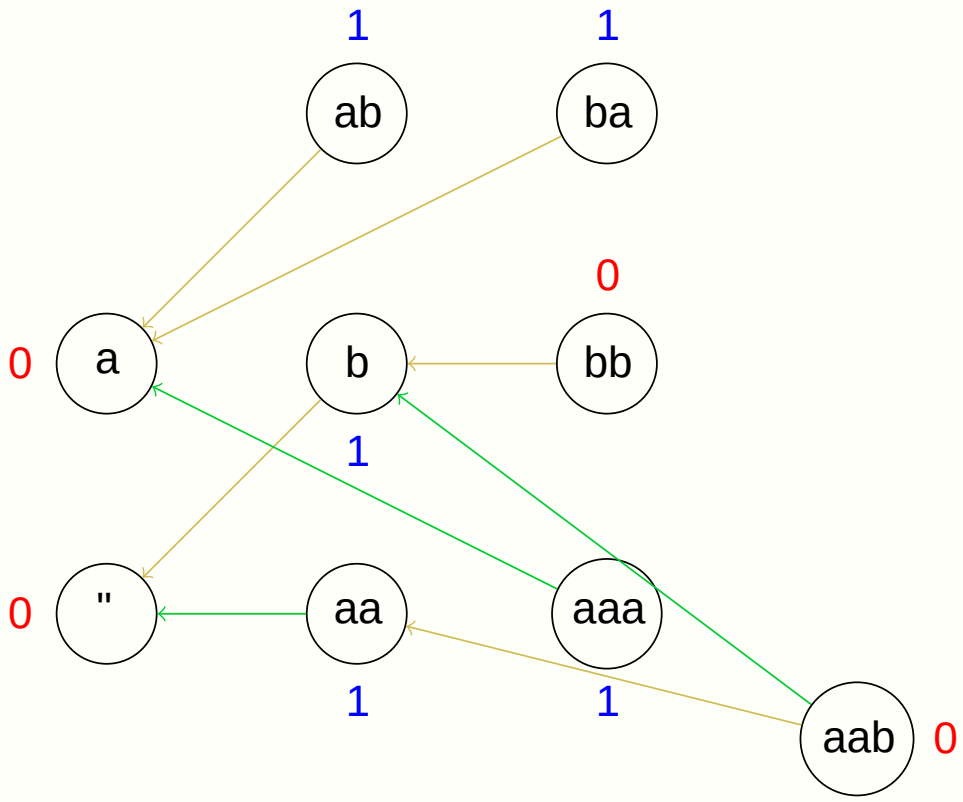

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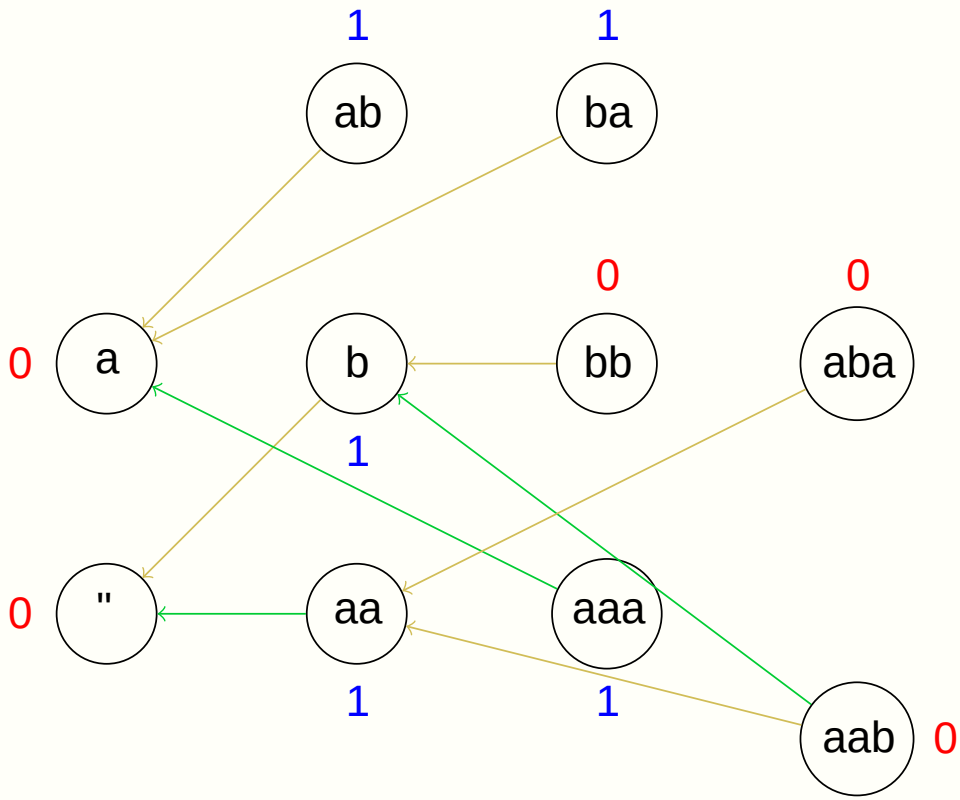

