An aperiodic monotile

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An aperiodic monotile

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Joseph Samuel Myers
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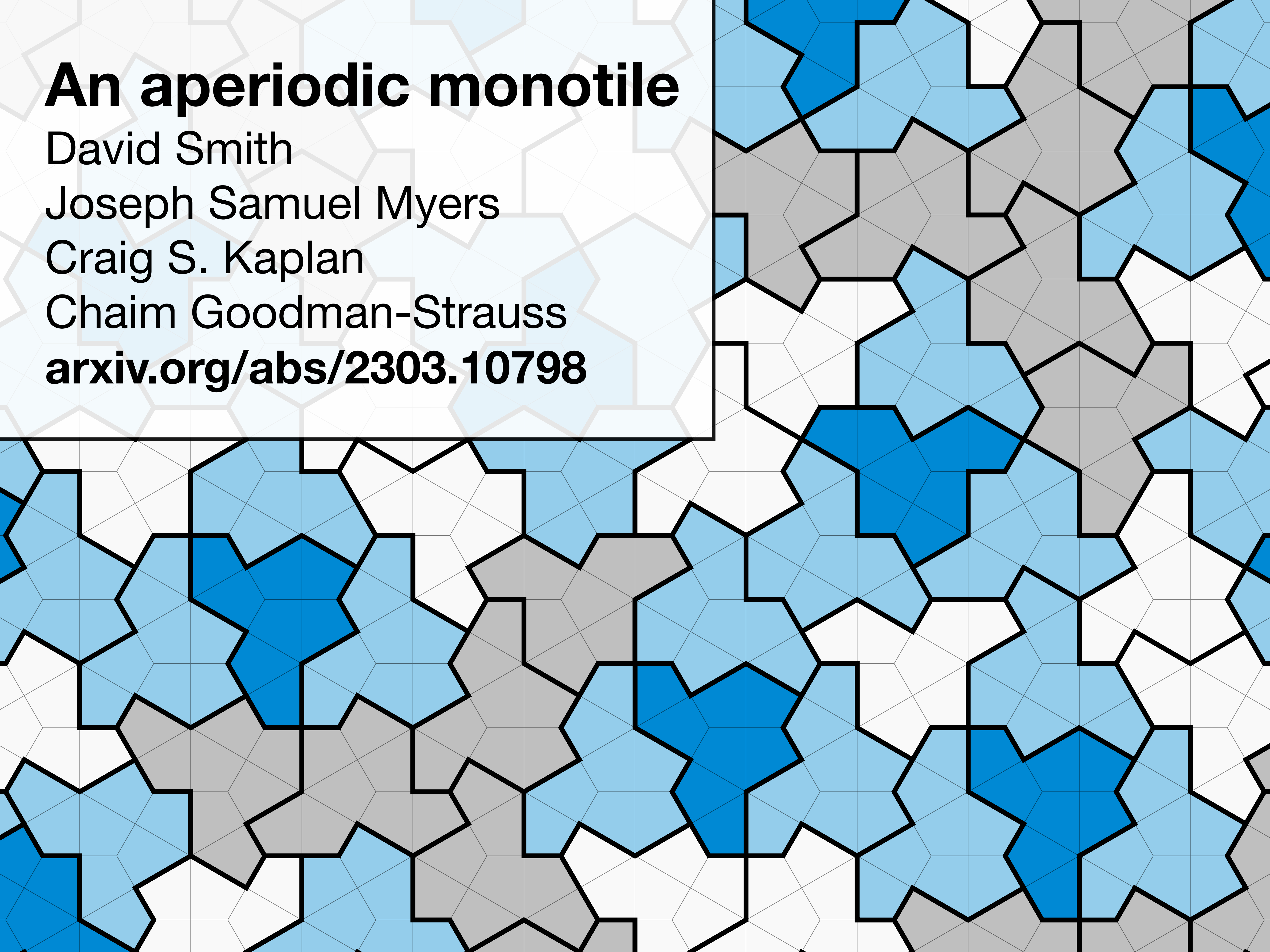
Chaim Goodman-Strauss

arxiv.org/abs/2303.10798



Basic terminology

A **tile** is a closed topological disk.

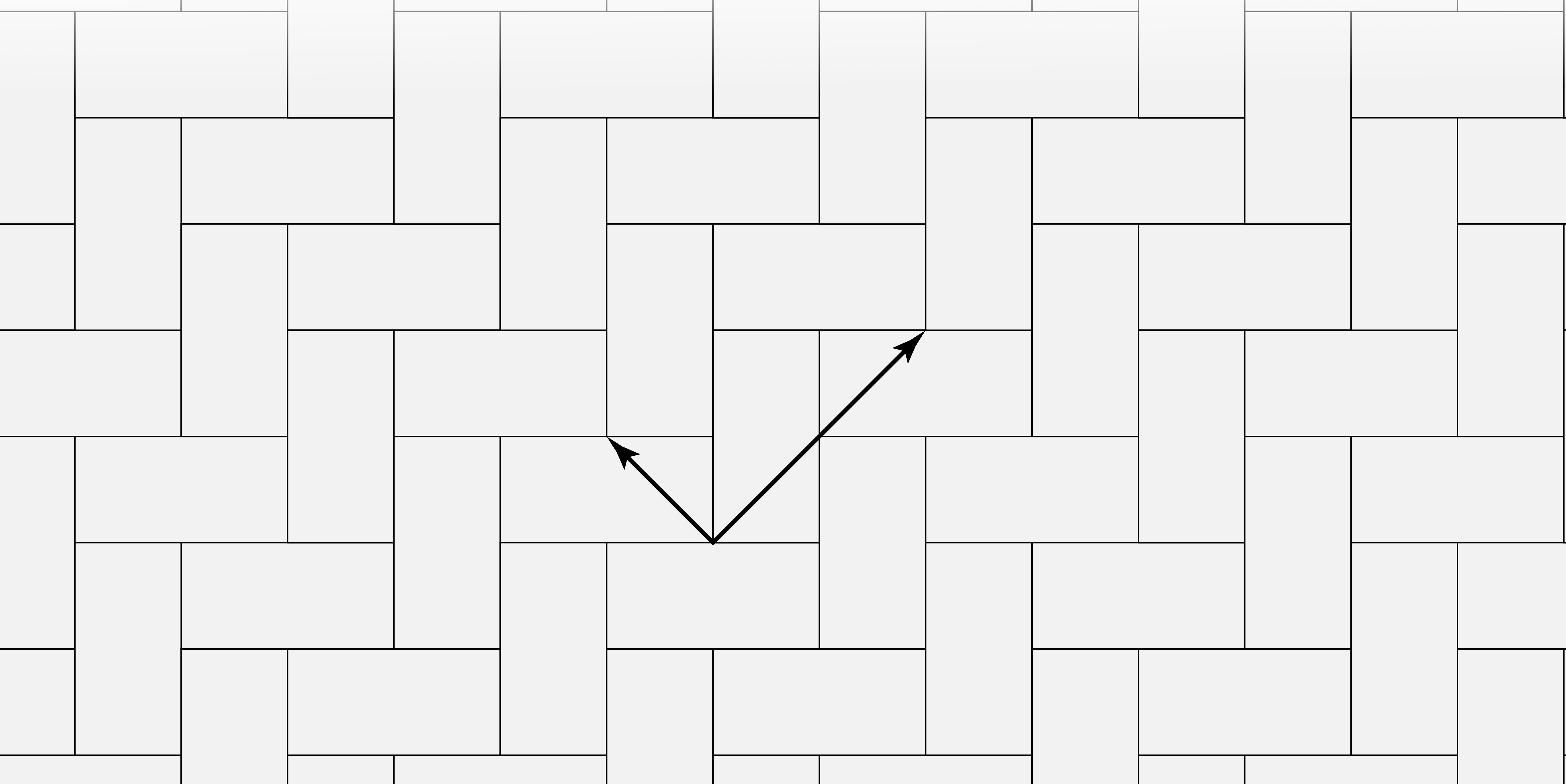
Let $\mathscr{S} = \{S_1, \ldots, S_n\}$ be a finite set of tiles. A **tiling from** $\mathscr{S}$ is a countable set $\mathscr{T} = \{T_1, T_2, \ldots\}$ such that

1. Every T_i is congruent to a member of $\mathscr{S}$;
2. The interiors of the T_i are pairwise disjoint; and
3. The union of the T_i is the whole plane

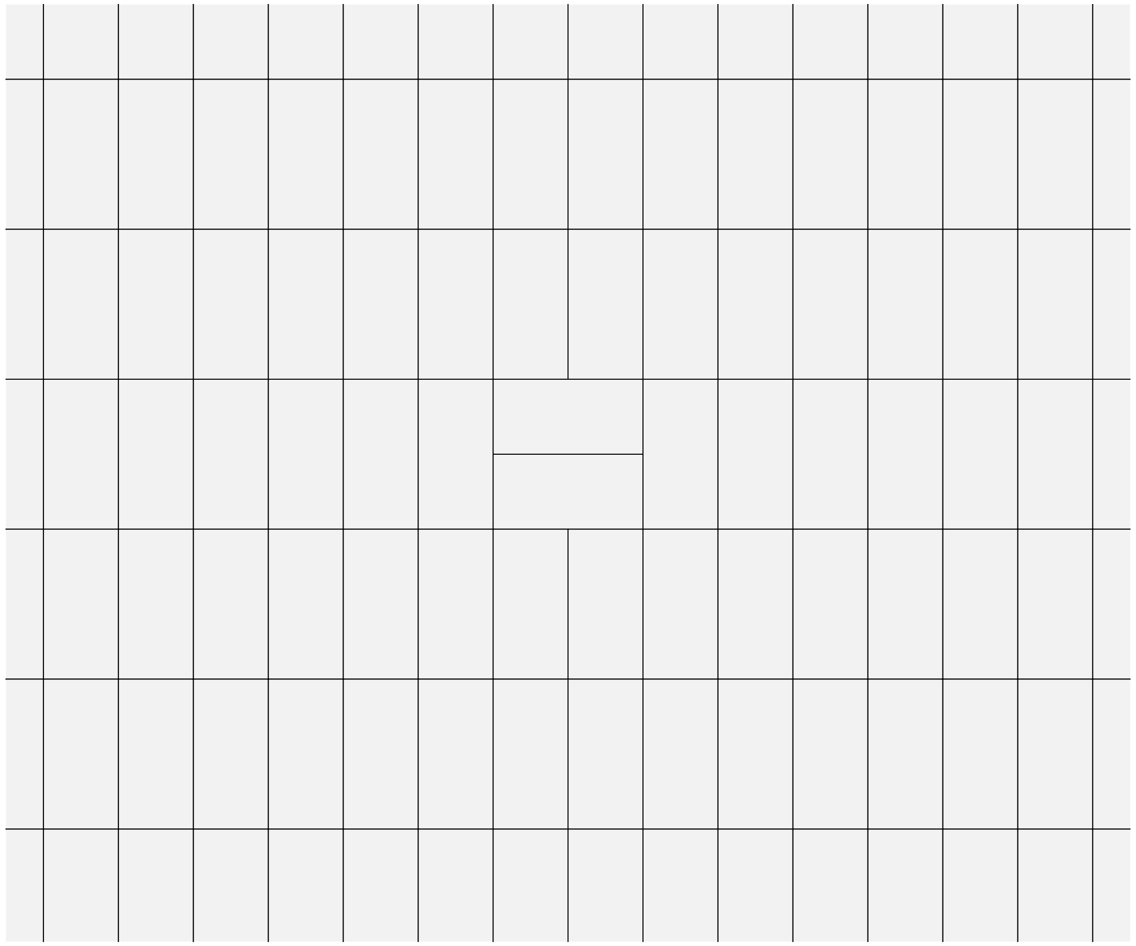
We say that $\mathscr{S}$ **admits** $\mathscr{T}$. If $\mathscr{S}$ consists of a single shape, the tiling is called **monohedral**.

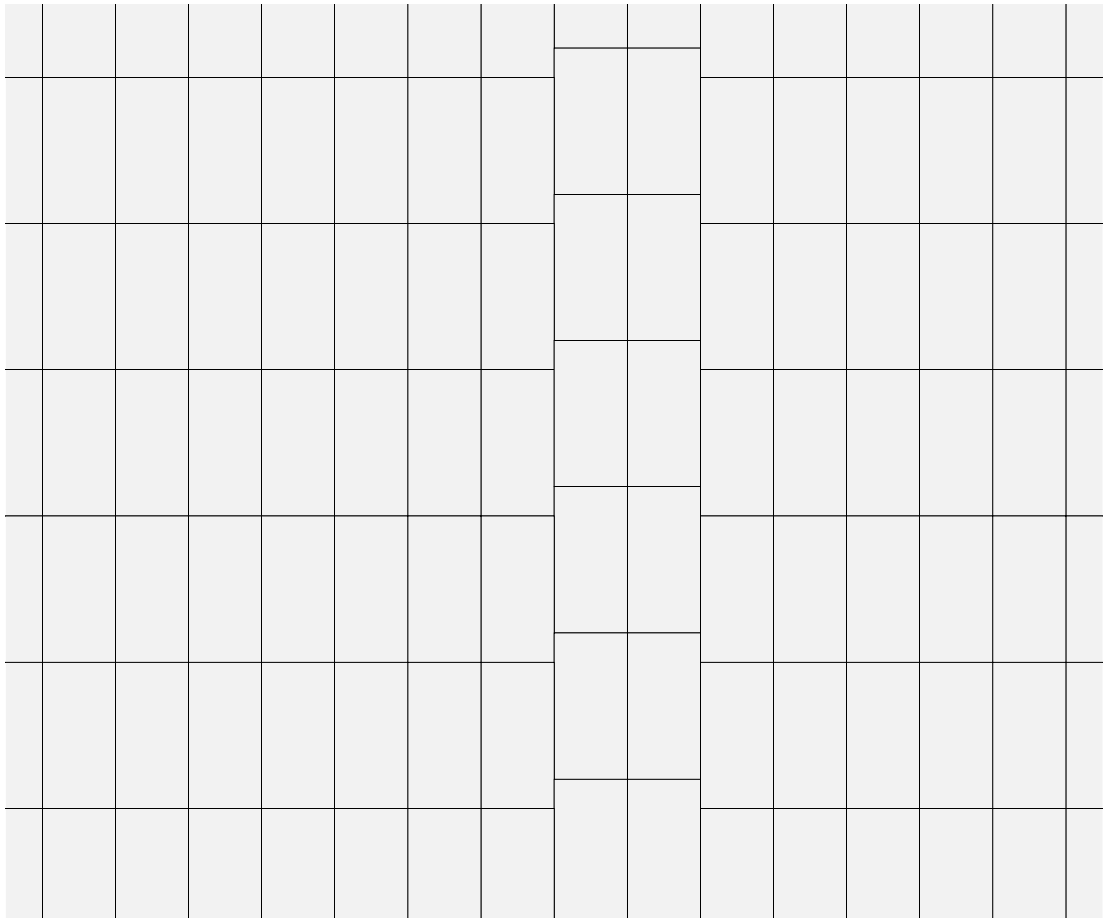
Periodicity

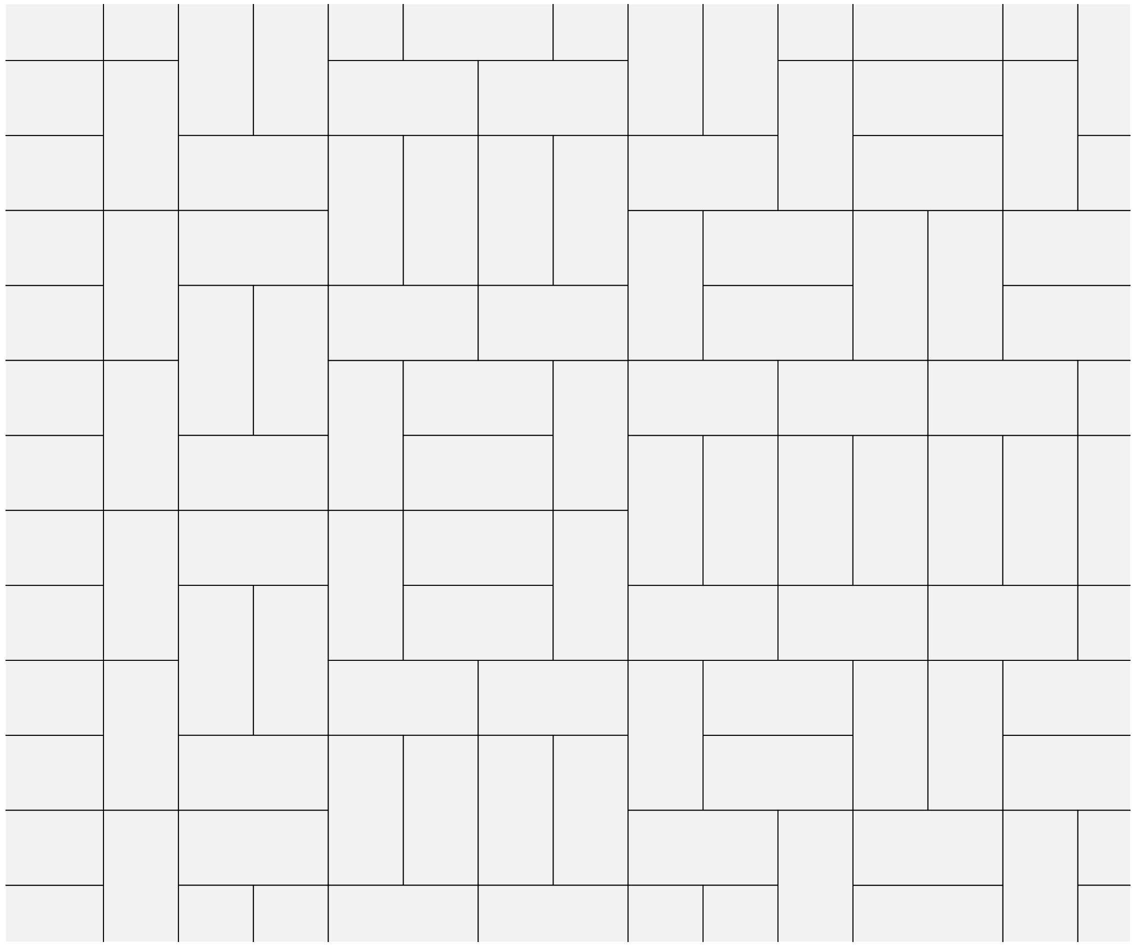
In the Euclidean plane, a tiling is **periodic** if its symmetry group includes at least two non-parallel translations, and **non-periodic** otherwise.



Non-periodicity is common: many sets of shapes admit both periodic and non-periodic tilings.







Aperiodicity

A set of tiles is **aperiodic** if it admits tilings, but none that are periodic.

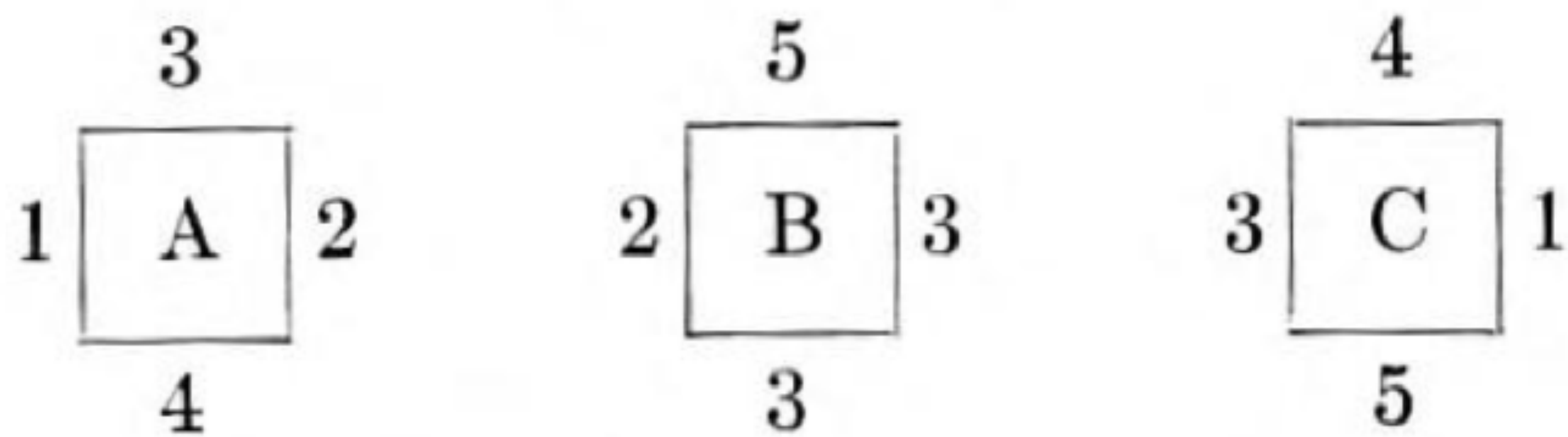
Aperiodicity is a property of a set of tiles, and not of a tiling! The tiles conspire to prevent periodicity.

Proving aperiodicity:

- Exhibit a non-periodic tiling
- Show that no tiling can be periodic

Wang tiles

Wang [1961] conjectured that there are no aperiodic sets of (Wang) tiles.



Clearly, a sufficient condition for a set of plates to have a solution is that there exists a cyclic rectangle of the plates.

