On a Faithful Representation of Sturmian Monoid

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Representation of a monoid

Let $(\mathcal{M}, *)$ be a monoid
Let $L(V)$ be the set of all linear operators on a vector space V.

A **representation of the monoid** $(\mathcal{M}, *)$ is a mapping $\mathcal{R} : \mathcal{M} \to L(V)$ such that

$$\mathcal{R}(x * y) = \mathcal{R}(x) \circ \mathcal{R}(y) \qquad \text{for every } x, y \in \mathcal{M}.$$

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Example: Let $\mathcal{A}$ be a finite alphabet. Consider the monoid of all morphisms over $\mathcal{A}^*$ with composition $\circ$:

$\mathcal{R} : \varphi \mapsto M_\varphi$, where M_φ is the incidence matrix of φ, is a representation.
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A representation $\mathcal{R}$ is **faithful** if $\mathcal{R}$ is injective, i.e. if $x \neq y$, then $\mathcal{R}(x) \neq \mathcal{R}(y)$.

Definition: A morphism $\varphi : \{0,1\}^* \mapsto \{0,1\}^*$ is Sturmian if the sequence $\varphi(\mathrm{u})$ is Sturmian for any Sturmian sequence u.

Monoid of Sturm is the set of all Sturmian morphisms with composition $\circ$, it is generated by

$$E : \begin{cases} 0 \to 1 \\ 1 \to 0 \end{cases}, \qquad G : \begin{cases} 0 \to 0 \\ 1 \to 01 \end{cases} \qquad \text{and} \qquad \widetilde{G} : \begin{cases} 0 \to 0 \\ 1 \to 10 \end{cases}$$

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Lower and upper Sturmian sequences

Lower and upper mechanical sequences:

$$s_{\alpha,\delta}(n) := \lfloor \alpha(n+1)+\delta \rfloor - \lfloor \alpha n + \delta \rfloor$$

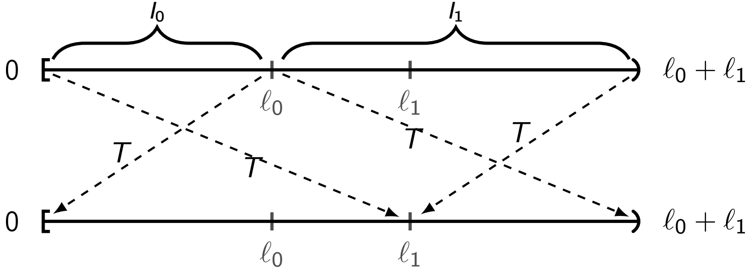
and

$$s'_{\alpha,\delta}(n) := \lceil \alpha(n+1)+\delta \rceil - \lceil \alpha n + \delta \rceil$$

for each $n \in \mathbb{N}$.

If $\alpha \in (0,1)$ is irrational, then
$s_{\alpha,\delta}$ is the lower Sturmian sequence with slope α and intercept δ;
$s'_{\alpha,\delta}$ is the upper Sturmian sequence with slope α and intercept δ.

Two interval exchange



$$T(x) = \begin{cases} x + \ell_1 & \text{if } x \in I_0, \\ x - \ell_0 & \text{if } x \in I_1. \end{cases}$$

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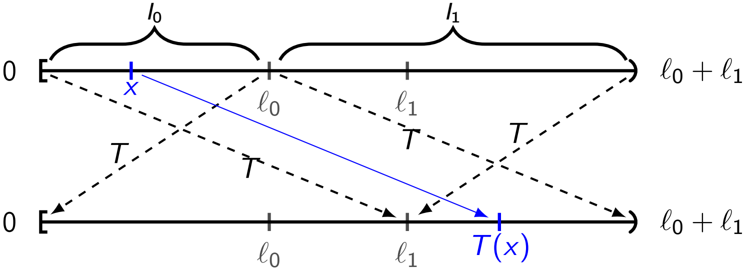

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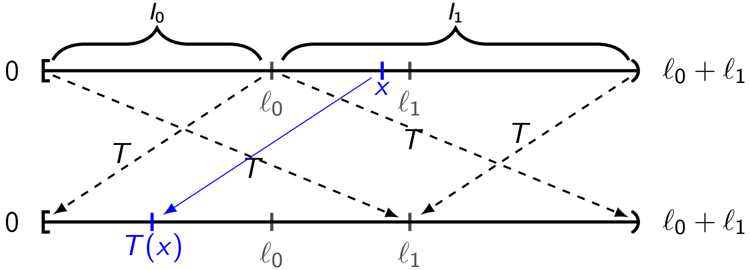