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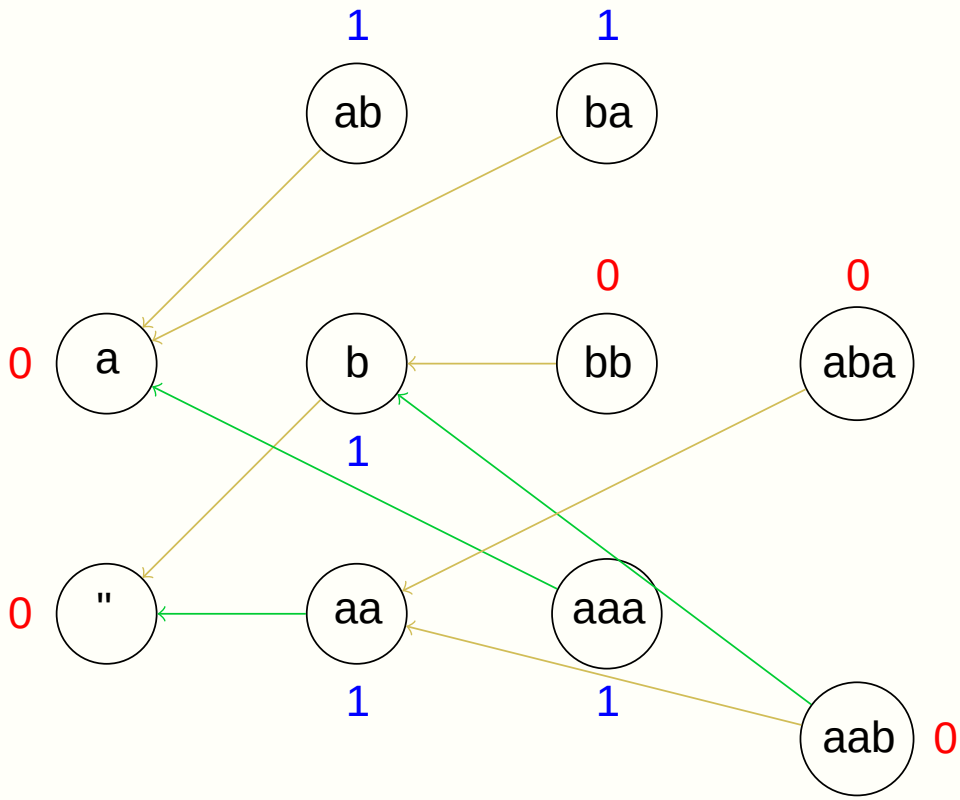

The quantity $|u|_a + 2|u|_b$ decreases by 2 after each move.