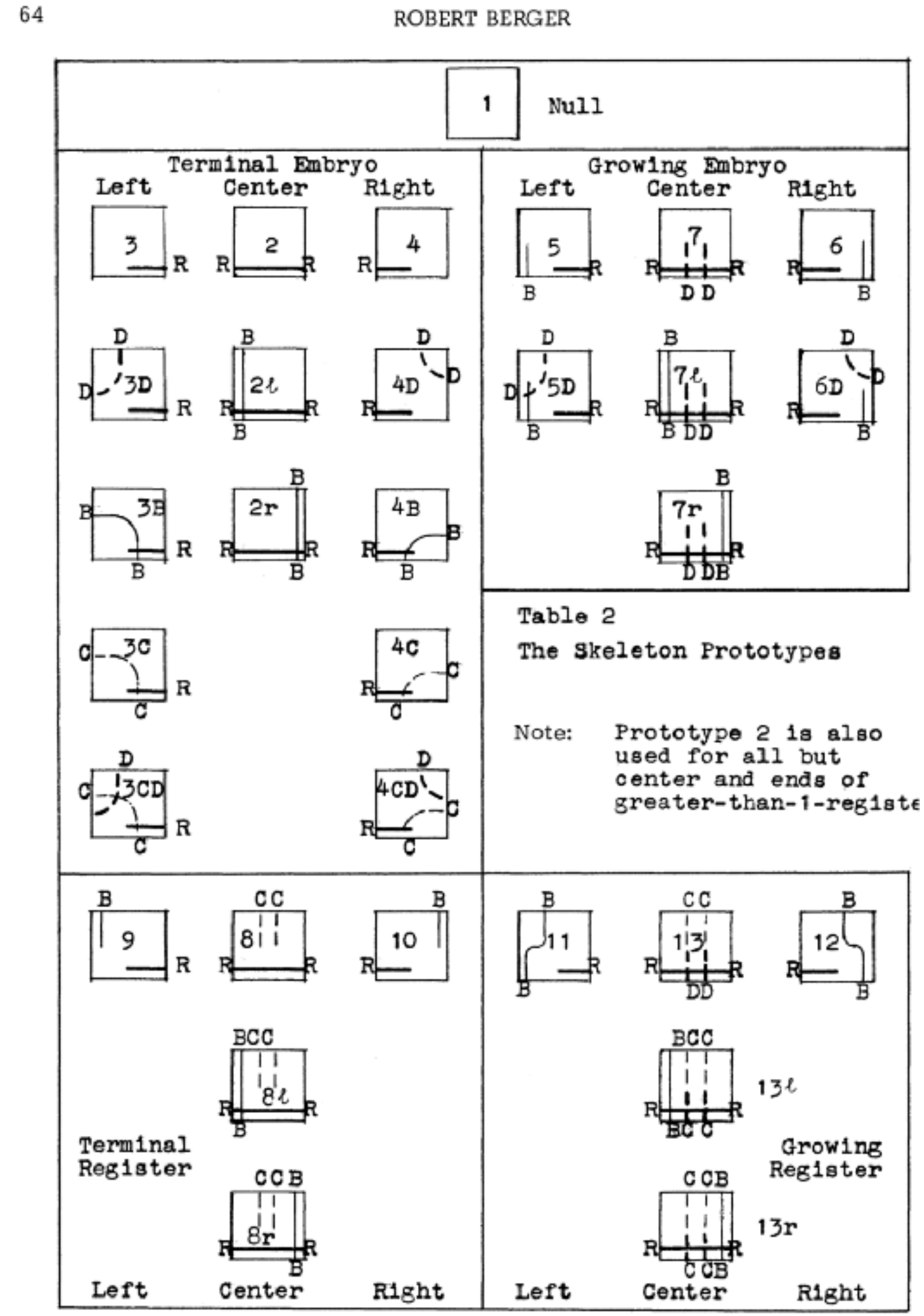
What appears to be a reasonable conjecture, which has resisted proof or disproof so far, is:

4.1.2 *The fundamental conjecture*: A finite set of plates is solvable (has at least one solution) if and only if there exists a cyclic rectangle of the plates; or, in other words, a finite set of plates is solvable if and only if it has at least one periodic solution.

Aperiodic Wang tiles

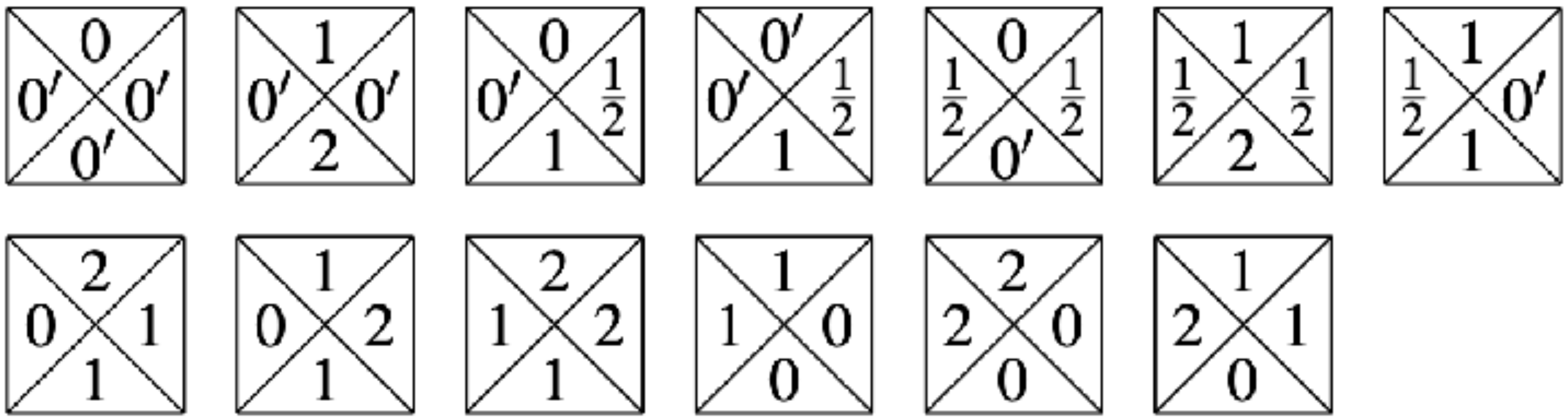
Berger [1966] exhibited an aperiodic set of 20426 Wang tiles (and remarked that smaller sets were possible).

Berger brought the total down to 104; Knuth [1968] managed 92.

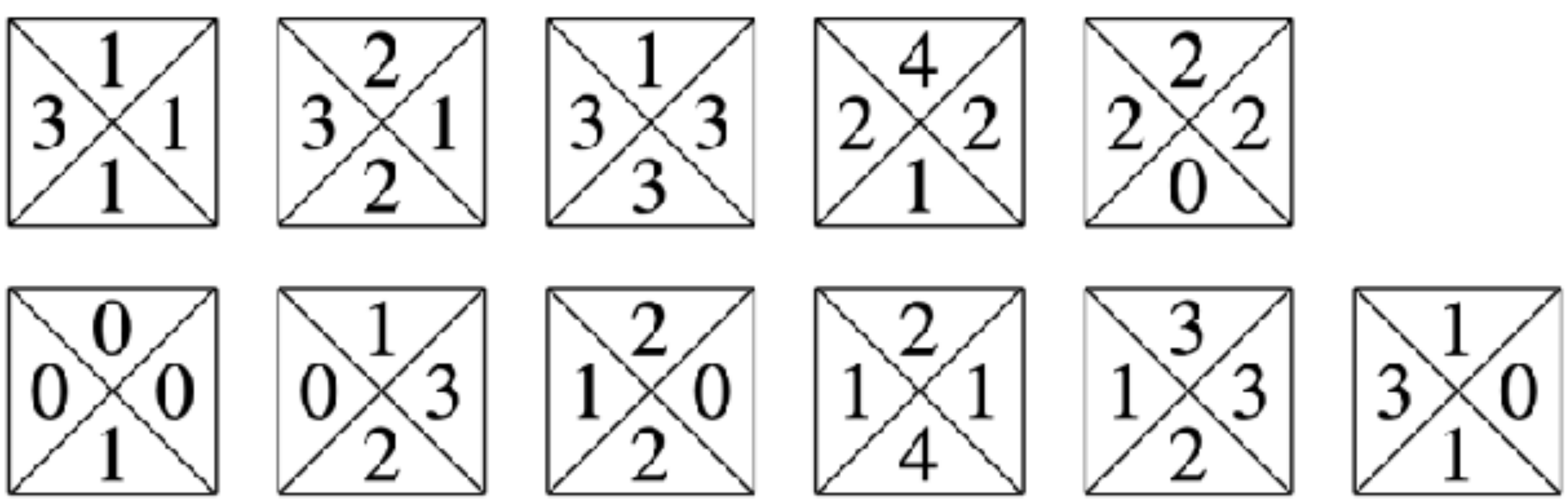


Minimal aperiodic Wang tiles

Culik [1996] exhibited a set of aperiodic Wang tiles of size 13.

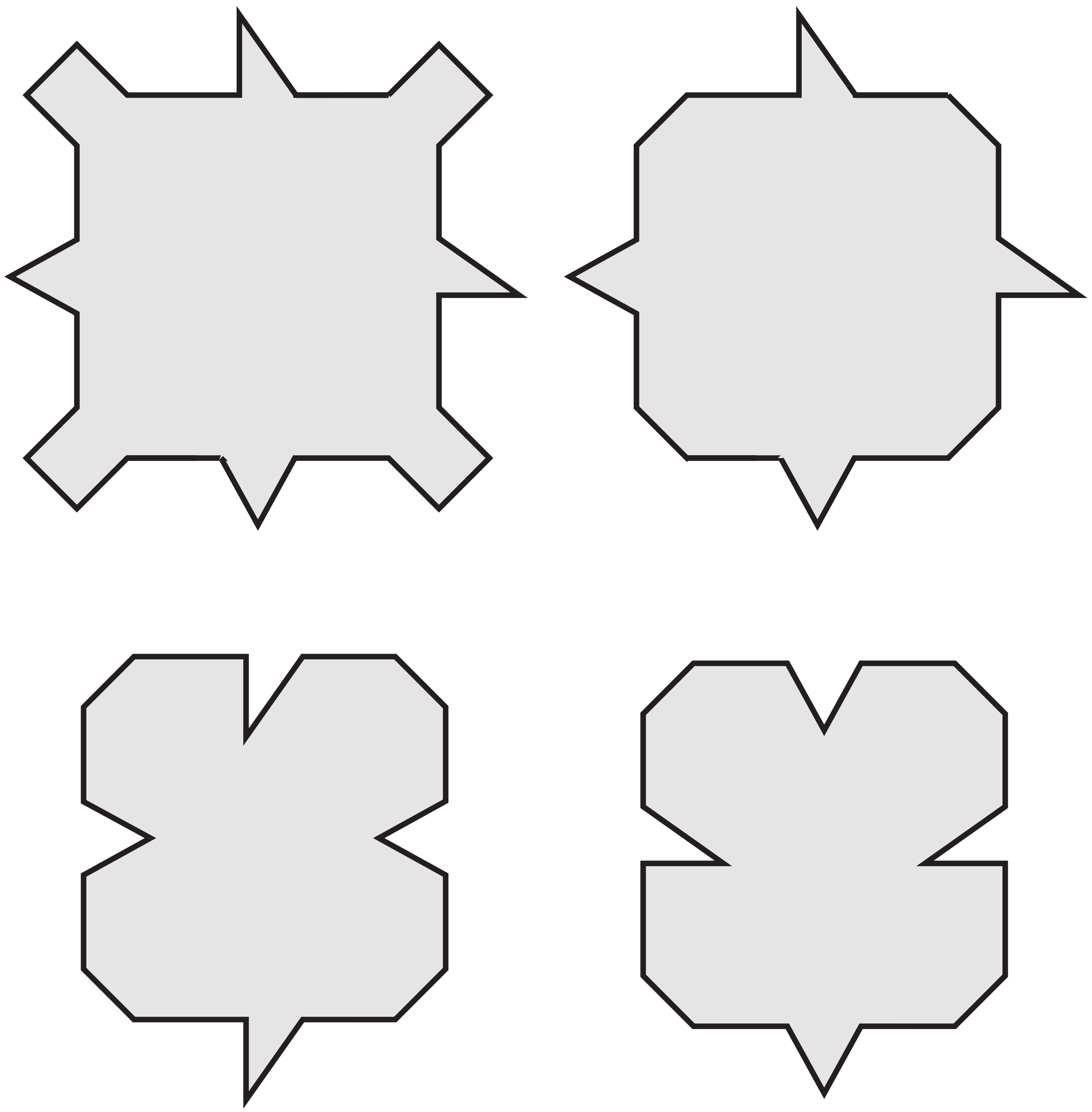


Jeandel and Rao [2021] found a set of size 11 and proved that this was minimal.



Robinson tiles

Robinson [1971] gave an aperiodic set of six tiles.



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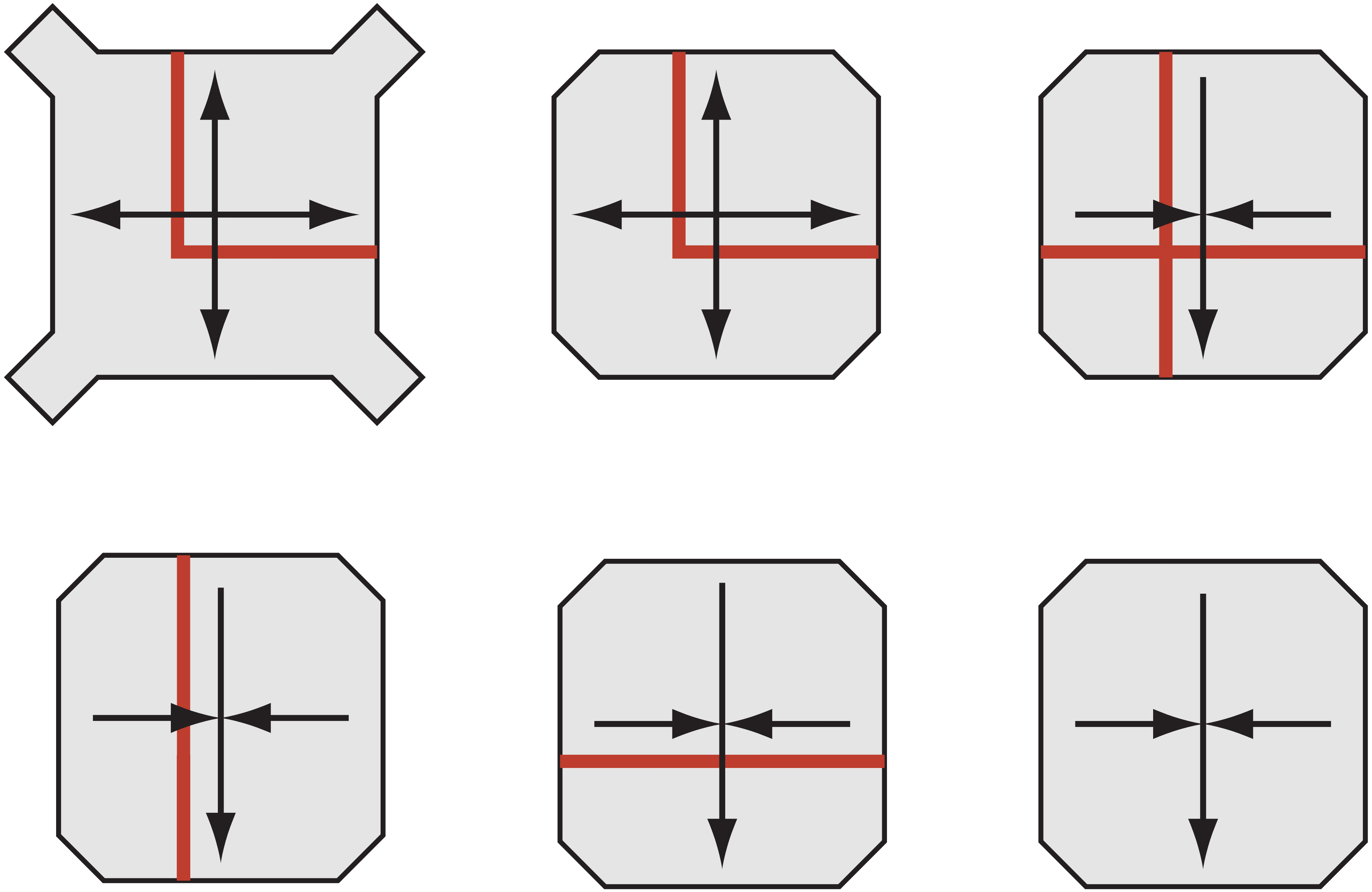
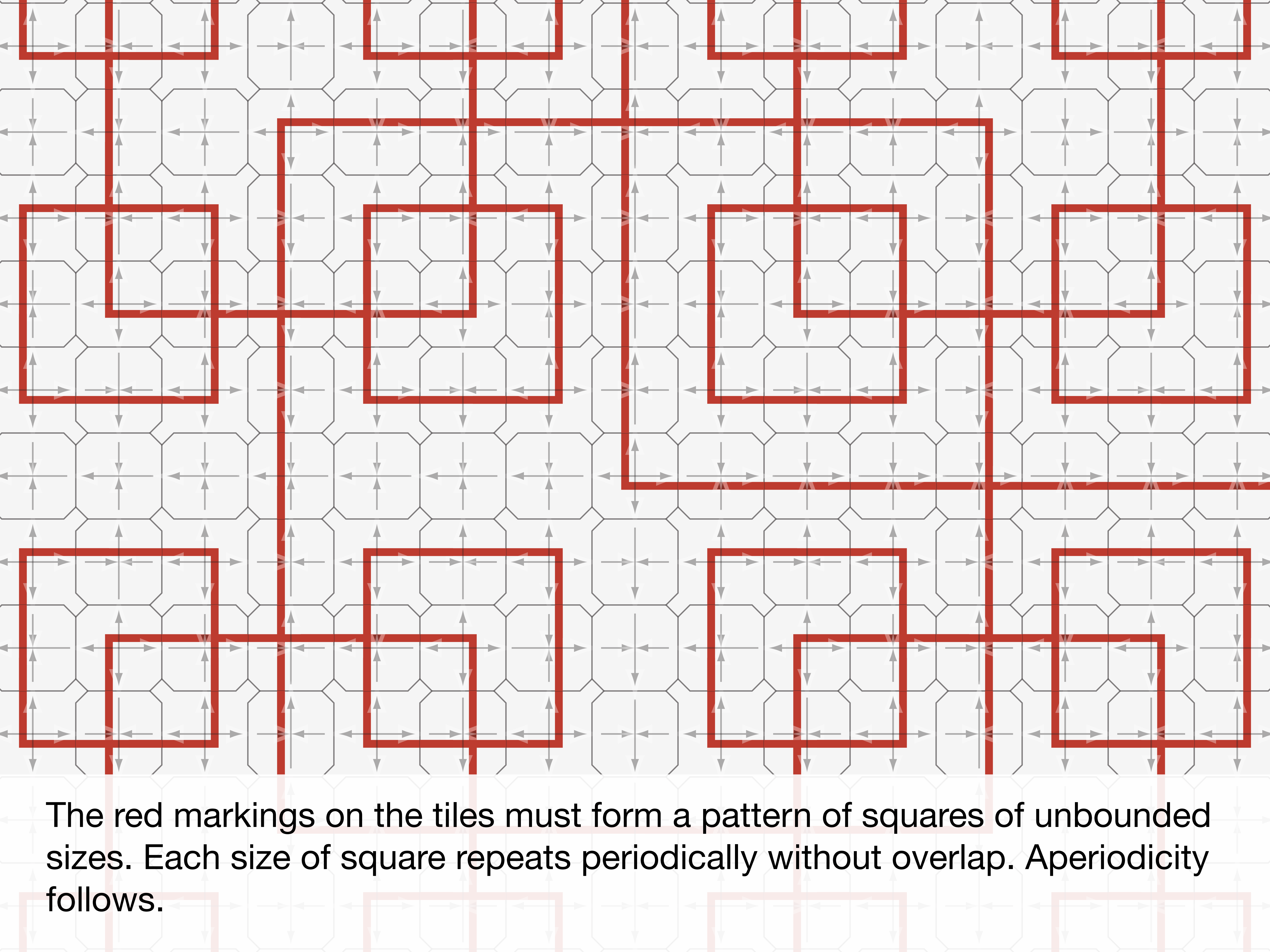
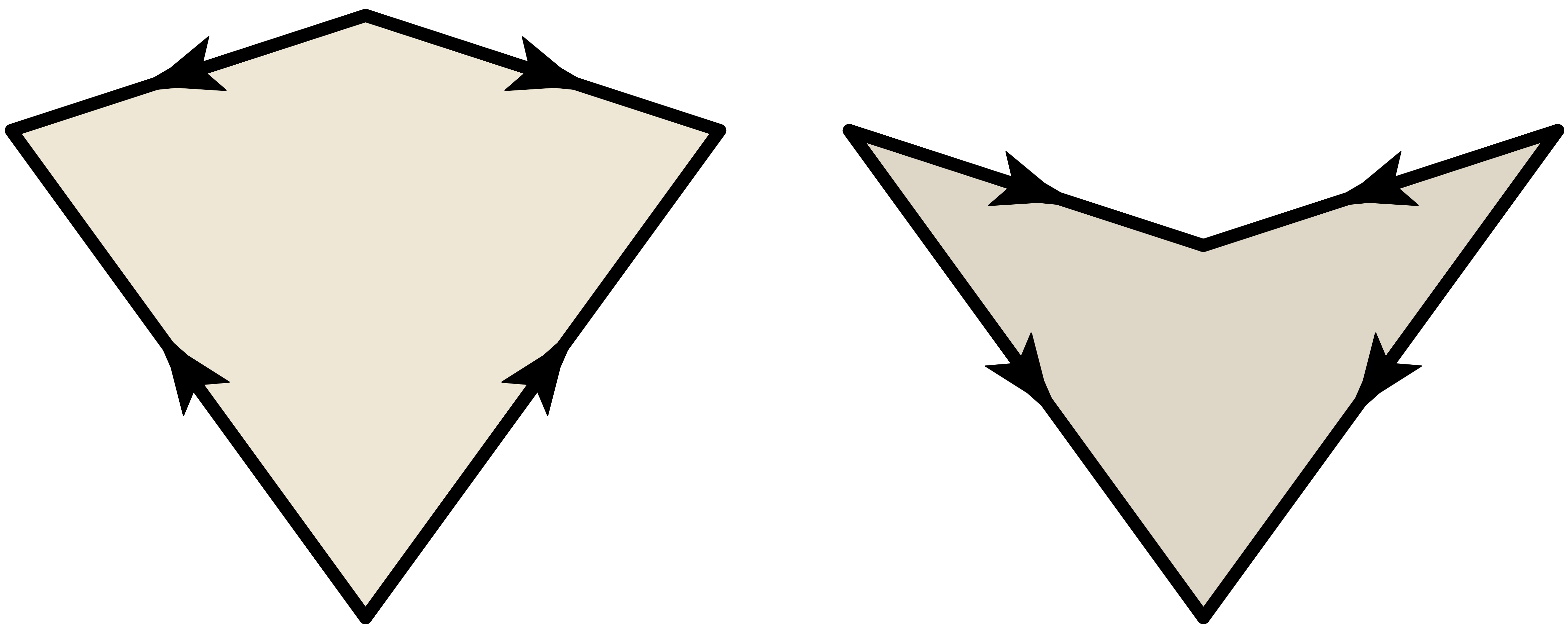




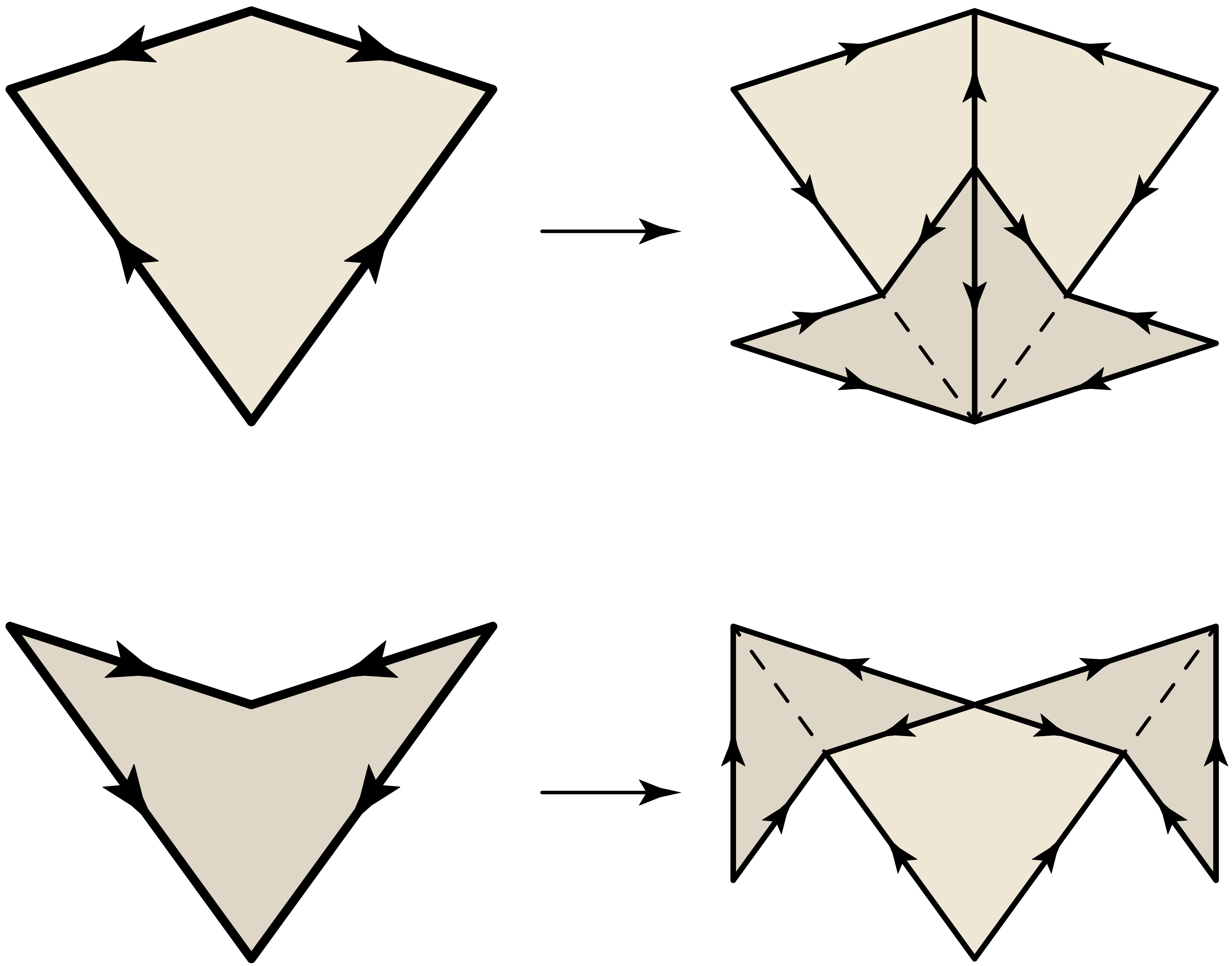
The red markings on the tiles must form a pattern of squares of unbounded sizes. Each size of square repeats periodically without overlap. Aperiodicity follows.