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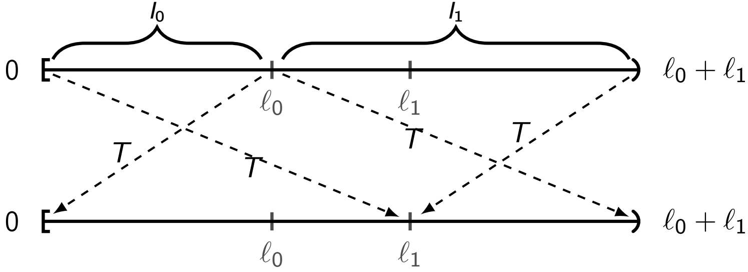

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Given $\rho \in I_0 \cup I_1$, the infinite sequence $\mathrm{u} = u_0 u_1 u_2 \ldots$ defined by

$$u_n = \begin{cases} 0 & \text{if } T^n(\rho) \in I_0, \\ 1 & \text{if } T^n(\rho) \in I_1, \end{cases}$$

equals the lower Sturmian sequence with slope $\frac{\ell_1}{\ell_0+\ell_1}$ and intercept $\frac{\rho}{\ell_0+\ell_1}$, i.e., $\mathrm{u} = \mathrm{s}_{\frac{\ell_1}{\ell_0+\ell_1}, \frac{\rho}{\ell_0+\ell_1}}$.

Vector of parameters of the sequence u

u is fully described by the vector of parameters (ℓ_0, ℓ_1, ρ) except for the fact whether it is lower or upper Sturmian sequence

u is also described by any $c(\ell_0, \ell_1, \rho)$ for $c > 0$.

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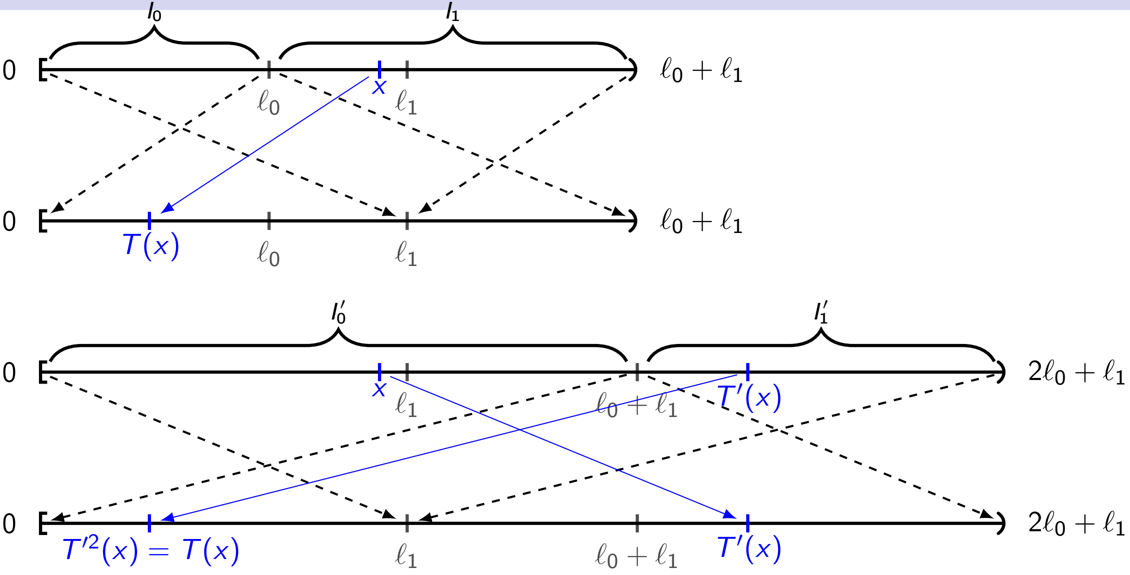
For $\delta \in (0, 1)$, $s_{\alpha,\delta}$ and $s'_{\alpha,\delta}$ have the same parameters;
$s_{\alpha,0}$ has parameters $(1 - \alpha, \alpha, 0)$;
$s'_{\alpha,0}$ has parameters $(1 - \alpha, \alpha, 1)$.