$! G\ (u) = 0 \iff |u|_a + 2|u|_b \bmod 4 \in \{0; 1\}$

The game $\{a^2, b\}$ is rational

Theorem D., Marsault, P., Rigo, 2020

The game $\{a^2, b\}$ has two classes of Grundy values L_0 and L_1, each forming a rational language.

DFA computing $S(u) = (|u|_a + 2|u|_b) \bmod 4$:

State $\overset{g:s}{\bigcirc}$ stands for Grundy value g and $S(u) = s$

Rationality of rewriting games?

Theorem D., Marsault, P., Rigo, 2020

Let G be the rewriting game $\{a^{k_1}, a^{k_2}, \ldots, b^{\ell_1}, b^{\ell_2}, \ldots\}$, where $1 < k_1 < k_2 < \cdots$ and $1 < \ell_1 < \ell_2 < \cdots$.
The language L_0 formed by the P-positions of G is not rational.

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Proof idea:

Intersection of L_0 with

$$L = b^{\ell_1 - 1}(ab^{\ell_1 - 1}) (ba^{k_1 - 1}) \cdots$$

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Proof idea:

Intersection of L_0 with

$$L = b^{\ell_1 - 1}(ab^{\ell_1 - 1})(ba^{k_1 - 1}) \cdots$$

By induction: $b^{\ell_1 - 1}(ab^{\ell_1 - 1})^i (ba^{k_1 - 1})^j$ from L is $\mathcal{P}$ i i j.

The intersection of L and L_0 is not rational, and thus L_0 is not rational.

Games with two rules

Theorem D.,Marsault, P.,Rigo, 2020

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Remark: If $\ell = 1$ (or $k = 1$), the stack is not longer needed and L_0, L_1 are rational. The number of moves is $|u|_b + b\frac{|u|_a}{k}c$.

Games with $a \to$ " and $b \to$ "

$\{a, a^{2k+1}, b\}$: G has 2 values, L_0 and L_1 are rational.

Games with $a!$ " and $b!$ "

$\{a, a^{2k+1}, b\}$: G has 2 values, L_0 and L_1 are rational.

$\{a, a^2, b\}$: G has 4 values, computed by a DFA with 12 states.

Games with $a \to \varepsilon$ and $b \to \varepsilon$

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$\{a, a^2, a^3, b\}$:

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Games with $a\ !\ ''$ and $b\ !\ ''$

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$\{a; a^2; a^3; b\}$: G has 4 values, computed by a DFA with 8 states.

$\{a; a^2; a^3; a^4; b\}$: open

$$(\max G(u))_{|u|=0;1;2;\ldots} =$$

$$0; 1; 2; 3; 4; 5; 5; 6; 7; 7; 7; 7; 7; 8; 9; 9; 10; 11; 11; 12; 13; 13; 13; 14$$

Question: Are the values for this game bounded?

Computing the rationality more easily

Need to examine all the L_i's ?

Theorem Waldmann, 2002

For an octal rewriting game:
L_0 rational $\Rightarrow$ Grundy is bounded + all L_i are rational.

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- Question : For the game $a; a^2; b; b^2$, is G bounded? Is there an i such that L_i is not rational?

Deciding P -position of a rewriting game is not decidable

Theorem DMPR,2020

Given a rewriting game and a rational language L_R of starting positions, it is undecidable to determine if there exists a P -position in L_R.

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Conjecture

Given a rewriting game, it is undecidable to determine if the P-positions are rational.

Conclusion

Rewriting games generalize a large set of combinatorial games with a nice equivalence between periodicity and regularity.

Some very simple games are not solved yet like $f\mathsf{a}; \mathsf{a}^4; \mathsf{b}g$.

What happens if we add the rule $\mathsf{a} \; ! \; \mathsf{b}$?

Is there some relation between the DAG, the rules and the DFA ?

Which games are (not) rewriting games?

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Thank you !