Penrose tiles

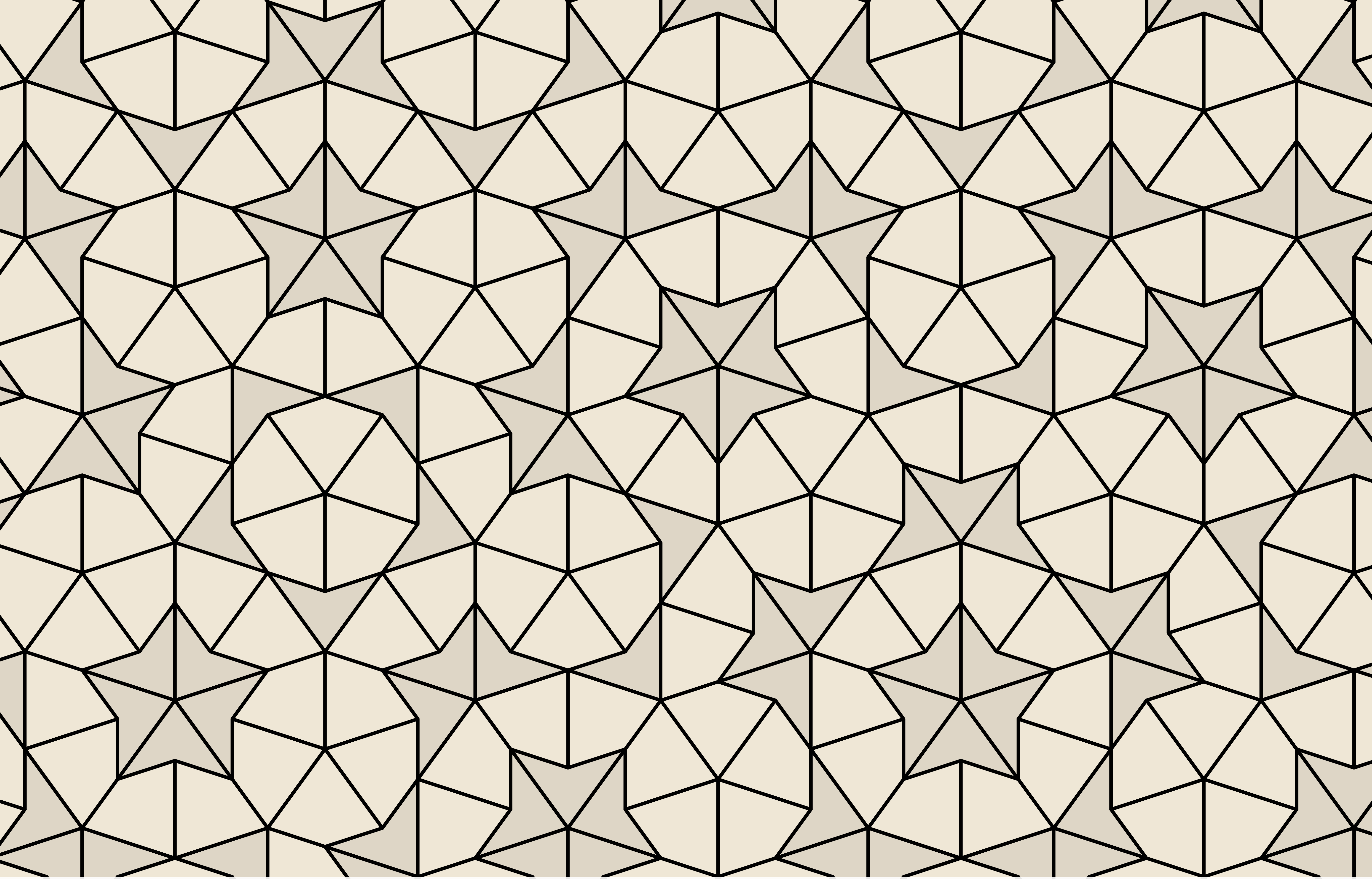
Penrose [1974] gave two aperiodic sets of size two.



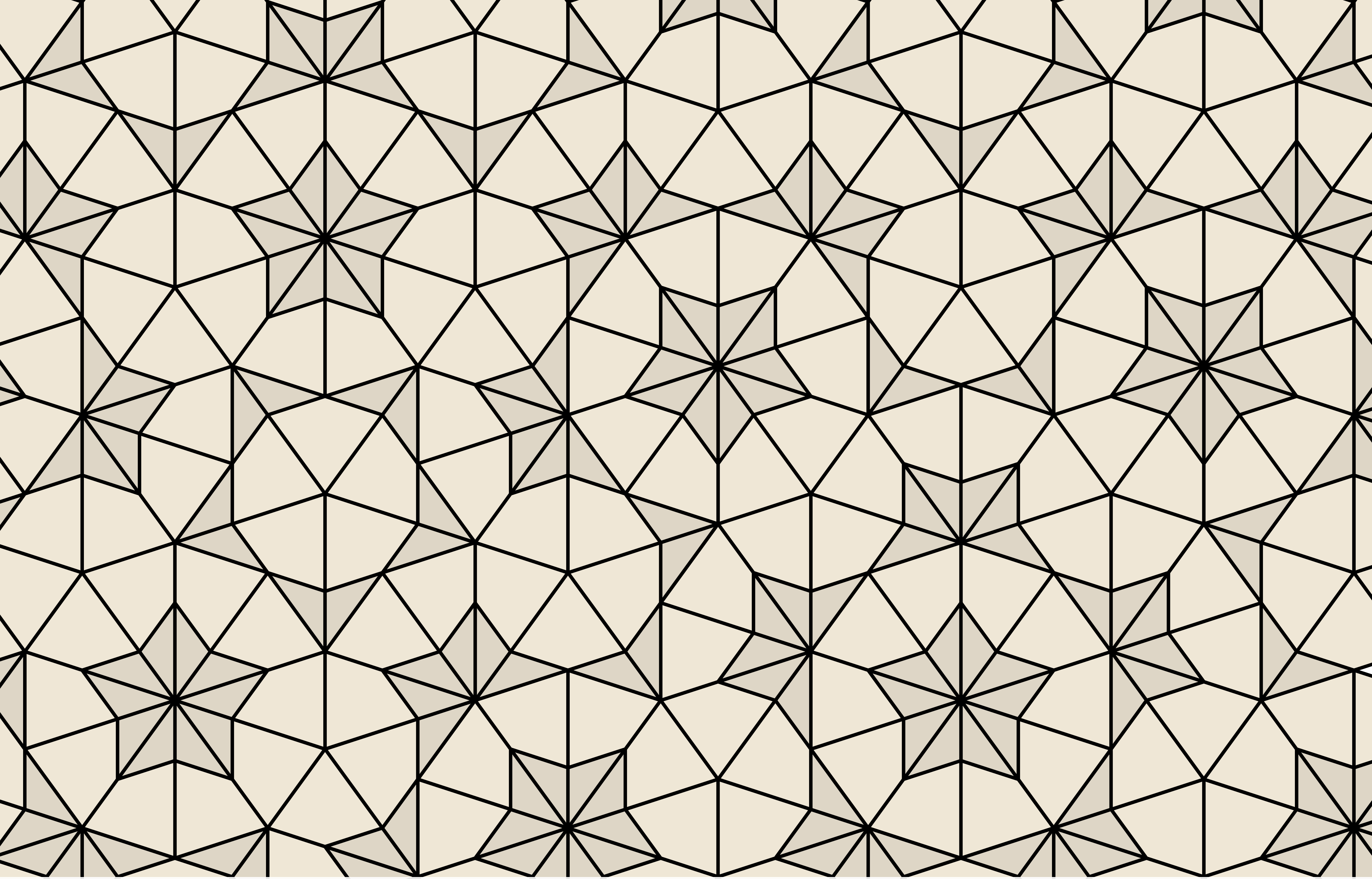
P2, the "kite and dart"



Substitution rules (and the Extension Theorem) show that the kite and dart admit (non-periodic) tilings of the plane.



To show that **every** tiling by kites and darts is non-periodic, show that tiles can uniquely be composed into "supertiles".



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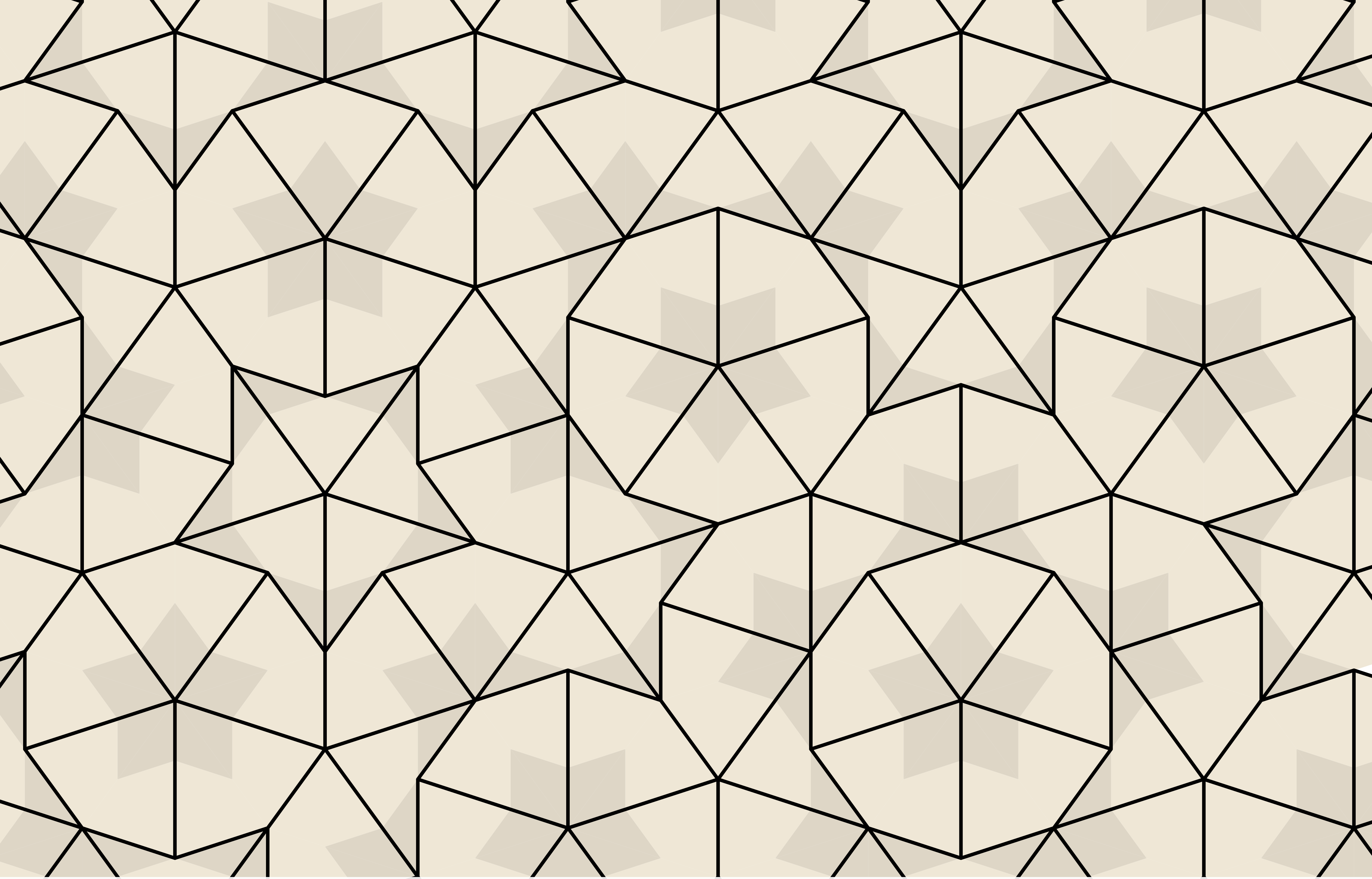

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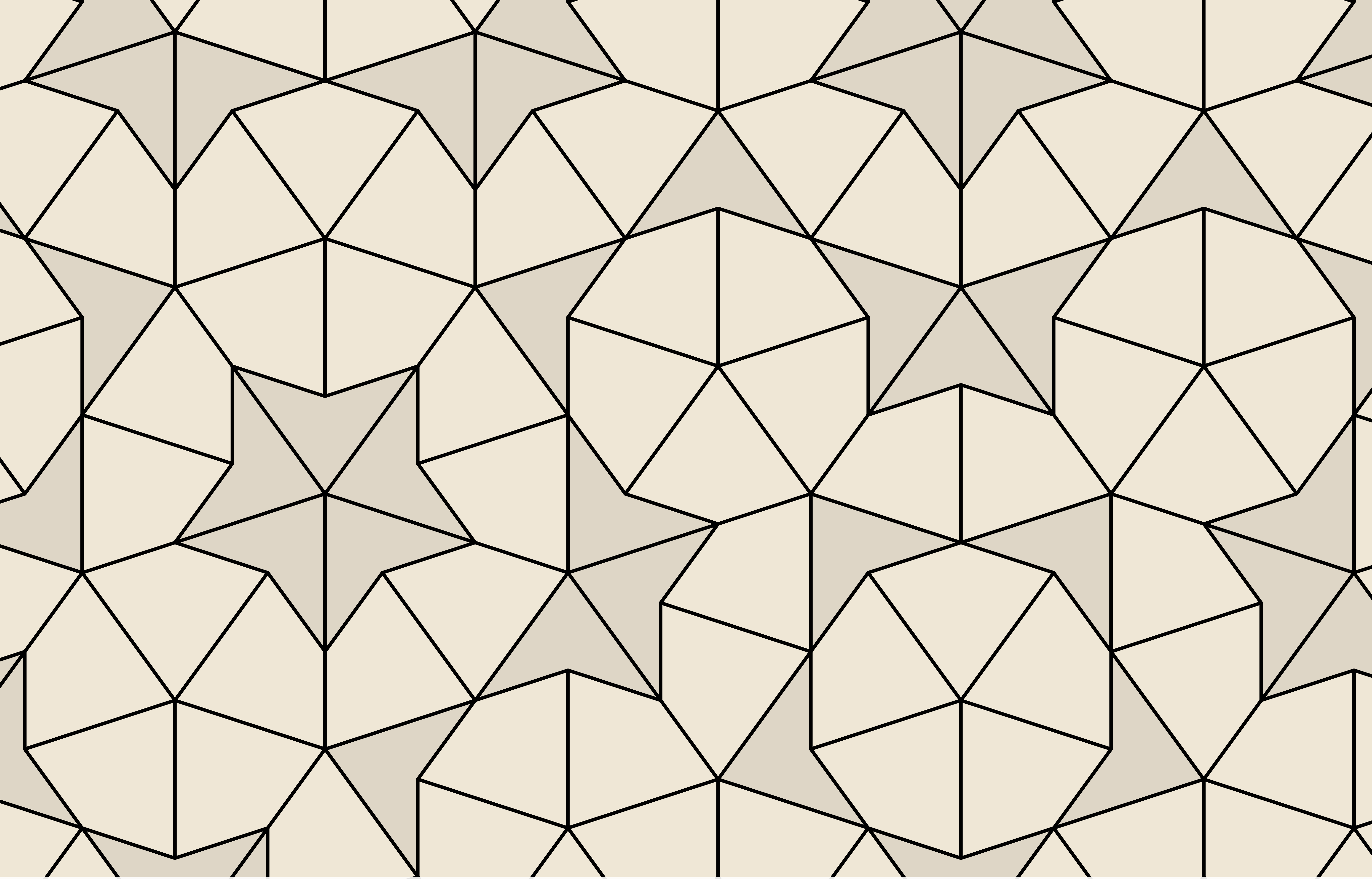

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The quest for an einstein

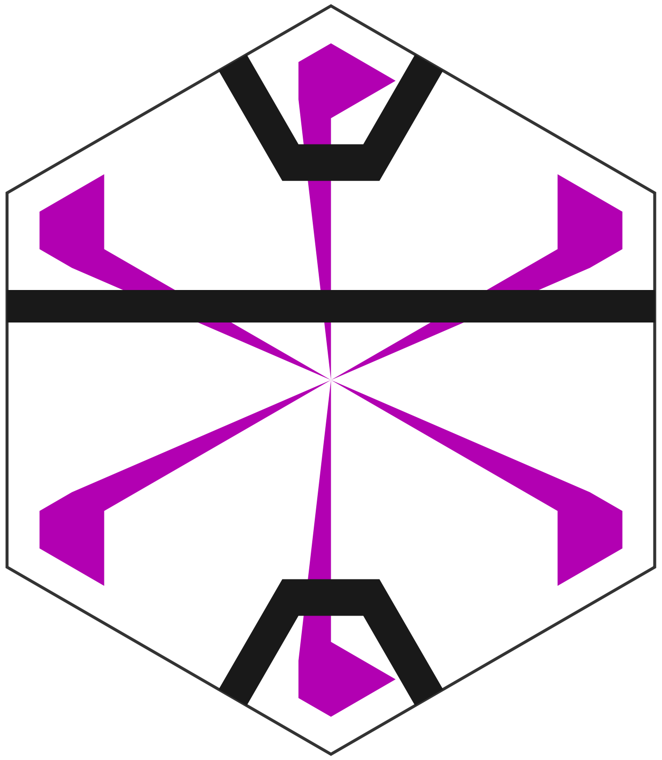
Since the 1970s, several other small aperiodic sets were discovered by Ammann, Goodman-Strauss, and others.

Does there exist an aperiodic set of size one, AKA an aperiodic monotile, AKA an "einstein"?

Grünbaum and Shephard [1987]: "Though the existence of such a tile may appear unlikely, one must remember that only a few years ago, the existence of aperiodic sets containing just two tiles seemed essentially impossible."

The Socolar-Taylor tile

Socolar and Taylor [2011] presented an aperiodic hexagon!