Action of $G : 0 \mapsto 0, 1 \mapsto 01$ on u



Action of generators of $\mathcal{M}$ on u

[Parvaix, 1997]

$$G(\mathsf{s}_{\alpha,\delta}) = \mathsf{s}_{\frac{\alpha}{1+\alpha},\frac{\delta}{1+\alpha}}, \quad \widetilde{G}(\mathsf{s}_{\alpha,\delta}) = \mathsf{s}_{\frac{\alpha}{1+\alpha},\frac{\alpha+\delta}{1+\alpha}} \quad \text{and} \quad E(\mathsf{s}_{\alpha,\delta}) = \mathsf{s}'_{1-\alpha,1-\delta}.$$

Let u be a lower (upper) Sturmian sequence with parameters (ℓ_0, ℓ_1, ρ). The lower (upper) Sturmian sequence

- $G(\mathsf{u})$ has parameters $(\ell_0 + \ell_1, \ell_1, \rho)$;
- $\widetilde{G}(\mathsf{u})$ has parameters $(\ell_0 + \ell_1, \ell_1, \rho + \ell_1)$;
- $D(\mathsf{u})$ has parameters $(\ell_0, \ell_0 + \ell_1, \rho + \ell_0)$;
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- $\widetilde{D}(\mathrm{u})$ has parameters $(\ell_0, \ell_0 + \ell_1, \rho)$.

$$R_G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{\widetilde{G}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad R_{\widetilde{D}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad R_D = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

In particular, if $\begin{pmatrix} \ell_0 \\ \ell_1 \\ \rho \end{pmatrix}$ are parameters of u, then $\mathcal{R}(\psi) \begin{pmatrix} \ell_0 \\ \ell_1 \\ \rho \end{pmatrix}$ are parameters of $\psi(\mathrm{u})$ for $\psi \in \{G, \widetilde{G}, D, \widetilde{D}\}$.

Representation $\mathcal{R} : \mathcal{M} \mapsto \mathbb{R}^{3\times 3}$

$\mathcal{M}$ is not free: its presentation is for any $k \in \mathbb{N}$

$$GD^k\widetilde{G} = \widetilde{G}\widetilde{D}^kG \quad \text{and} \quad DG^k\widetilde{D} = \widetilde{D}\widetilde{G}^kD.$$

[Theorem 2.3.14, Lothaire: Algebraic combinatorics on words]

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$$\mathcal{R}(GD^k\widetilde{G}) = \mathcal{R}(\widetilde{G}\widetilde{D}^kG) \quad \text{and} \quad \mathcal{R}(DG^k\widetilde{D}) = \mathcal{R}(\widetilde{D}\widetilde{G}^kD).$$

For $\psi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n \in \mathcal{M} = \langle G, \widetilde{G}, D, \widetilde{D}\rangle$, $\varphi_i \in \left\{G, \widetilde{G}, D, \widetilde{D}\right\}$ set

$$\mathcal{R}(\psi) = R_{\varphi_1}R_{\varphi_2}\cdots R_{\varphi_n}.$$

$\Psi \in \mathcal{M}$ maps a lower Sturmian sequence to a lower Sturmian sequence, and an upper Sturmian sequence to an upper Sturmian sequences.

$\mathcal{R}$ is a faithful representation of $\mathcal{M}$.

Fixed points

$$\mathcal{R}(\psi) = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix}, \quad \text{with } a, b, c, d, e, f \in \mathbb{N} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = M_\psi.$$

Theorem 1 (Lepšová, Pelantová, S., 2022)

Let $\psi \in \mathcal{M}$ be a primitive morphism and u be a Sturmian sequence with the vector of parameters (ℓ_0, ℓ_1, ρ). The sequence u is fixed by ψ if and only if $(\ell_0, \ell_1, \rho)^\top$ is an eigenvector to the dominant eigenvalue of $\mathcal{R}(\psi)$.

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We could work with the whole Sturmian monoid with

$$R_E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

However, that makes things more complicated, and we know that if ψ is Sturmian, then $\psi^2 \in \mathcal{M}$.

Characterization of $\mathcal{R}(\mathcal{M})$ by convex closed cones

$$\mathcal{R}(\mathcal{M}) \subset Sl(\mathbb{N},3) = \{M \in \mathbb{N}^{3\times 3} : \det R = 1\} \subset Sl(\mathbb{Z},3)$$

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Put

$$C_1 := \{(x,y,z)^\top \in \mathbb{R}^3 \colon 0 \leq x, 0 \leq y, 0 \leq z \leq x+y\},$$
$$C_2 := \{(x,y,z)^\top \in \mathbb{R}^3 \colon 0 \leq x, 0 \geq y, y \leq z \leq x\}, \text{ and}$$
$$C_3 := \{(0,0,z)^\top \in \mathbb{R}^3 \colon 0 \leq z\}$$

If $\psi \in \mathcal{M}$, then

$$\mathcal{R}(\psi)(C_1) \subset C_1, \quad \left(\mathcal{R}(\psi)\right)^{-1}(C_2) \subset C_2, \quad \text{and} \quad \mathcal{R}(\psi)(C_3) = C_3.$$

Theorem 2 (Lepšová, Pelantová, S., 2022)

$$\mathcal{R}(\mathcal{M}) = \{R \in Sl(\mathbb{Z},3) : R(C_1) \subset C_1,\ R^{-1}(C_2) \subset (C_2),\ R(C_3) = C_3\}.$$

Application 1

Which parameters ℓ_0, ℓ_1, ρ allow the relevant Sturmian sequence to be fixed by a primitive substitution?

Theorem 3 (Yasutomi, 1999)

Let u *be a Sturmian sequence with the parameters* $(1-\alpha, \alpha, \rho)$, *where* $\alpha \in (0,1)$ *is irrational and* $\rho \in [0,1)$. *The sequence* u *is fixed by a primitive morphism if and only if*

1. α *and* ρ *belong to the same quadratic field* $\mathbb{Q}(\sqrt{m})$;
2. $\overline{\alpha} \notin (0,1)$;
3. $\min\{\overline{\alpha}, 1-\overline{\alpha}\} \leq \overline{\rho} \leq \max\{\overline{\alpha}, 1-\overline{\alpha}\}$,

where the mapping $x \mapsto \overline{x}$ *is the non-trivial field automorphism on* $\mathbb{Q}(\sqrt{m})$ *induced by* $\sqrt{m} \mapsto -\sqrt{m}$.

The faithful representation $\mathcal{R}$ allows for a short proof of the implication $(\Longrightarrow)$.

Proof of [Yasutomi, 1999] ($\Longrightarrow$) I

Assume u is a Sturmian sequence with the parameters $(1-\alpha, \alpha, \rho)^\top = \vec{p}$, $\alpha \in (0,1)$ is irrational and $\rho \in [0,1)$, $\Psi(\mathrm{u}) = \mathrm{u}$ for a primitive morphism Ψ.

By Theorem 1: $\vec{p}$ is an eigenvector of the dominant eigenvalue Λ of $\mathcal{R}(\psi) = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix}$

In detail: Λ is an eigenvalue of the primitive matrix $M_\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with eigenvector $(1-\alpha, \alpha)^\top$, hence

1. Λ is quadratic, say $\mathbb{Q}(\Lambda) = \mathbb{Q}(\sqrt{m})$;
2. $1-\alpha, \alpha \in \mathbb{Q}(\Lambda)$;
3. $e(1-\alpha) + f\alpha + 1\rho = \Lambda\rho$ implies $\rho \in \mathbb{Q}(\Lambda)$.

Proof of [Yasutomi, 1999] ($\Longrightarrow$) II

$\overline{\Lambda}$ is another eigenvalue, its eigenvector is $\overline{\vec{p}} = (1 - \overline{\alpha}, \overline{\alpha}, \overline{\rho})^\top$ is eigenvector
The last eigenvector is $(0, 0, 1)^\top$.

By Theorem 2: $\mathcal{R}(\Psi)^{-1} C_2 \subset C_2$.
Therefore, by Brouwer's theorem, at least one eigenvector of $\mathcal{R}(\Psi)^{-1}$ belongs to C_2.

$\mathcal{R}(\Psi)$ shares eigenvectors with $\mathcal{R}(\Psi)^{-1}$.

Proof of [Yasutomi, 1999] ($\Longrightarrow$) III

An eigenvector of $\mathcal{R}(\Psi)$ belongs to
$C_2 = \{(x, y, z)^\top \in \mathbb{R}^3 \colon 0 \leq x, 0 \geq y, y \leq z \leq x\}$

$(0, 0, 1)^\top \notin C_2$
Since $\alpha \in (0, 1)$, $\vec{p} = (1 - \alpha, \alpha, \rho)^\top \notin C_2$
Hence: $\overline{\vec{p}}$ or $-\overline{\vec{p}}$ is in C_2.