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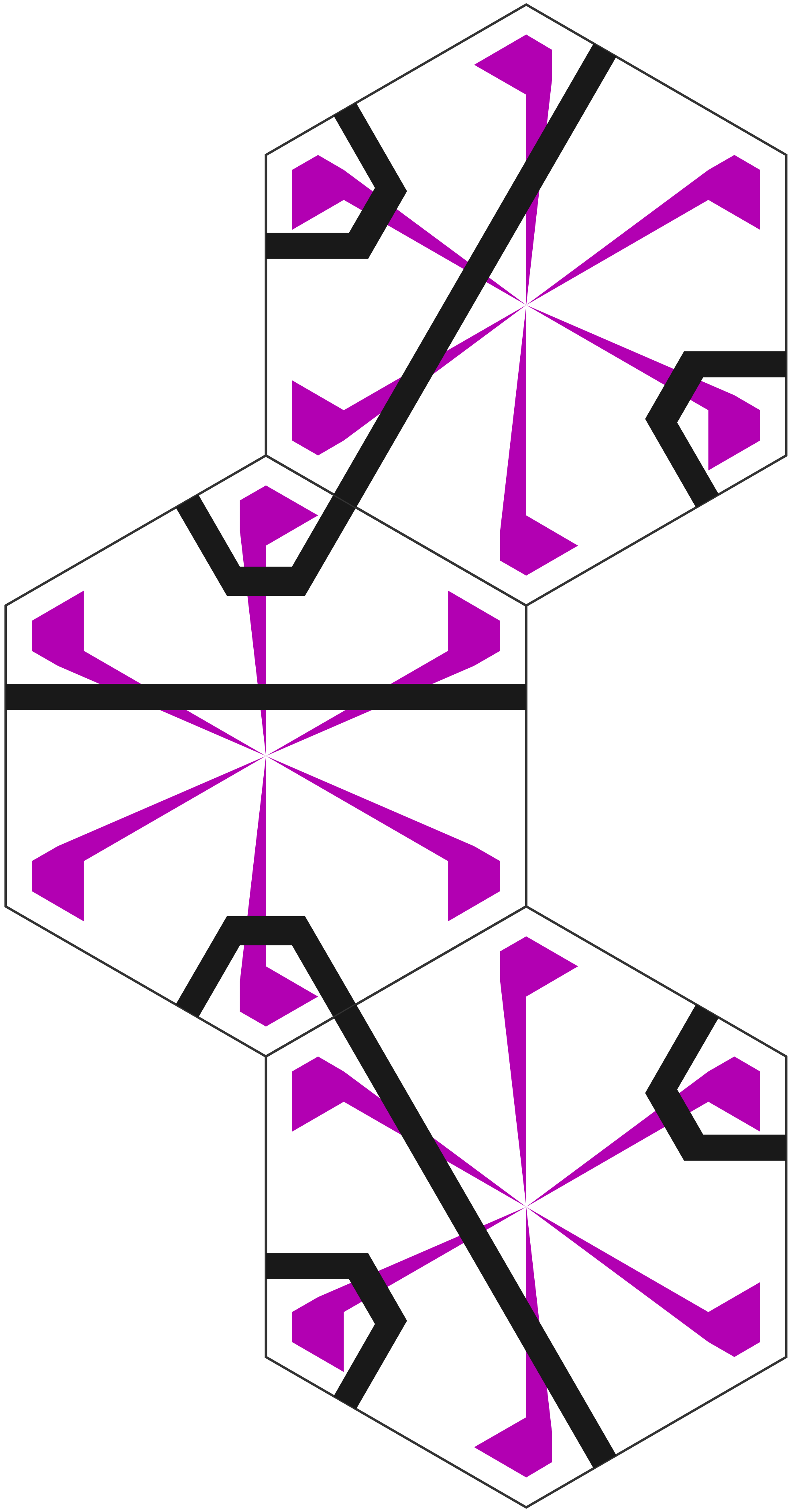

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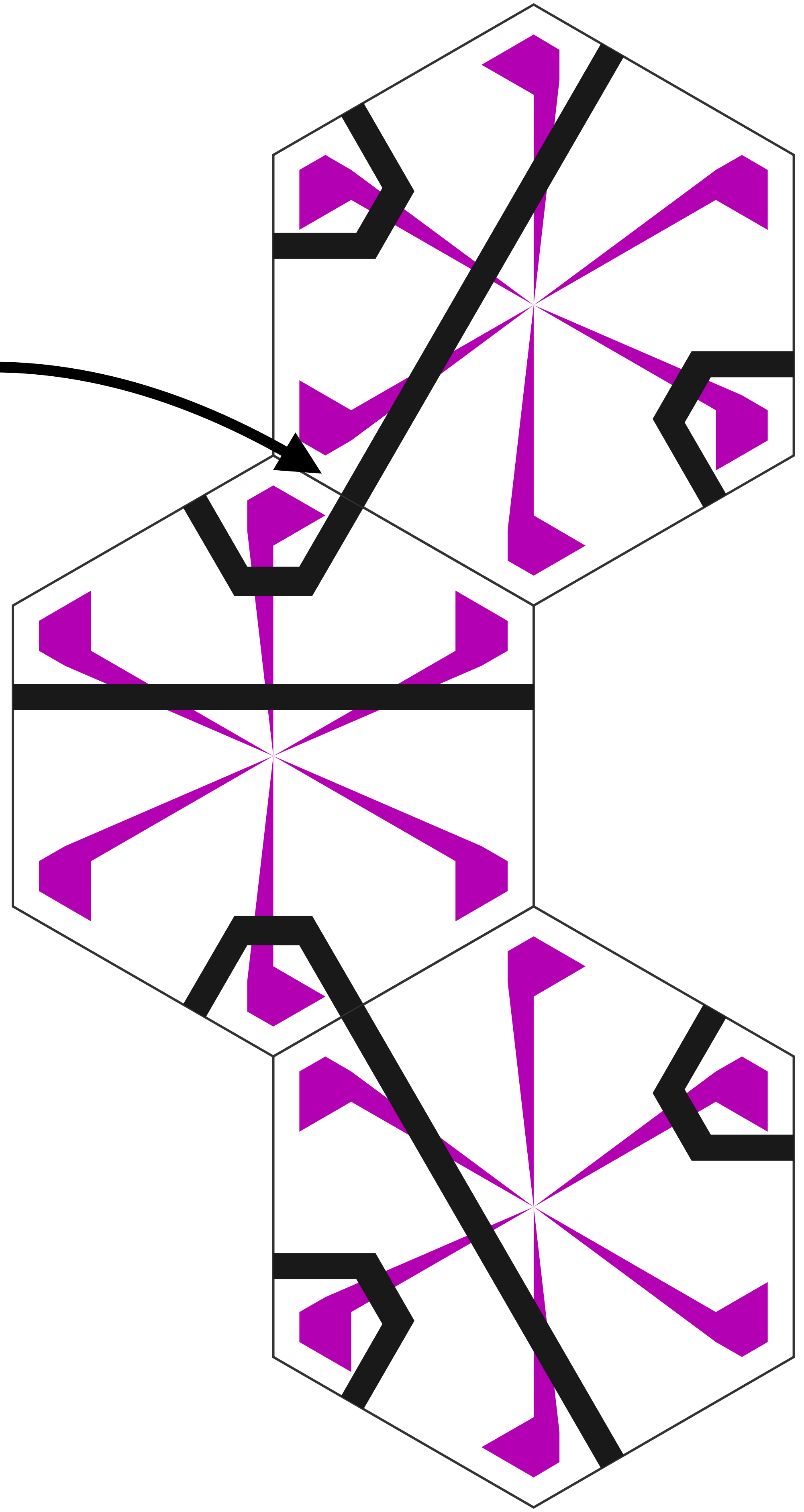
Black markings must pass continuously across tile edges



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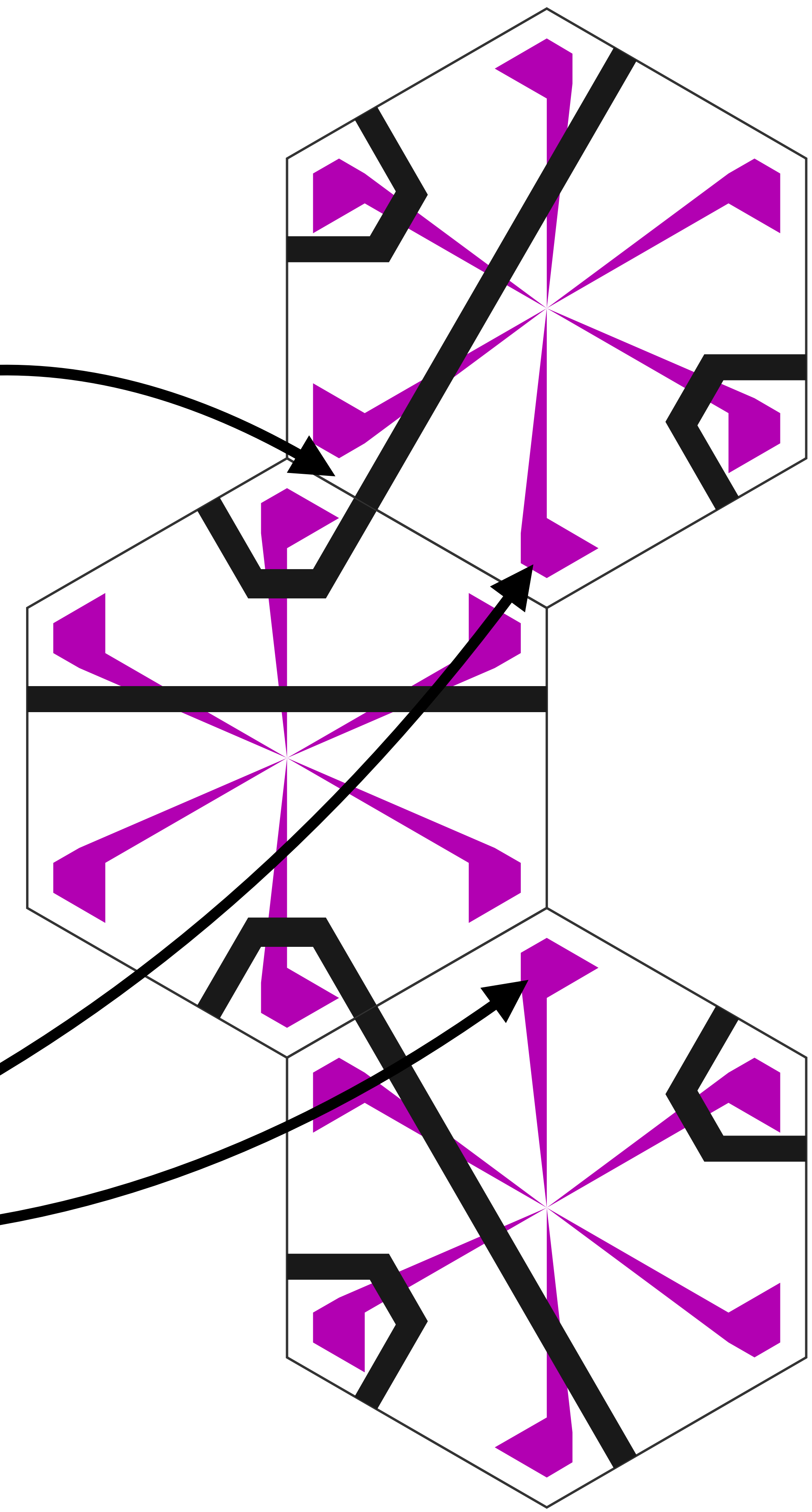
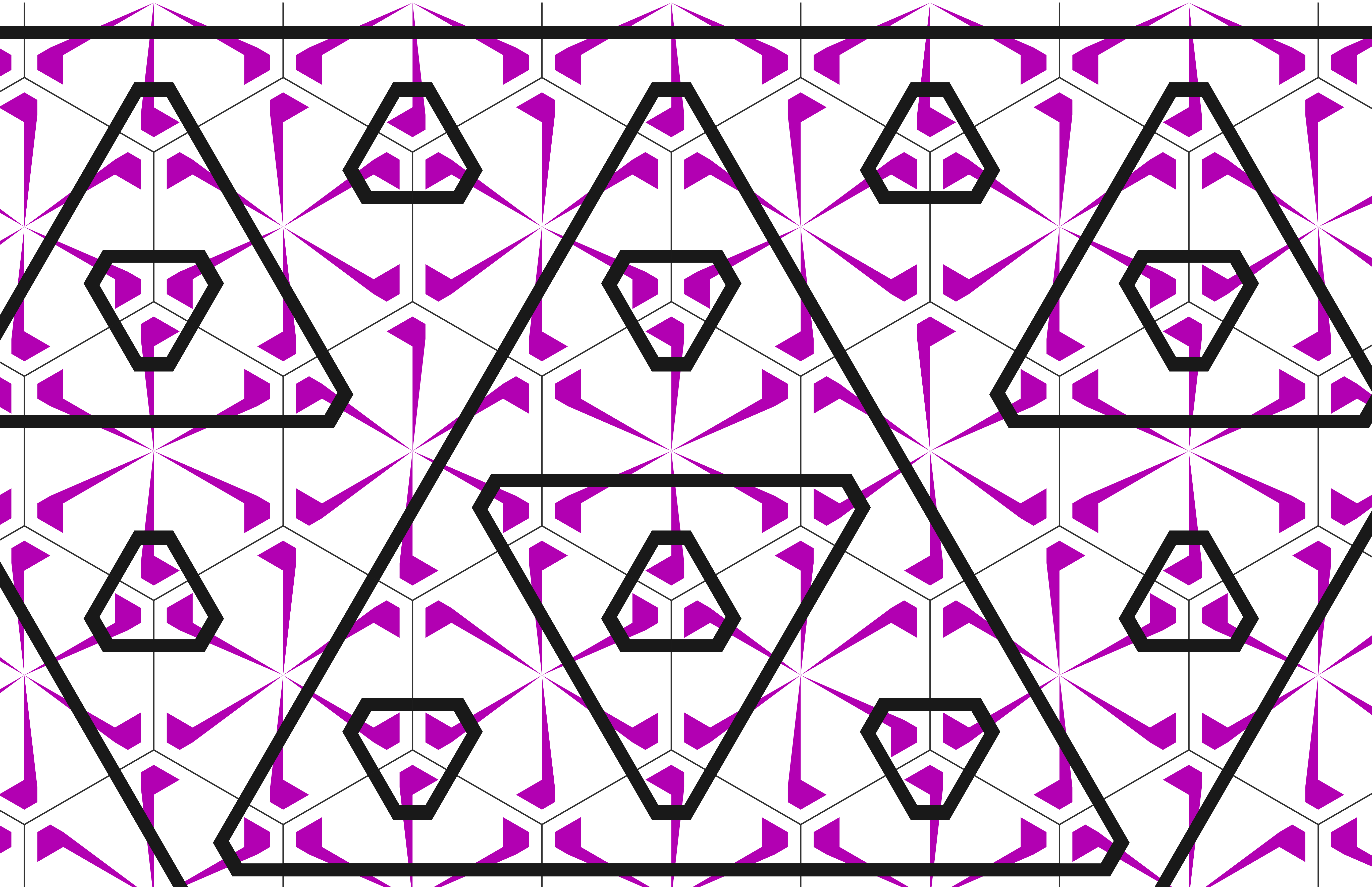


Black markings must pass continuously across tile edges

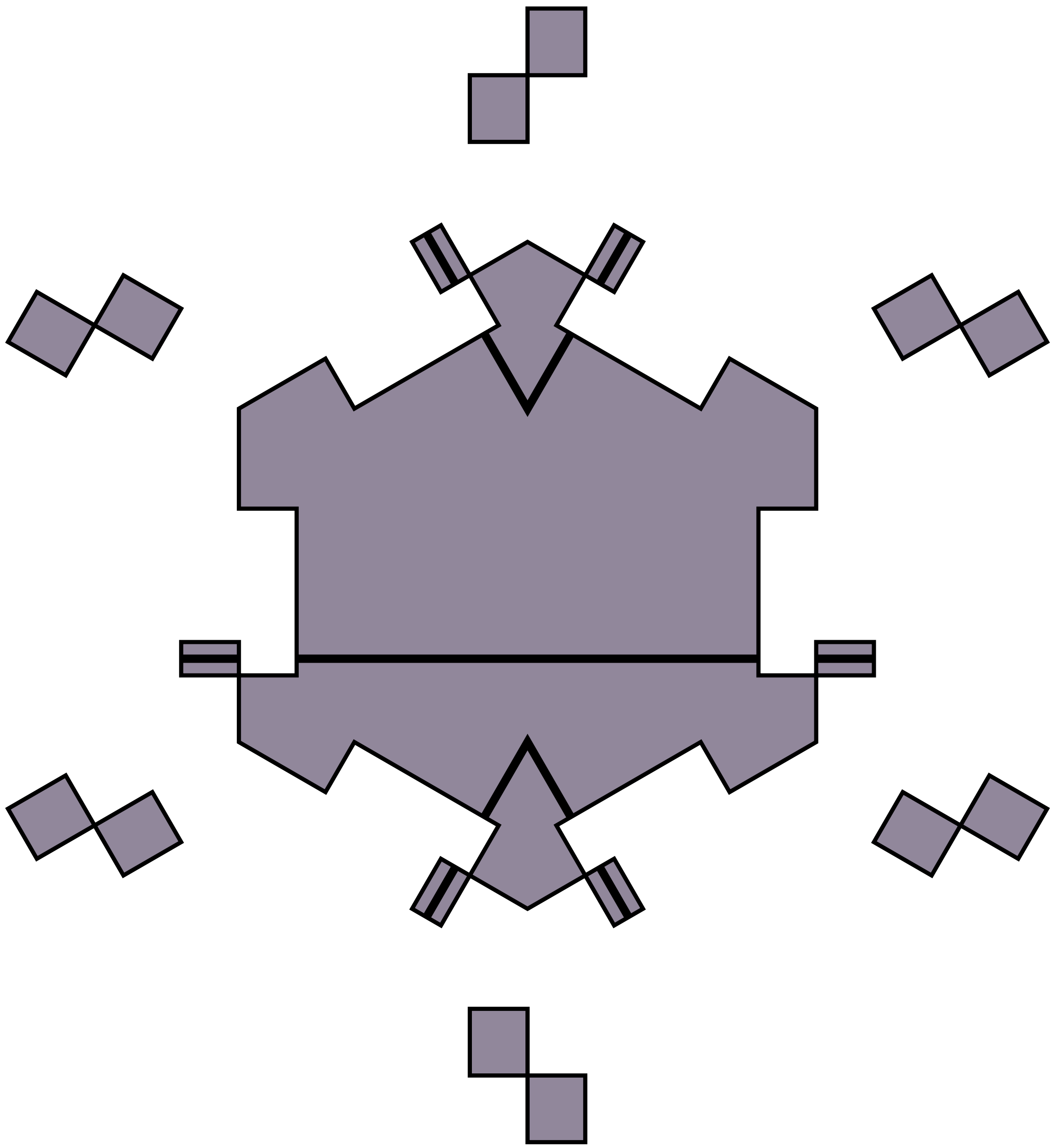
Flag orientations must agree at an edge's endpoints

The Socolar-Taylor tile

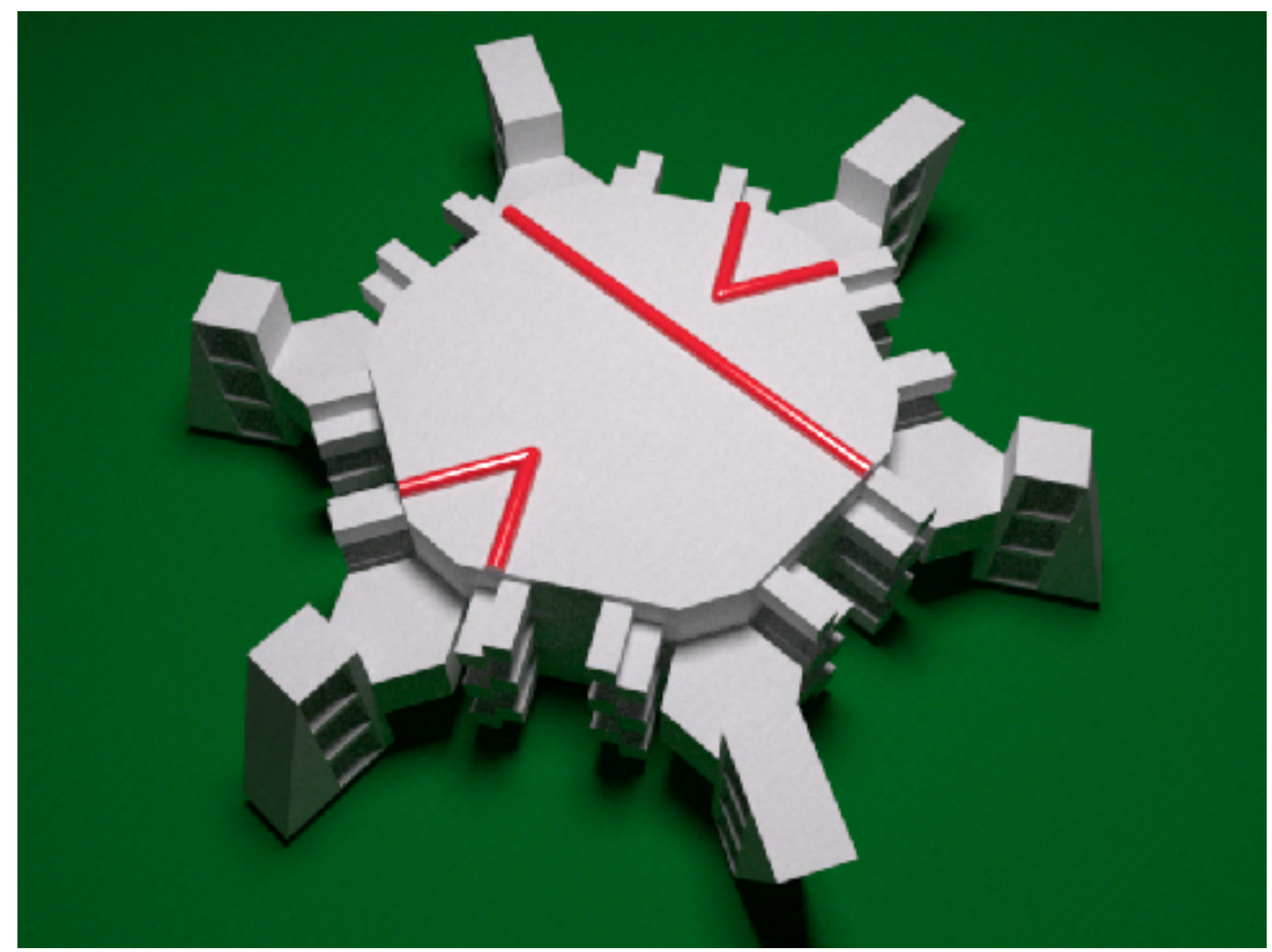


The Socolar-Taylor tile

Variations of the Socolar-Taylor tile can express its matching conditions geometrically.