Both cases imply from the definition of C_2 what is to show:
$\overline{\alpha} \notin (0, 1)$
$\min\{\overline{\alpha}, 1 - \overline{\alpha}\} \leq \overline{\rho} \leq \max\{\overline{\alpha}, 1 - \overline{\alpha}\}$

Application 2

Recall that for each $n \in \mathbb{N}$:

$$\mathsf{s}_{\alpha,\delta}(n) := \lfloor \alpha(n+1) + \delta \rfloor - \lfloor \alpha n + \delta \rfloor$$

$$\mathsf{s}'_{\alpha,\delta}(n) := \lceil \alpha(n+1) + \delta \rceil - \lceil \alpha n + \delta \rceil$$

[Dekking,2018]:
Find slope α and intercept δ such that

1. $\mathsf{s}'_{\alpha,\delta} \neq \mathsf{s}_{\alpha,\delta}$,
2. $\mathsf{s}'_{\alpha,\delta}$ is fixed by a primitive morphism, and
3. $\mathsf{s}_{\alpha,\delta}$ is fixed by a primitive morphism.

Note:

$$\mathsf{s}_{\alpha,\delta}(n) \neq \mathsf{s}'_{\alpha,\delta}(n) \qquad \text{for at most two } n \in \mathbb{N}$$

Application 2

The faithful representation $\mathcal{R}$ allows for an alternative of the following:

Theorem 4 (Dekking,2018)

Let $\alpha \in (0,1)$, α irrational, and $\delta \in [0,1)$. Assume that both sequences $\mathsf{s}_{\alpha,\delta}$ and $\mathsf{s'}_{\alpha,\delta}$ are fixed by primitive morphisms and $\mathsf{s}_{\alpha,\delta} \neq \mathsf{s'}_{\alpha,\delta}$. Either

1. *$\delta = 1 - \alpha$, in which case $\mathsf{s}_{\alpha,\delta}$ and $\mathsf{s'}_{\alpha,\delta}$ are distinct fixed points of the same primitive morphism $\psi \in \langle \widetilde{G}, \widetilde{D} \rangle$; or*
2. *$\delta = 0$, in which case $\mathsf{s}_{\alpha,\delta}$ is fixed by a morphism $\psi \in \langle G, \widetilde{D} \rangle$ and $\mathsf{s'}_{\alpha,\delta}$ is fixed by a morphism $\eta \in \langle \widetilde{G}, D \rangle$. Moreover, if $\psi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n$ with $\varphi_i \in \{G, \widetilde{D}\}$, then*

$$\eta = \xi_1 \circ \xi_2 \circ \cdots \circ \xi_n, \quad \text{where} \quad \xi_i = \begin{cases} \widetilde{G} & \text{if} \quad \varphi_i = G, \\ D & \text{if} \quad \varphi_i = \widetilde{D}, \end{cases} \quad \text{for } i = 1, \ldots, n.$$

Application to fixed points of primitive Sturmian morphisms

We say that morphism ψ and φ over $\mathcal{A}$ are conjugate if there exists a $w \in \mathcal{A}^*$ such that
$\psi(a)w = w\varphi(a)$ for every $a \in \mathcal{A}$
or $w\psi(a) = \varphi(a)w$ for every $a \in \mathcal{A}$.

Note: If ψ is conjugate to φ, then $M_\psi = M_\varphi$.

The faithful representation $\mathcal{R}$ allows an alternative proof of:

Theorem 5 (Lothaire, 2002)

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(\mathbb{N}, 2)$*, then* M *is the incidence matrix of* $a + b + c + d - 1$ *mutually conjugate Sturmian morphisms.*

The square root of fixed point of a Sturmian morphism

[Saari (2010)]: for every Sturmian sequence u there exist 6 factors $w_1, \ldots, w_6$ such that

$$u = w_{i_1}^2 w_{i_2}^2 w_{i_3}^2 \ldots, \qquad \text{where } i_k \in \{1, \ldots, 6\} \text{ for each } k \in \mathbb{N},$$

and moreover, for each $k \in \mathbb{N}$, the shortest square prefix of the sequence $w_{i_k}^2 w_{i_{k+1}}^2 w_{i_{k+2}}^2 \ldots$ is $w_{i_k}^2$.