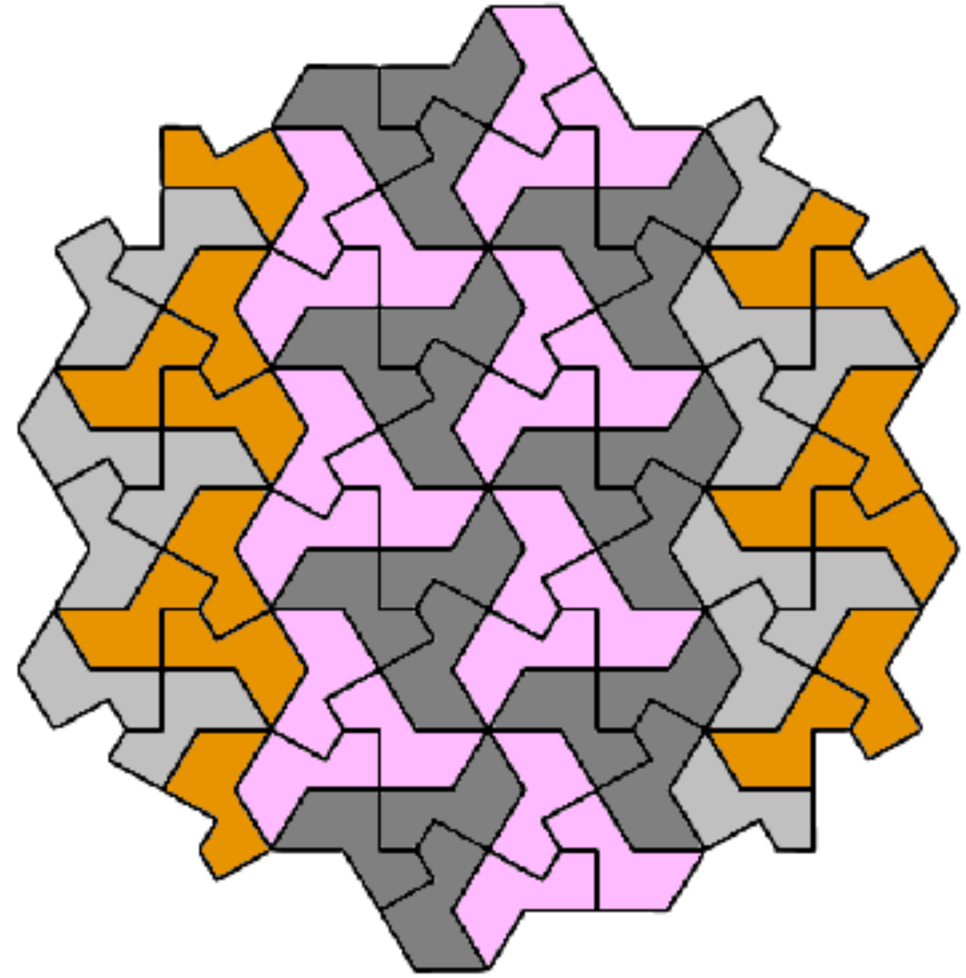
Disconnected tile

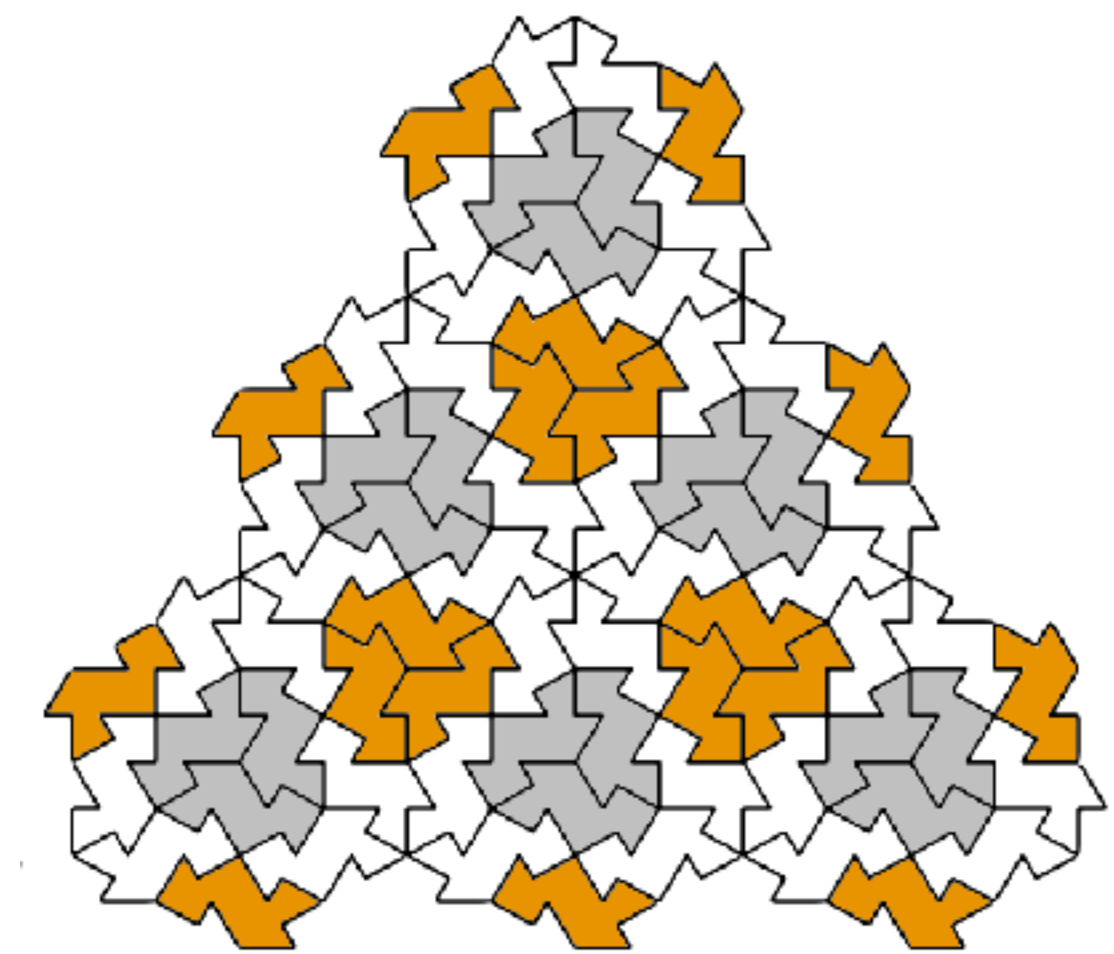


Connected 3D tile

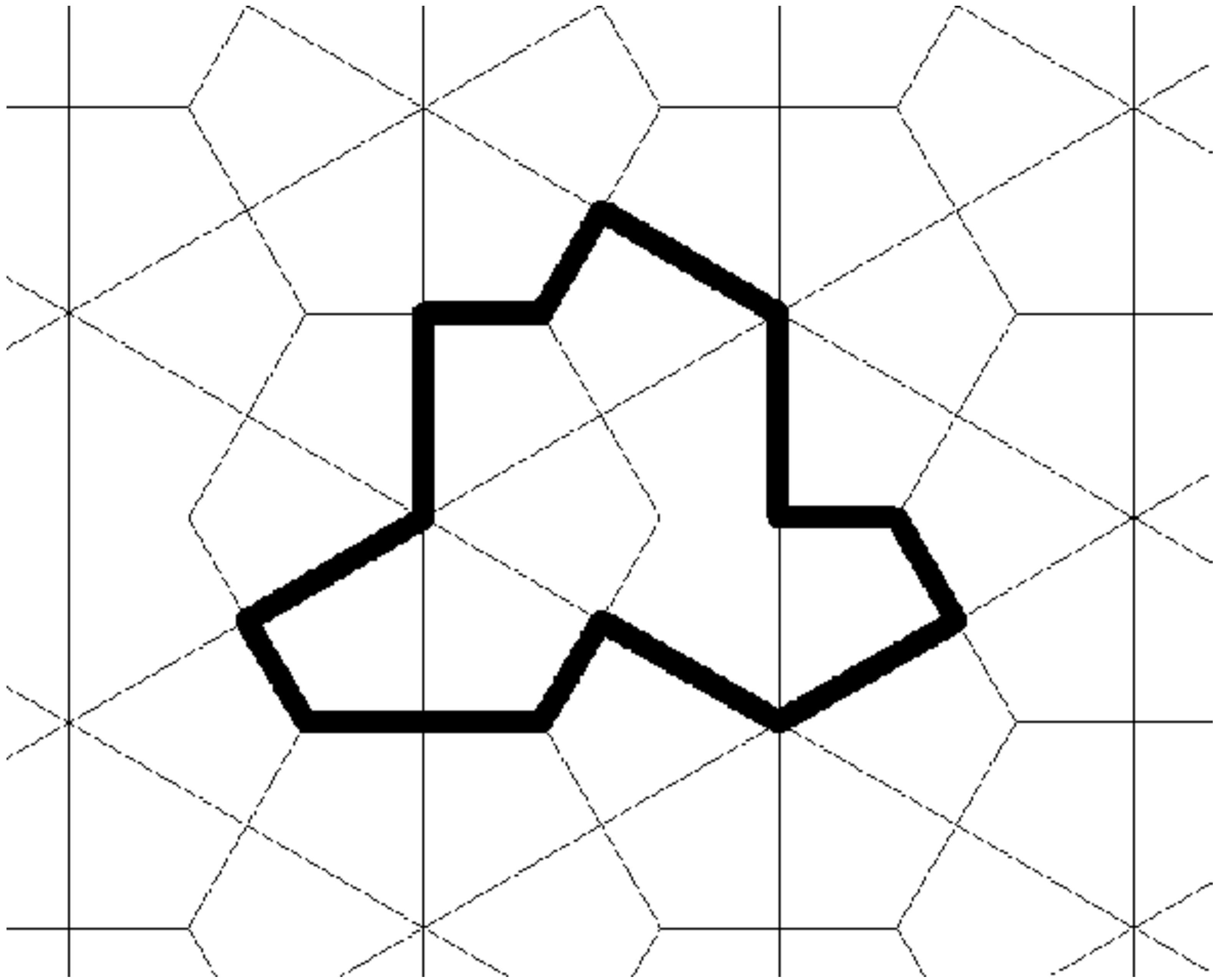
David Smith

...shape hobbyist





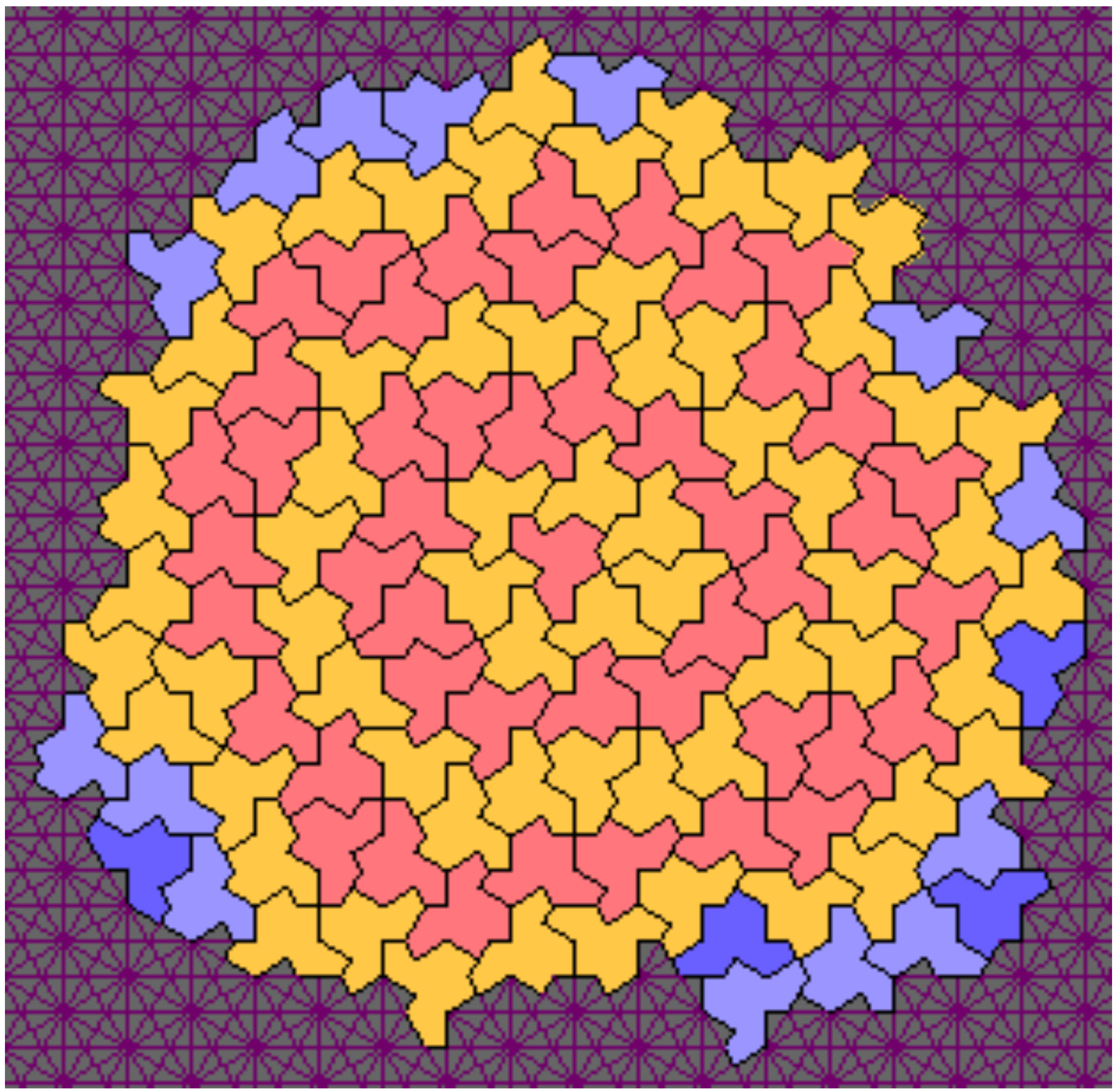
November 2022

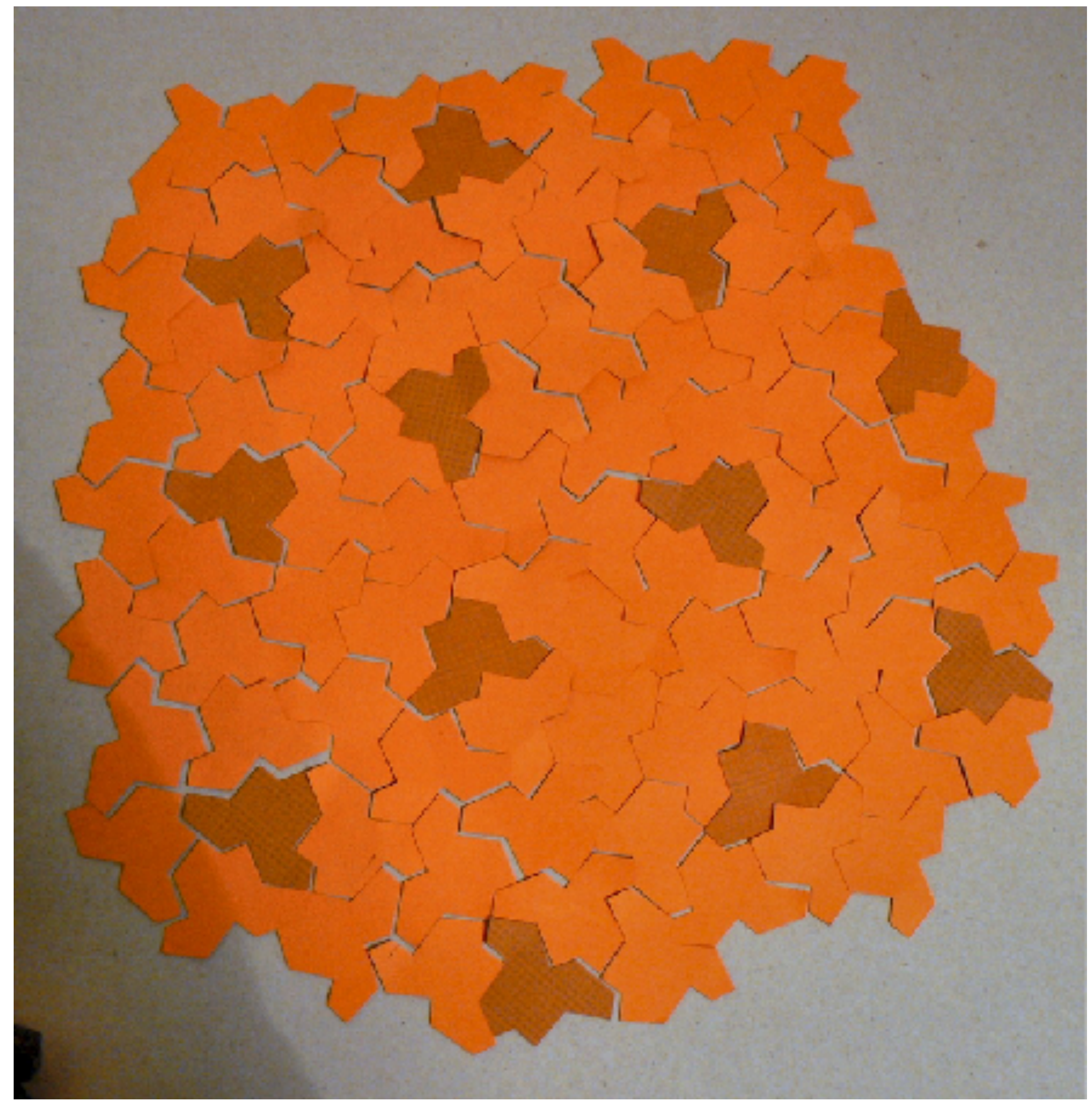


The "hat"

November 2022

David emailed me out of the blue: "It has a Heesch number of at least three, if it's a non-tiler (I couldn't get it to tile periodically)."





Measures of disorderliness

Let T be a tile.

If T admits periodic tilings, then the **isohedral number of** T is the minimum number of transitivity classes in any of those tilings.

⇒ A rough measure of a tiler's disorderliness.
⇒ Joseph Myers [1996–present] has computed isohedral numbers of unmarked polyforms, finding a record of 10.

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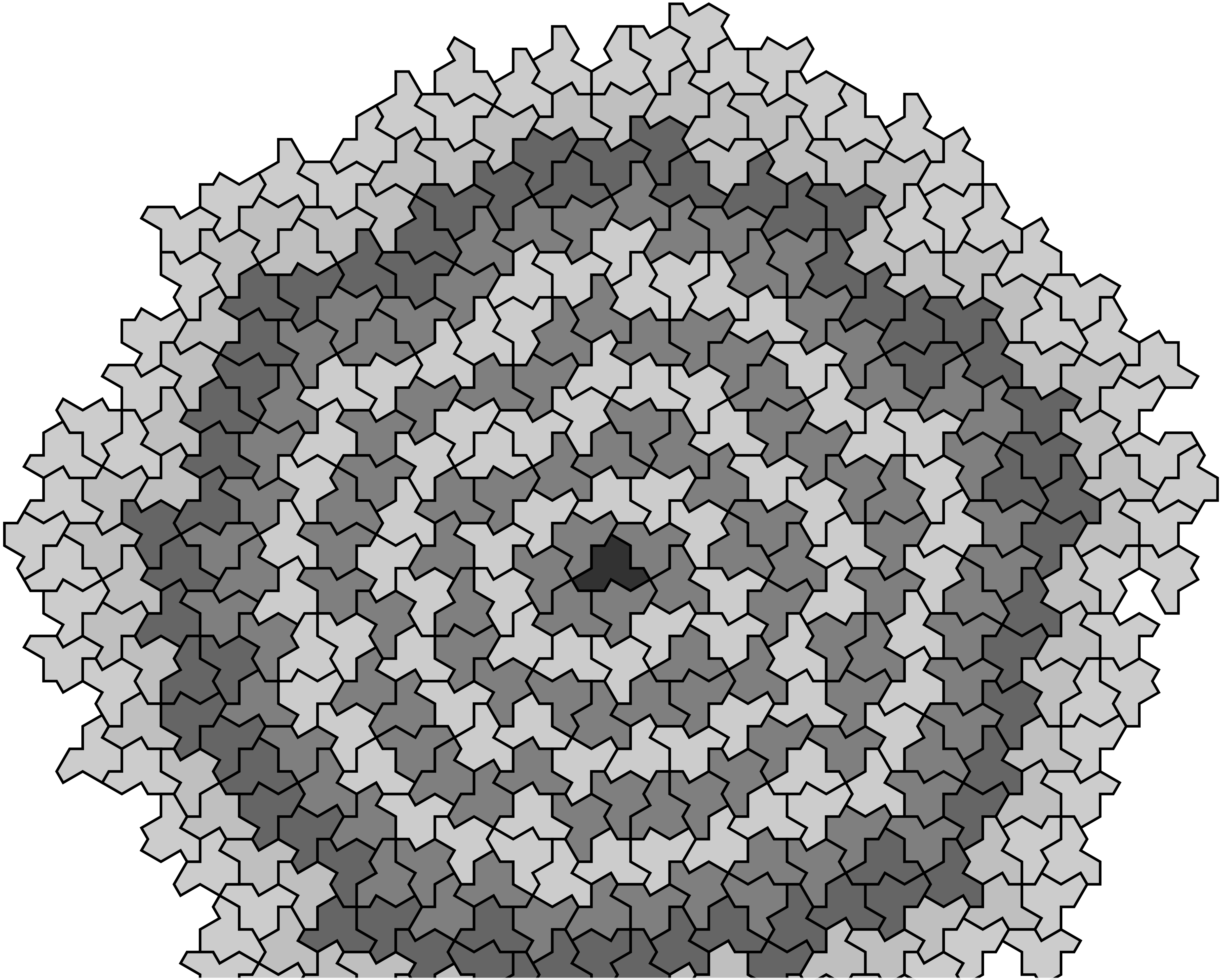
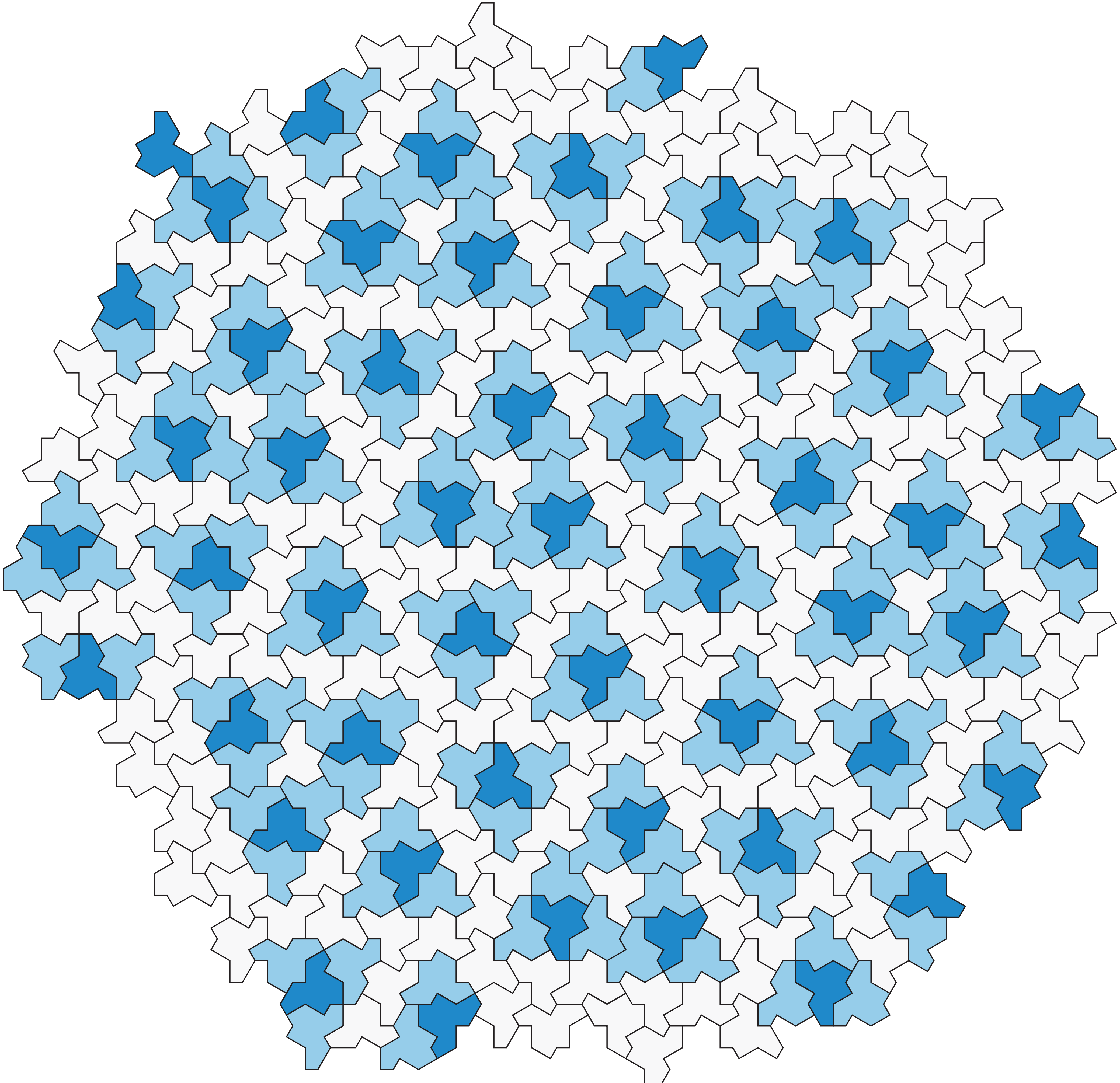
- ⇒ A rough measure of a tiler's disorderliness.
- ⇒ Joseph Myers [1996–present] has computed isohedral numbers of unmarked polyforms, finding a record of 10.

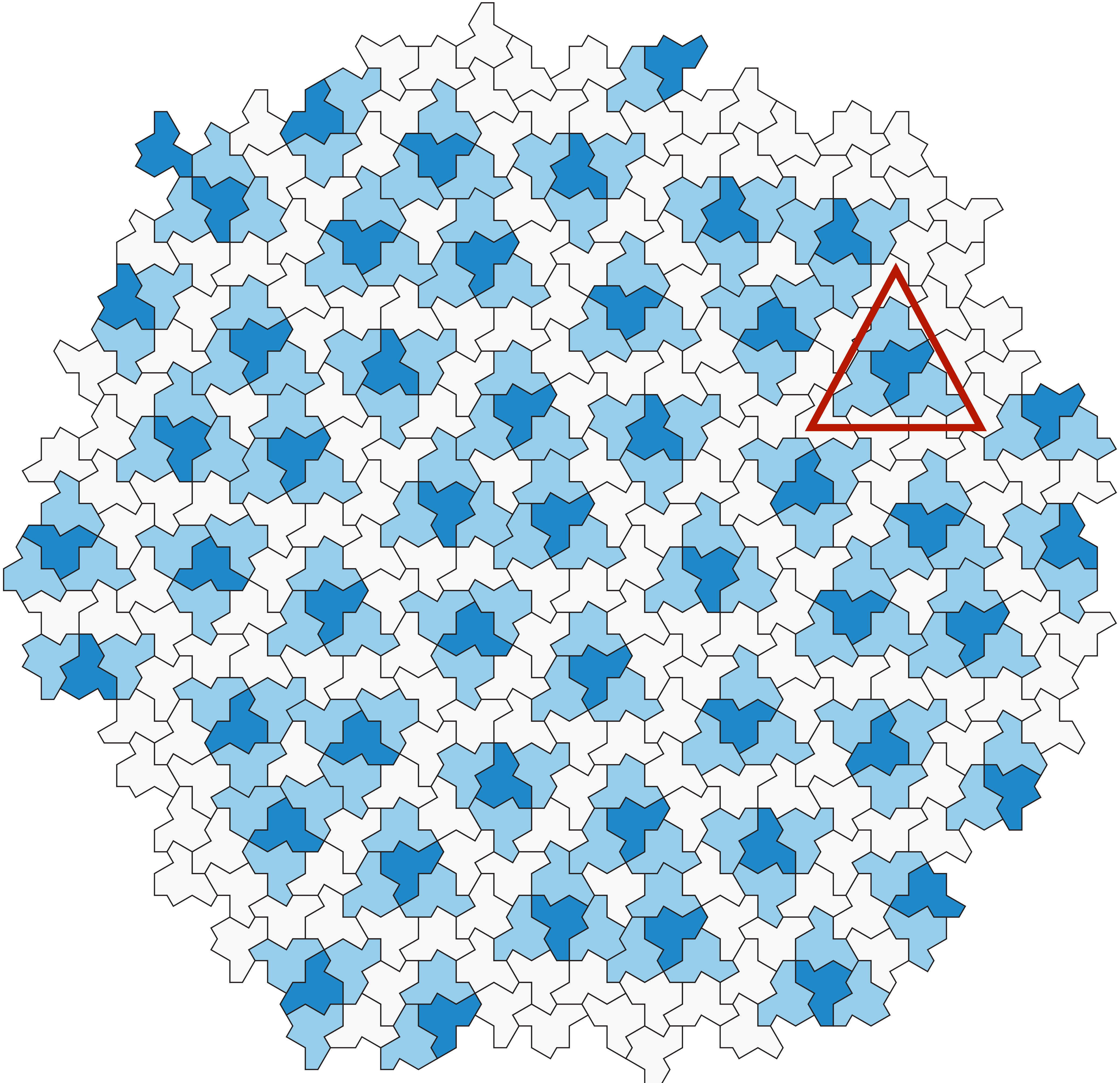
If T does not admit tilings, then the **Heesch number of T** is the maximum number of rings of copies of T that can surround it.

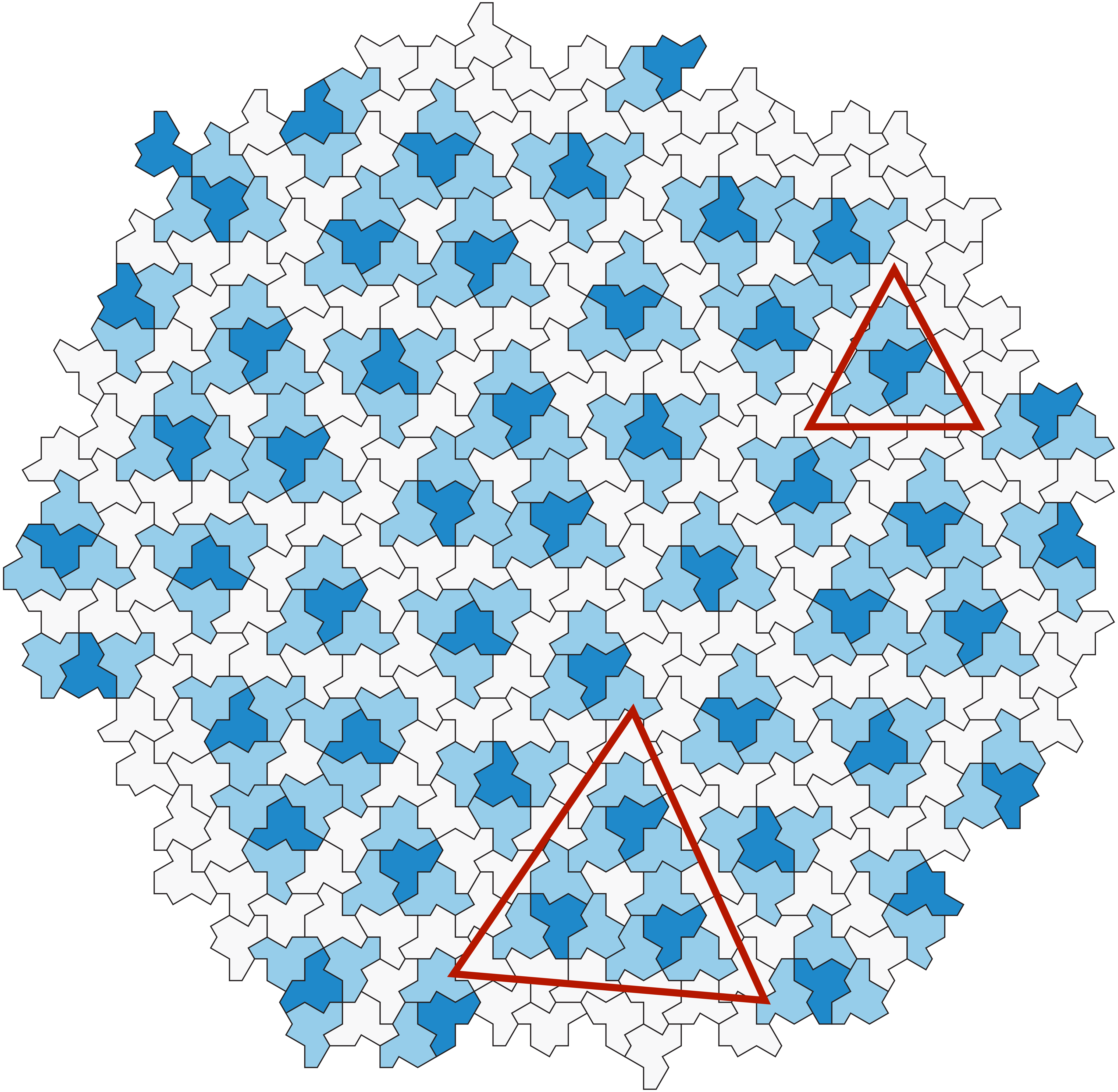
- ⇒ A rough measure of a non-tiler's disorderliness.
- ⇒ In my work [2022] I computed Heesch numbers of unmarked polyforms, finding examples up to 4. Current record is 6 [Bašić 2021]

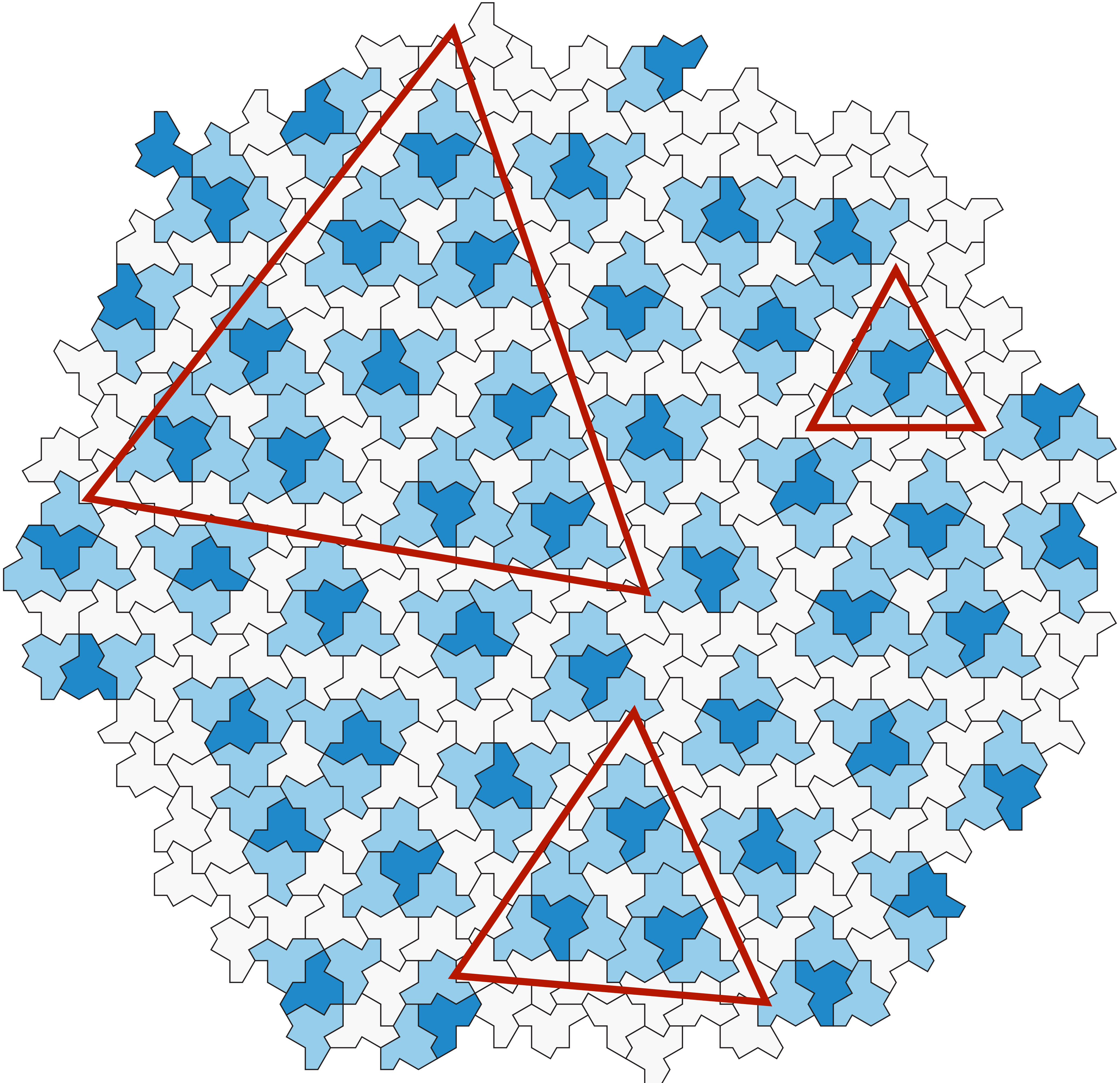


David asked whether my Heesch number software could work with kites (or drafters). Thanks to recent joint work with Ava Pun, it could.



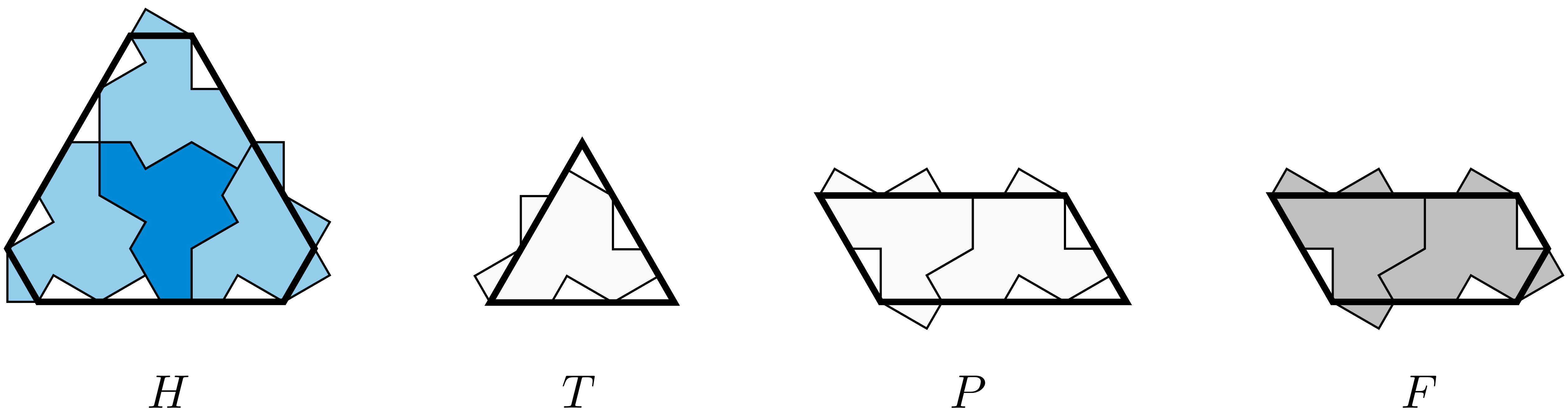






Metatiles

We define four **metatiles** by observation of recurring patterns in computer-generated patches.



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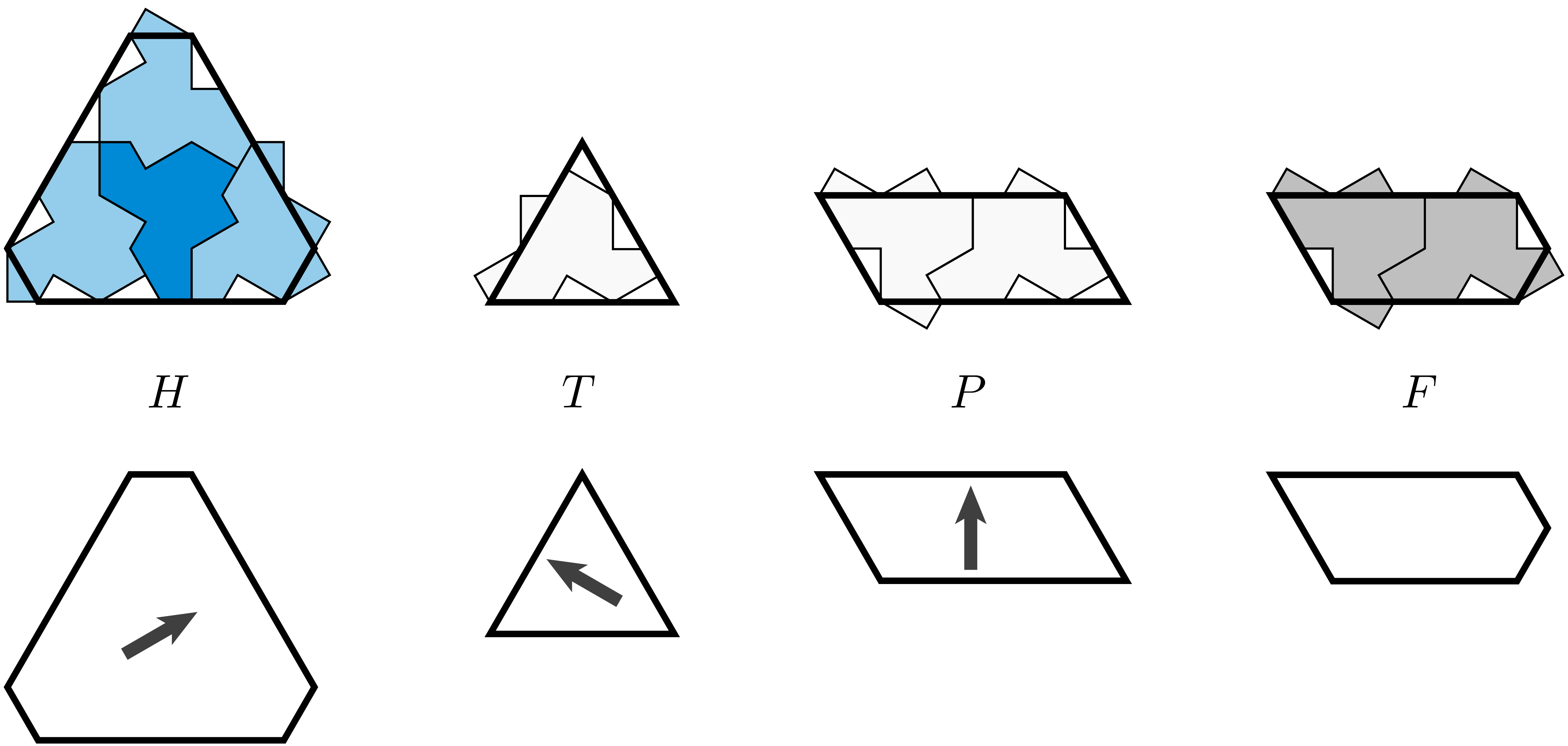
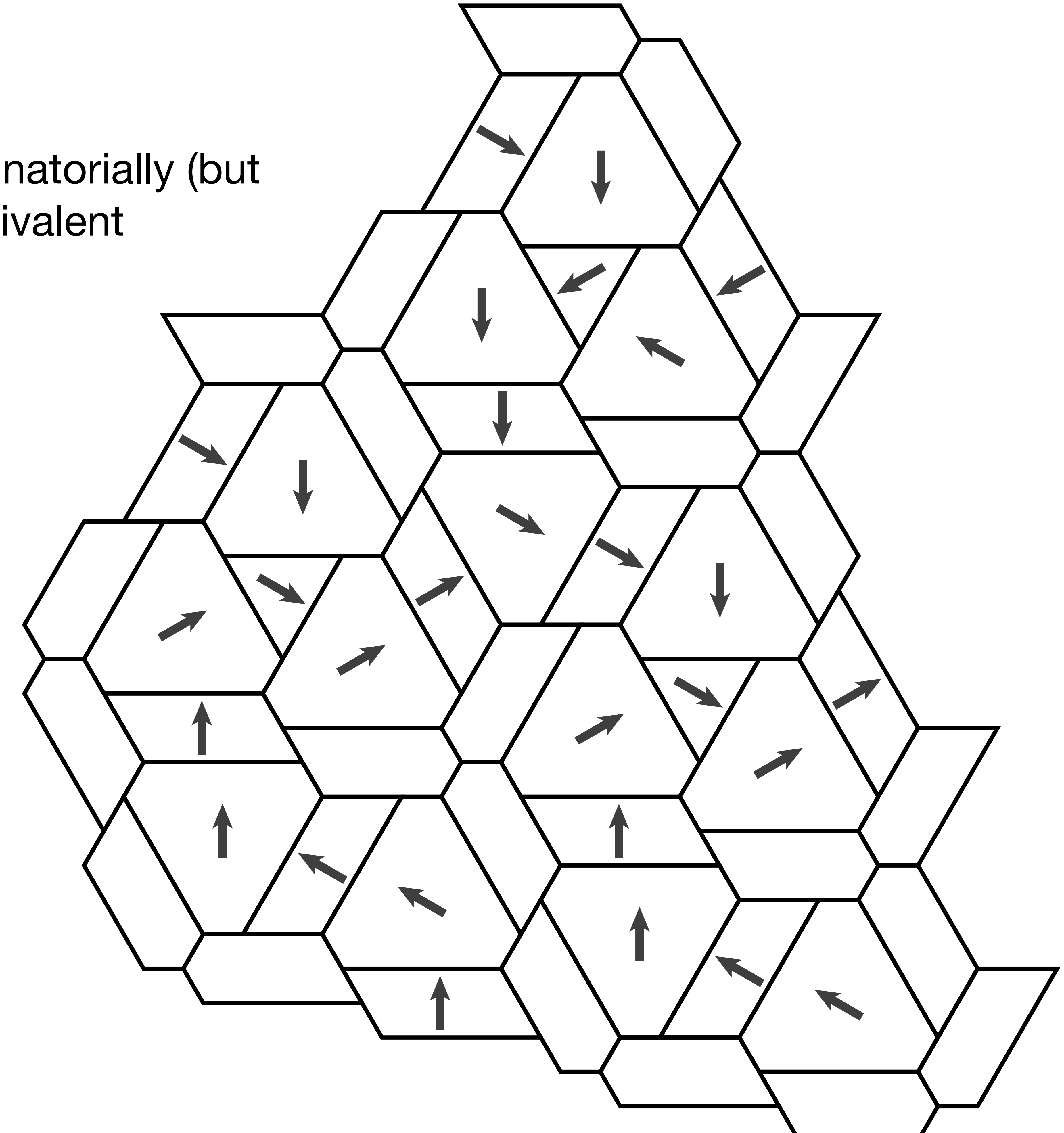


Supertiles

Metatiles beget combinatorially (but not geometrically) equivalent **supertiles.**