J. Peltomäki and M. Whiteland introduced

$$\sqrt{u} = w_{i_1} w_{i_2} w_{i_3} \ldots$$

Theorem 6 (J. Peltomäki and M. Whiteland, 2017)

If u is a Sturmian sequence with the slope α and the intercept δ, then $\sqrt{u}$ is a Sturmian sequence with the same slope α and the intercept $\frac{1-\alpha+\delta}{2}$.

Example: $\varphi : 0 \mapsto 10, 1 \mapsto 10101$

The fixed point of φ is

$$u = 10101101010110101011010110101011010101101010110101101010\ldots$$

u can be written as concatenation of the squares of these 6 factors:
$w_1 = 10, w_2 = 1, w_3 = 01, w_4 = 0110101, w_5 = 101, w_6 = 01101.$

$$u = 1010 \Big| 11 \Big| 0101 \Big| 01101010110101 \Big| 1010 \Big| 101101 \Big| 0101 \Big| 1010 \Big| 101101 \Big| 01101010 \ldots$$

$$\sqrt{u} = \underbrace{10}_{w_1}\underbrace{1}_{w_2}\underbrace{01}_{w_3}\underbrace{0110101}_{w_4}\underbrace{10}_{w_1}\underbrace{101}_{w_5}\underbrace{01}_{w_3}\underbrace{10}_{w_1}\underbrace{101}_{w_5}\underbrace{0110101}_{w_4}\underbrace{0110101}_{w_4}\underbrace{10}_{w_1}\ldots$$

A new result

Theorem 7 (Lepšová, Pelantová, S.)

Let $\mathrm{u} \in \{0,1\}^{\mathbb{N}}$ *be a Sturmian sequence fixed by a primitive morphism* $\varphi \in \mathcal{M}$*. The square root* $\sqrt{\mathrm{u}}$ *is fixed by a morphism* ψ *which is a conjugate of one of the morphisms* φ^k*,* $k \in \{1,2,3,4\}$*.*

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$$\varphi : 0 \mapsto 10, \quad 1 \mapsto 10101$$

$$\varphi^2 : 0 \mapsto 1010110, \quad 1 \mapsto 1010110101011010101$$

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$$\varphi : 0 \mapsto 10, \quad 1 \mapsto 10101$$

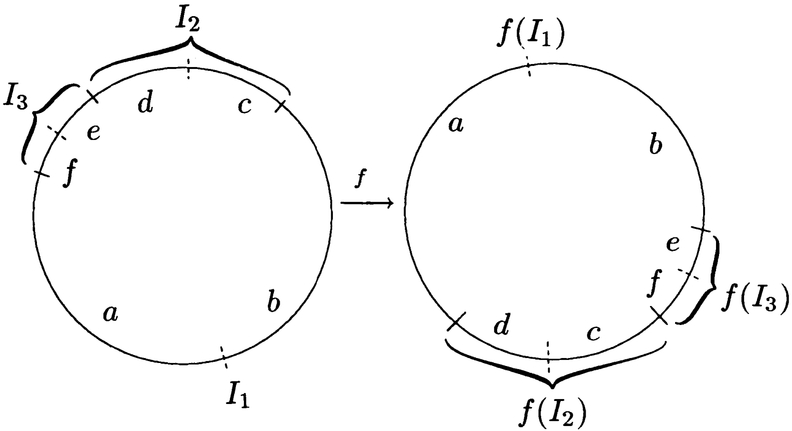
$$\varphi^2 : 0 \mapsto 1010110, \quad 1 \mapsto 10101101010110101 01$$

$$\psi : 0 \mapsto 1010101, 1 \mapsto 1010101101011010101,$$

$$\sqrt{\mathrm{u}} = \underbrace{10}_{w_1}\underbrace{1}_{w_2}\underbrace{01}_{w_3}\underbrace{0110101}_{w_4}\underbrace{10}_{w_1}\underbrace{101}_{w_5}\underbrace{01}_{w_3}\underbrace{10}_{w_1}\underbrace{101}_{w_5}\underbrace{0110101}_{w_4}\underbrace{0110101}_{w_4}\underbrace{10}_{w_1}\underbrace{101}_{w_5}\underbrace{01}_{w_3}\underbrace{1}_{w_2}\underbrace{10}_{w_1}\dots$$

Faithful representation of some other class of morphisms?

[P. Arnoux, G. Rauzy, 1991]:



Thank you for your attention