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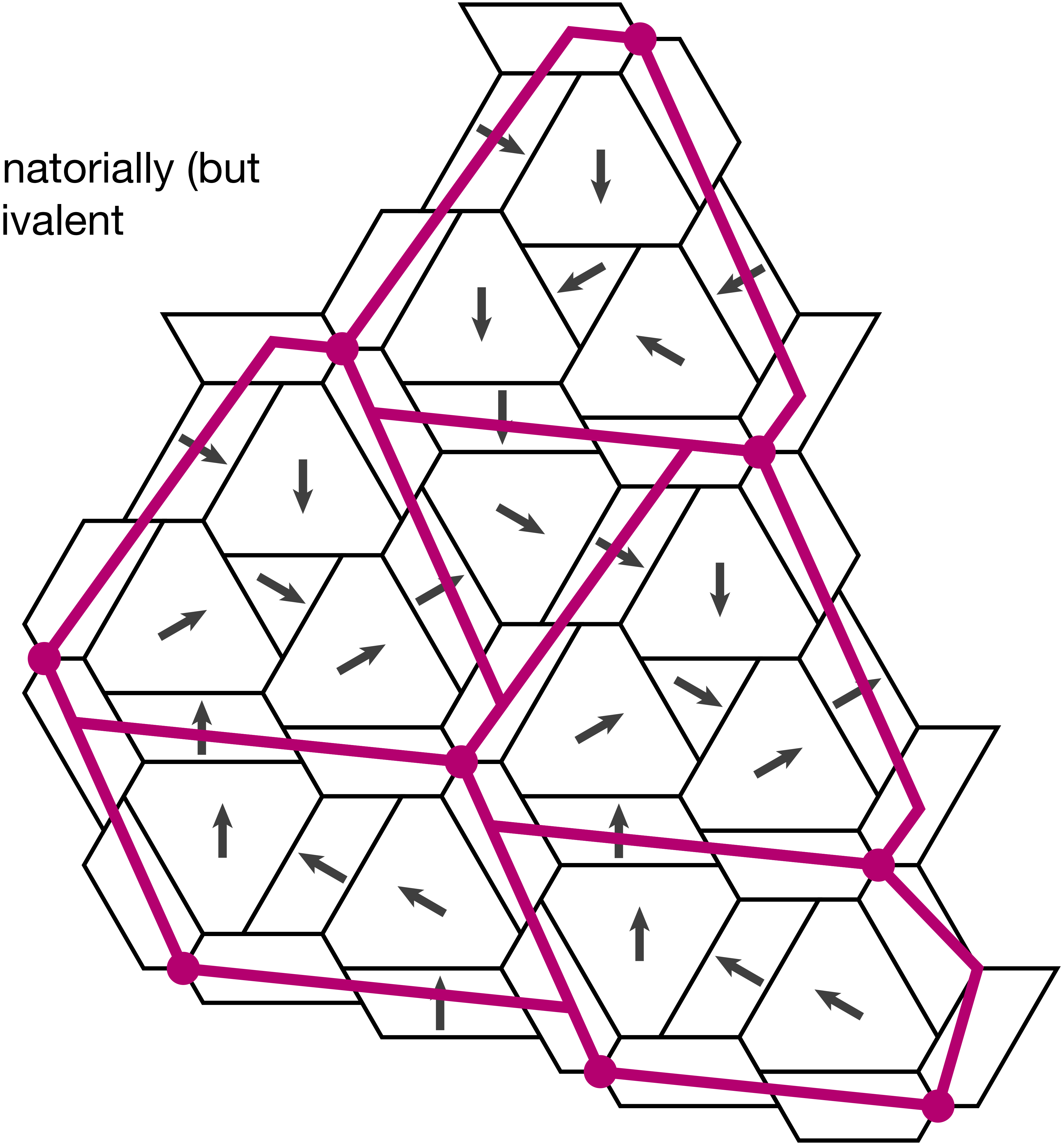

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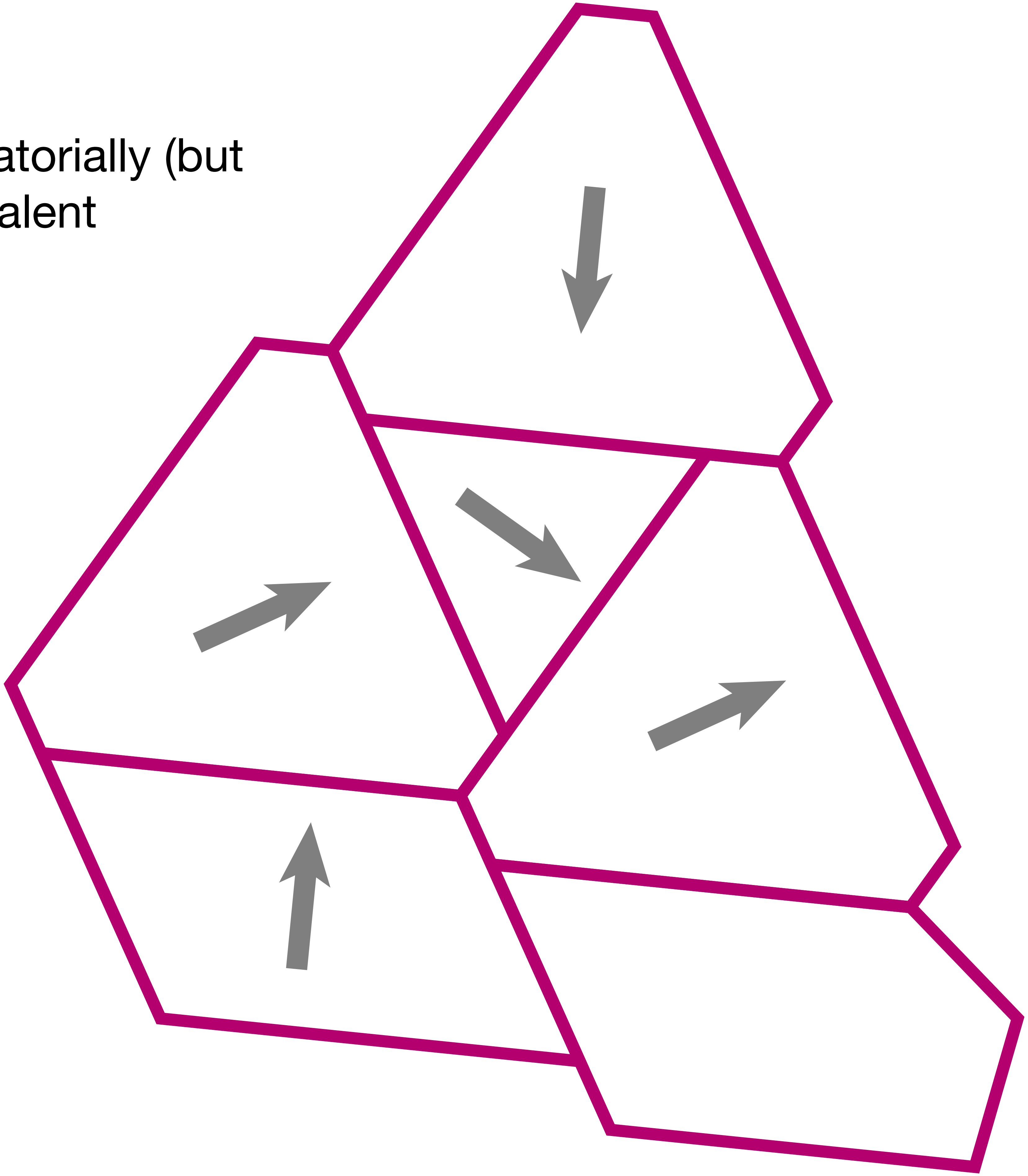

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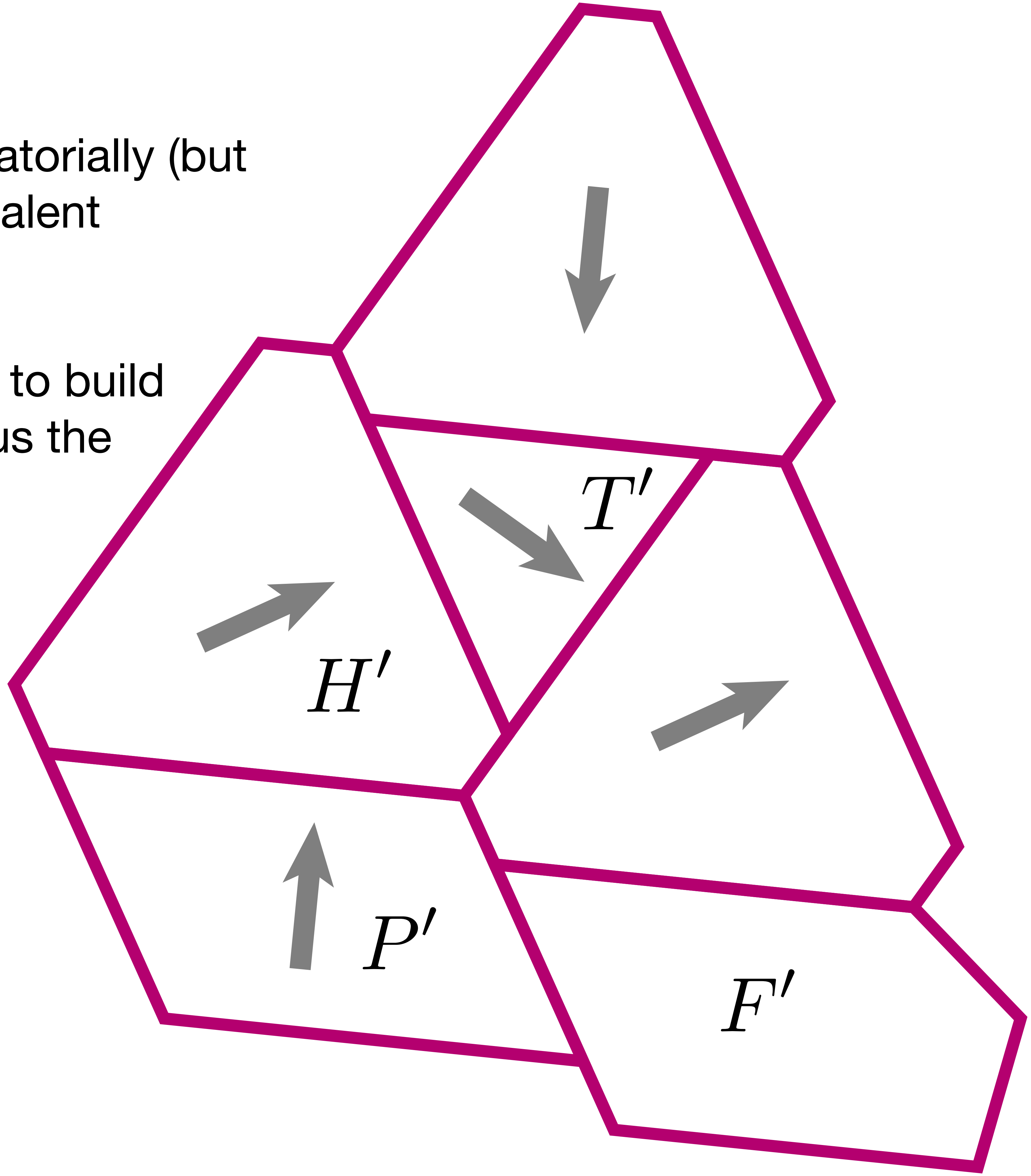
Iterate this construction to build patches of any size. Thus the hat tiles the plane!



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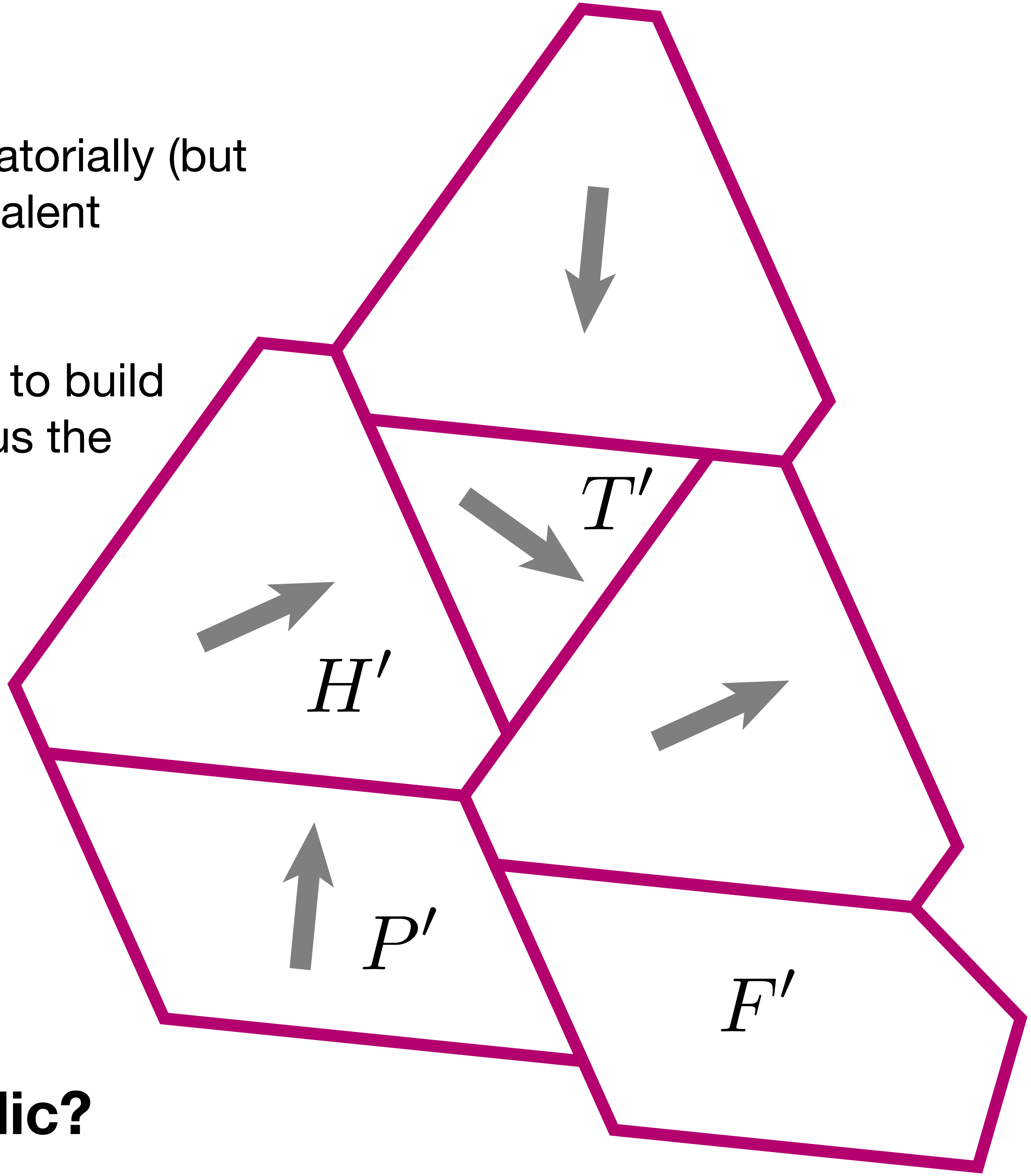


…but is it aperiodic?

Forcing non-periodicity

To complete a proof of aperiodicity, we must show that no tiling by the hat can be periodic.

Past aperiodic sets were generally engineered with matching conditions that facilitate a Berger-style proof of forced non-periodicity. The hat was found "in the wild".

With all the complexity baked into a single shape, a case-based analysis seems daunting.

Happily, the metatiles can help. So can Chaim and Joseph, who joined David and Craig in January!

Forcing non-periodicity

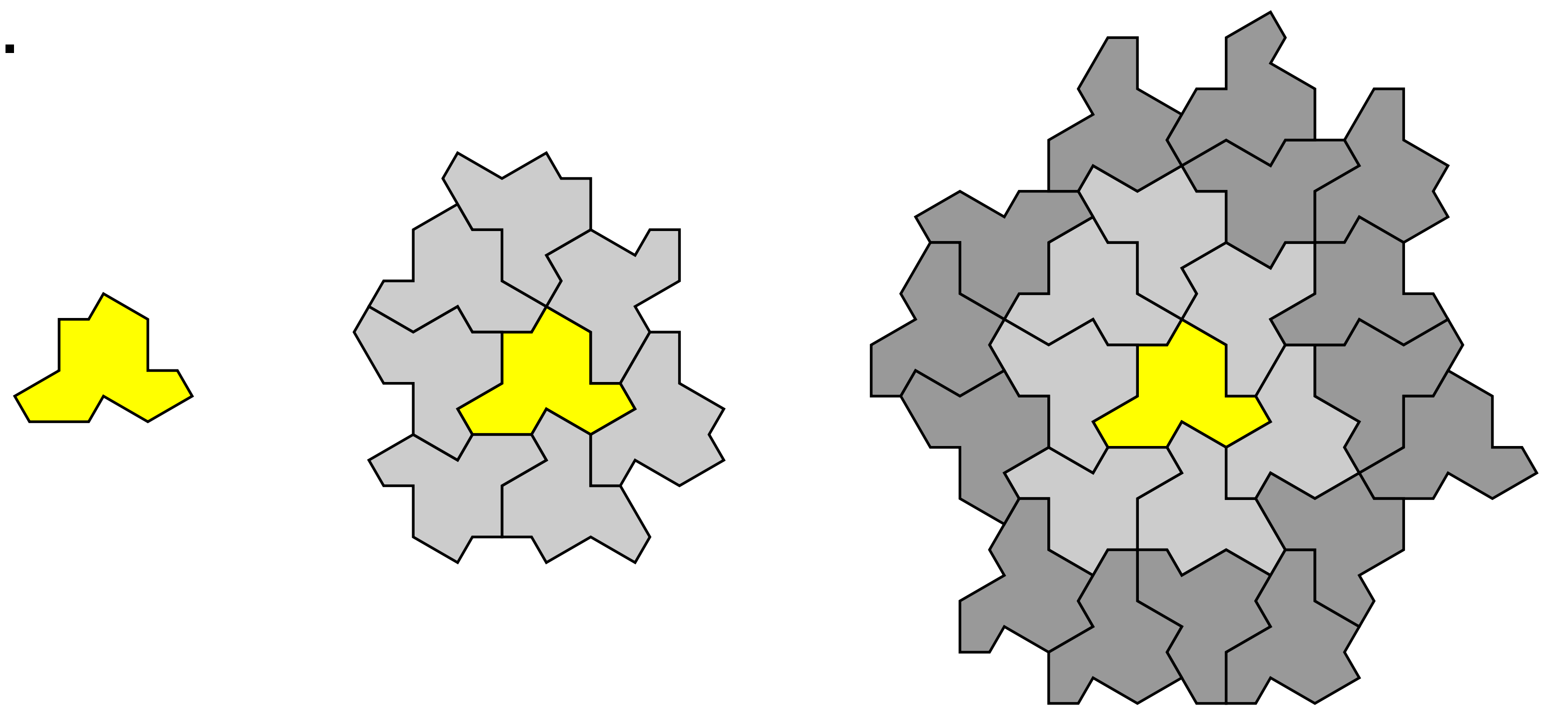
Use the metatiles as an intermediate step to manage complexity.

Step 1: Prove that in any tiling by hats, the tiles are forced to cluster into metatiles.

Step 2: Prove that the matching conditions on the metatiles force them to assemble into larger, combinatorially equivalent copies of themselves.

Surroundable 2-patches

An n**-patch** is a patch of tiles consisting of a tile surrounded by n rings of copies.



0-patch　　1-patch　　2-patch

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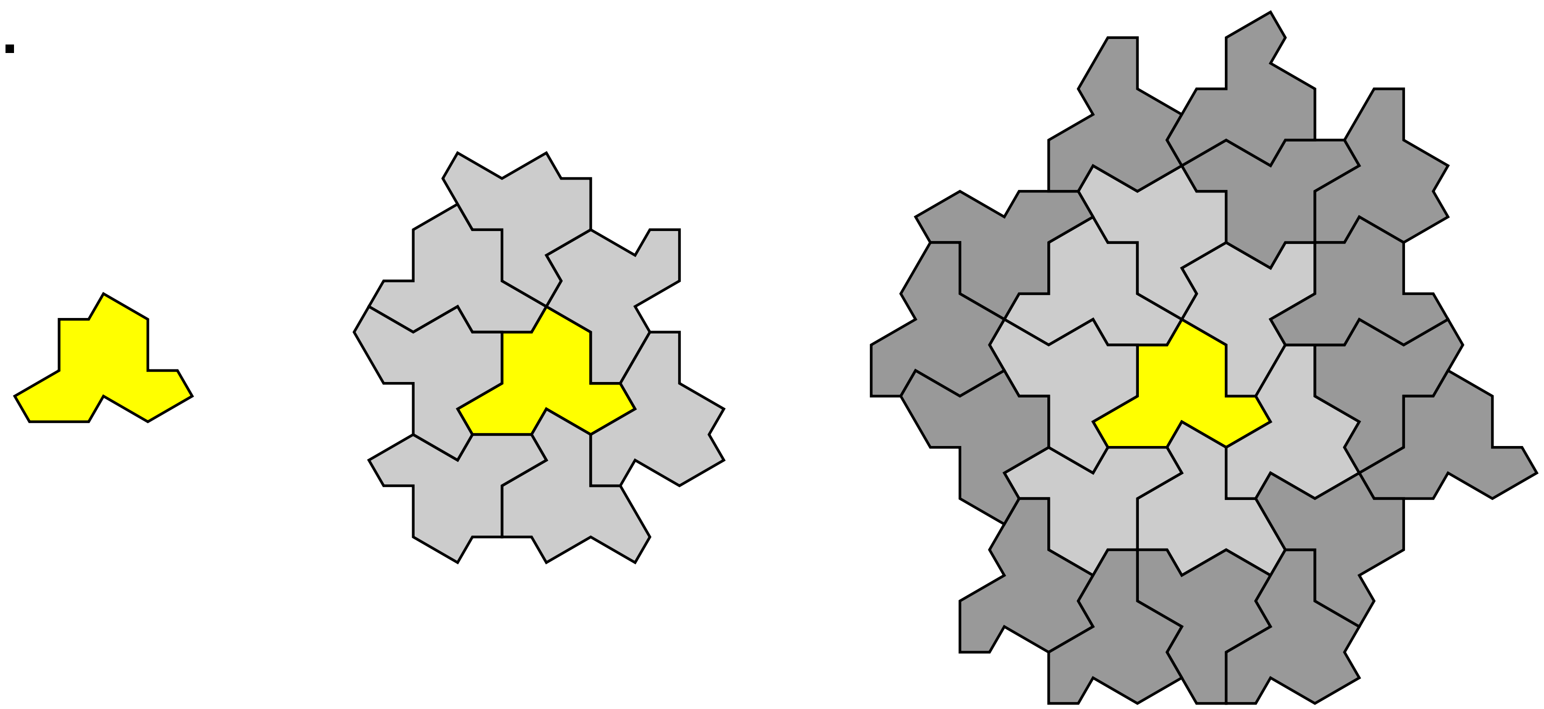
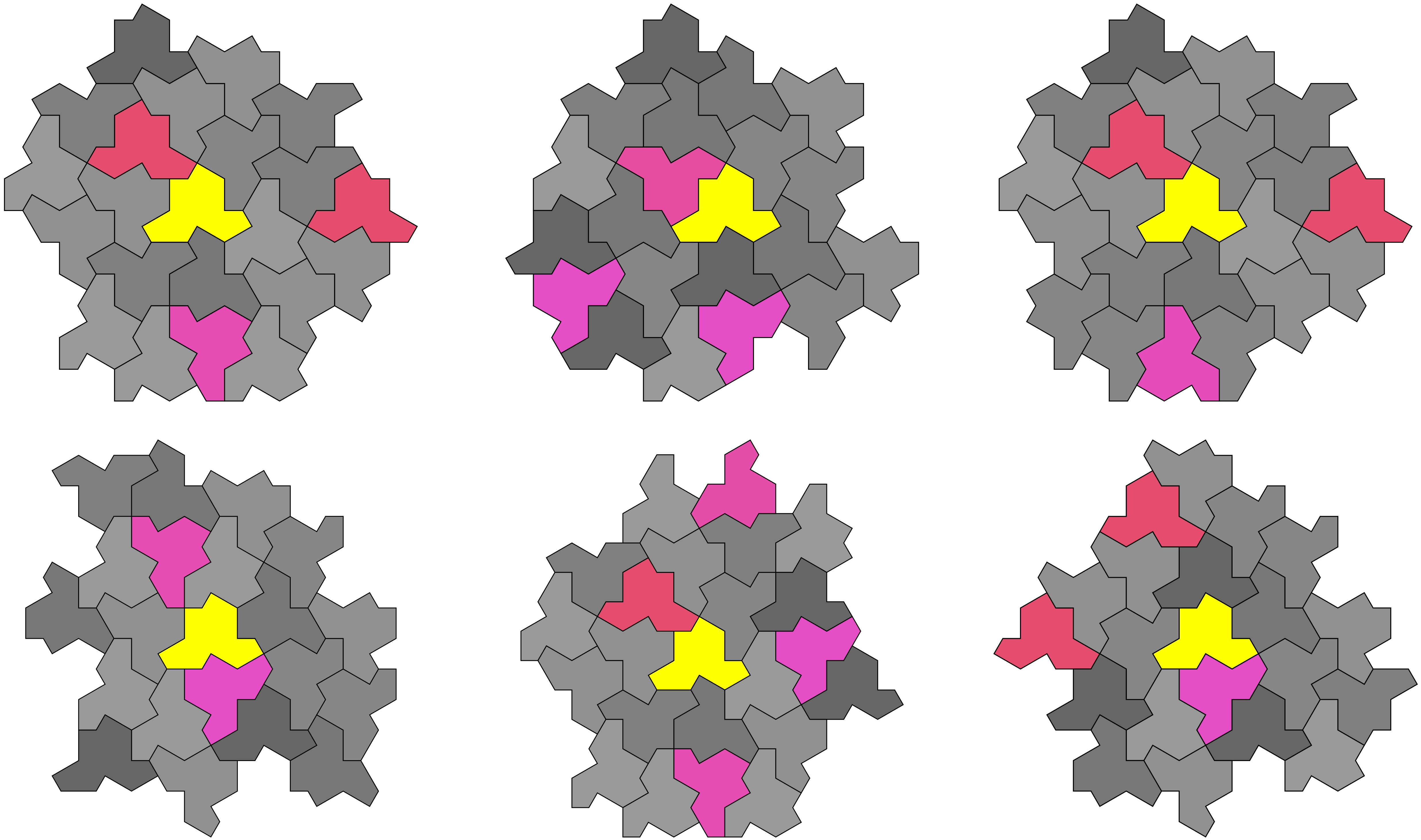


0-patch 1-patch 2-patch

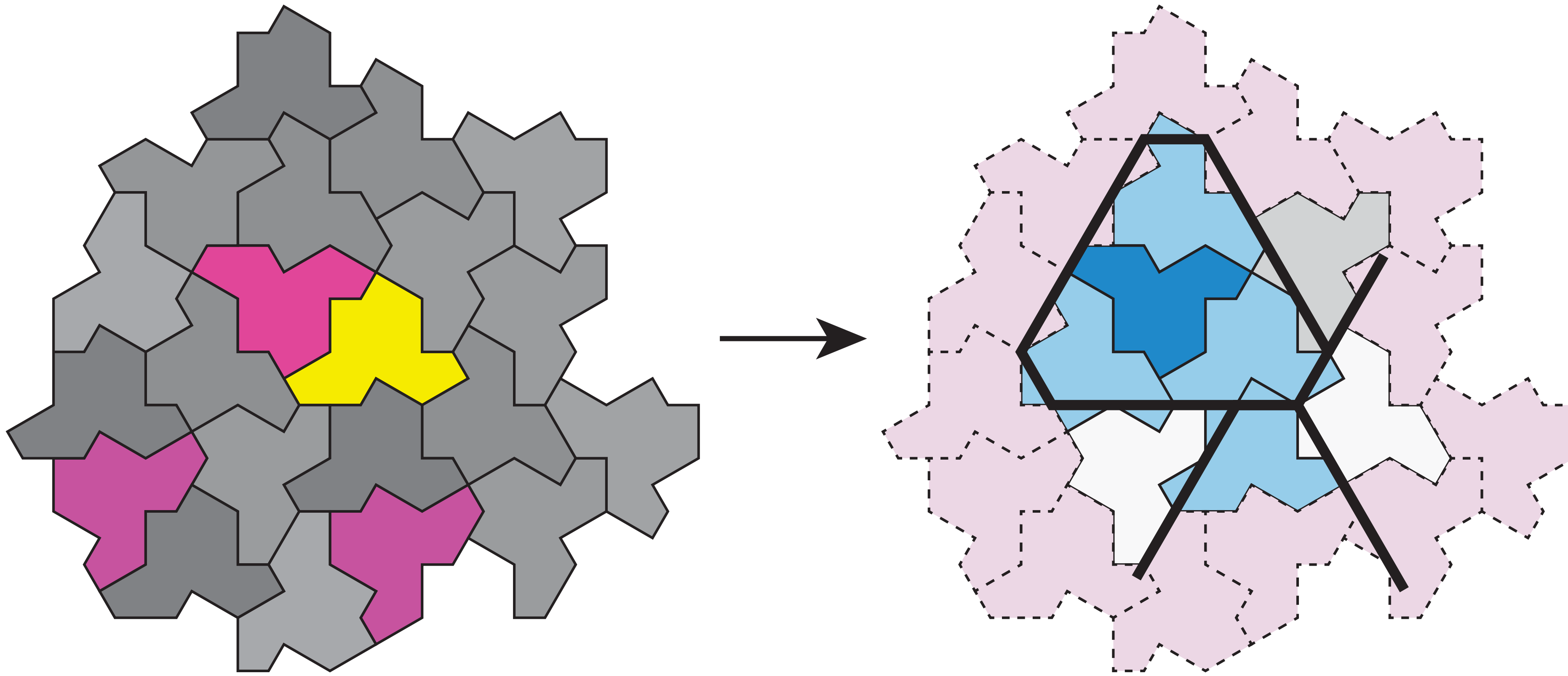
A **surroundable 2-patch** is a 2-patch that lies in the interior of any 3-patch.

Surroundable 2-patches



We use software to enumerate the 188 surroundable 2-patches of hats, which loosely simulate neighbourhoods of real tilings.

Validity of clustering

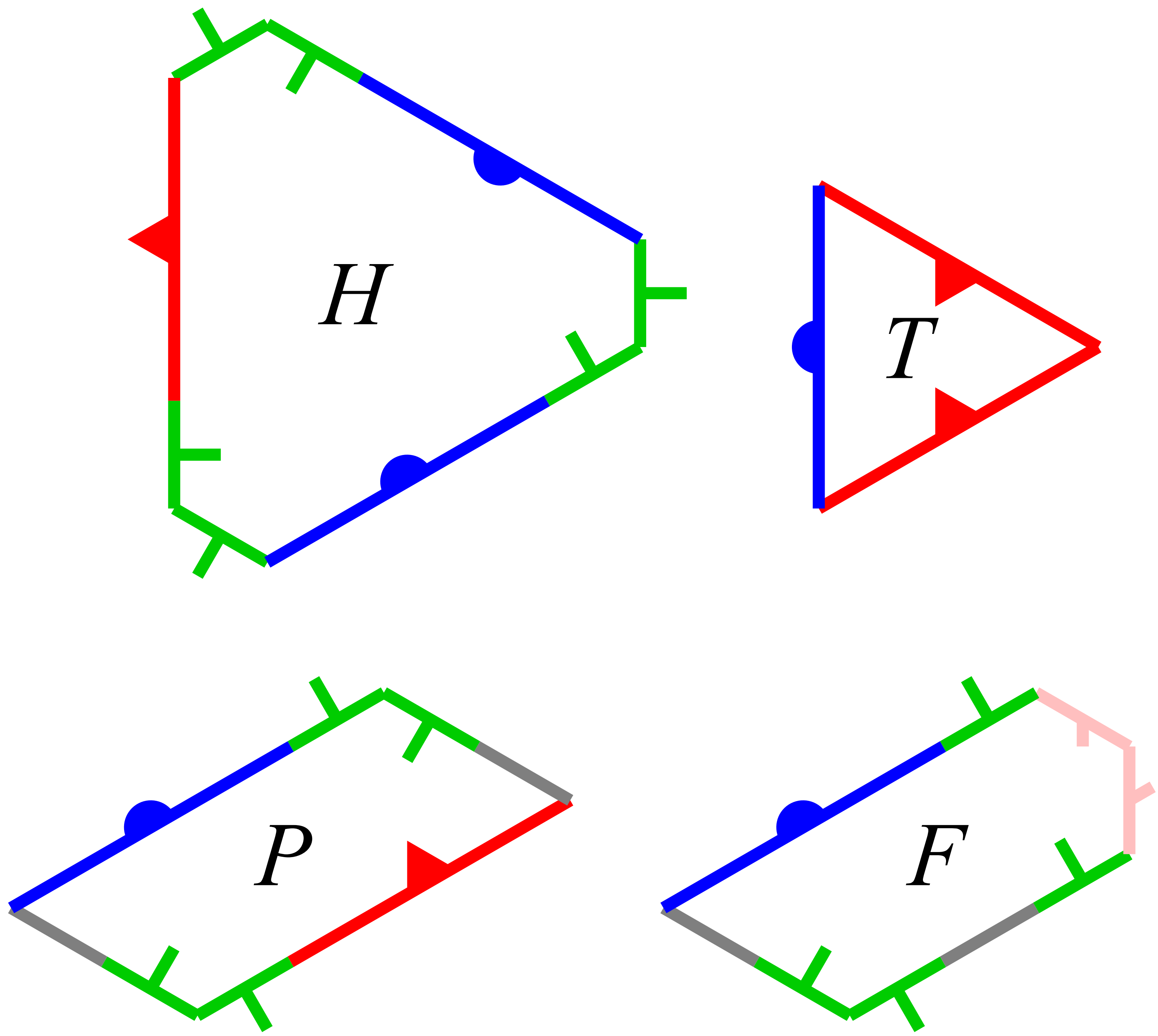


With software we can check that for every surroundable 2-patch,

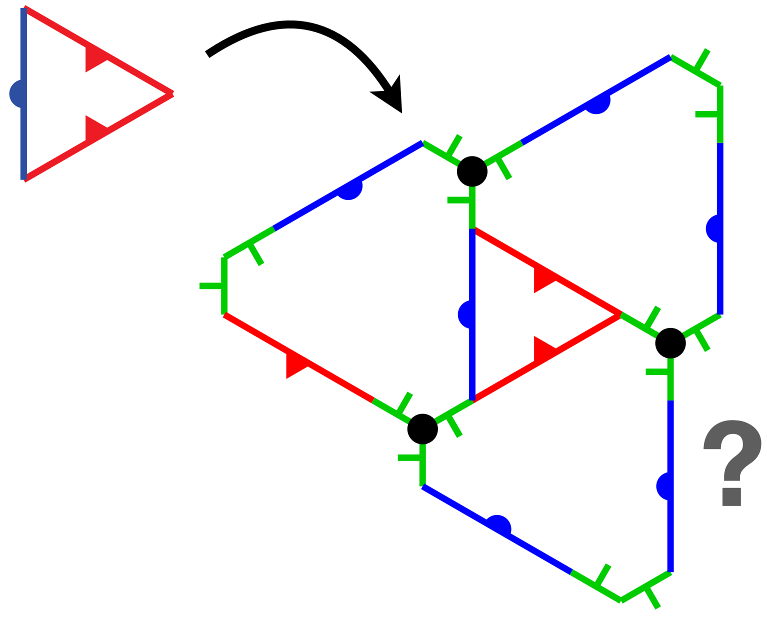
1. The interior hats can be assigned deterministic identities within metatiles; and
2. Implied metatile boundaries between interior tiles obey prescribed matching conditions.

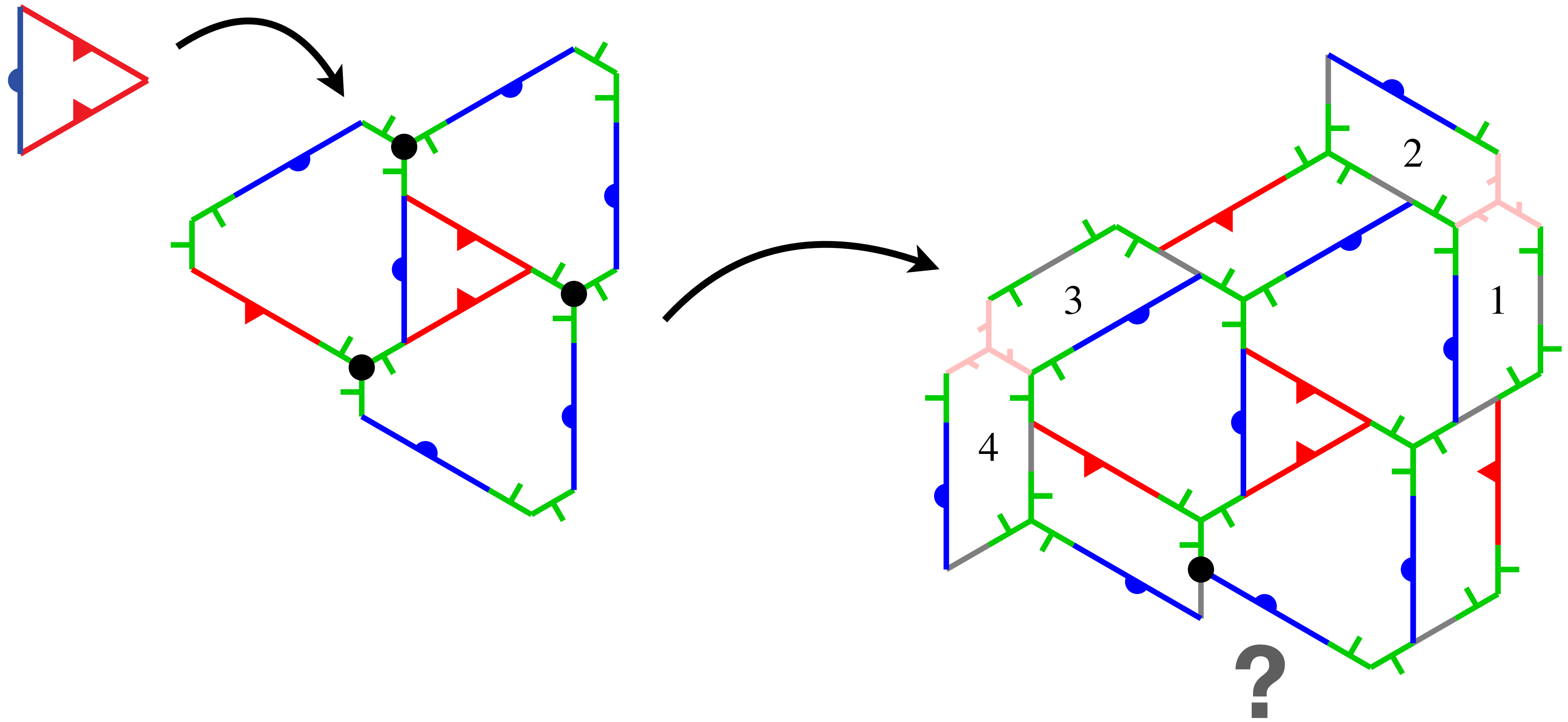
Thus any hat tiling has a legal decomposition into metatiles.

Assembly of supertiles



Build a big tree of clusters of metatiles, prove that the only legal results are supertiles with combinatorially equivalent matching conditions.





2
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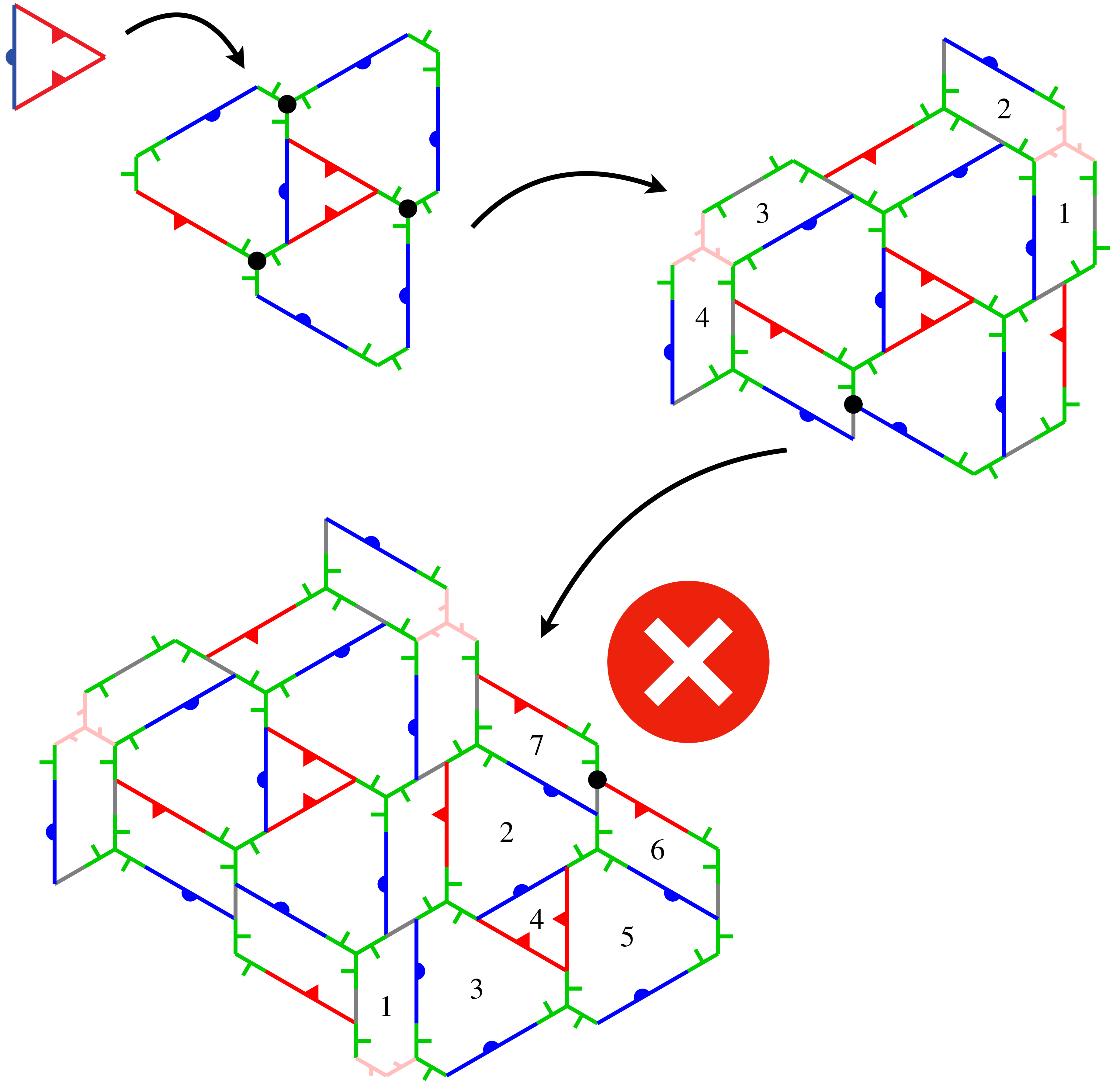
4
?



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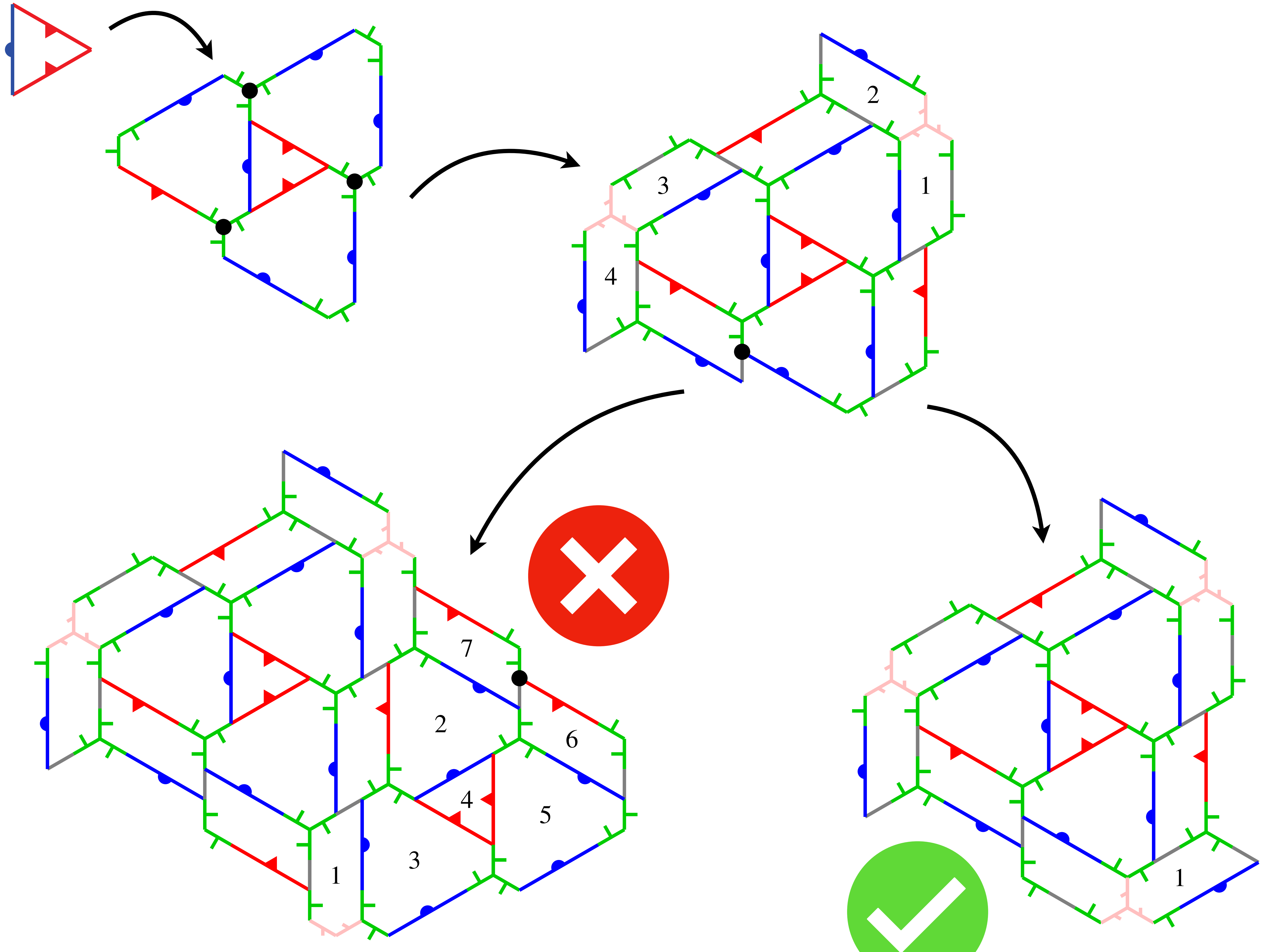

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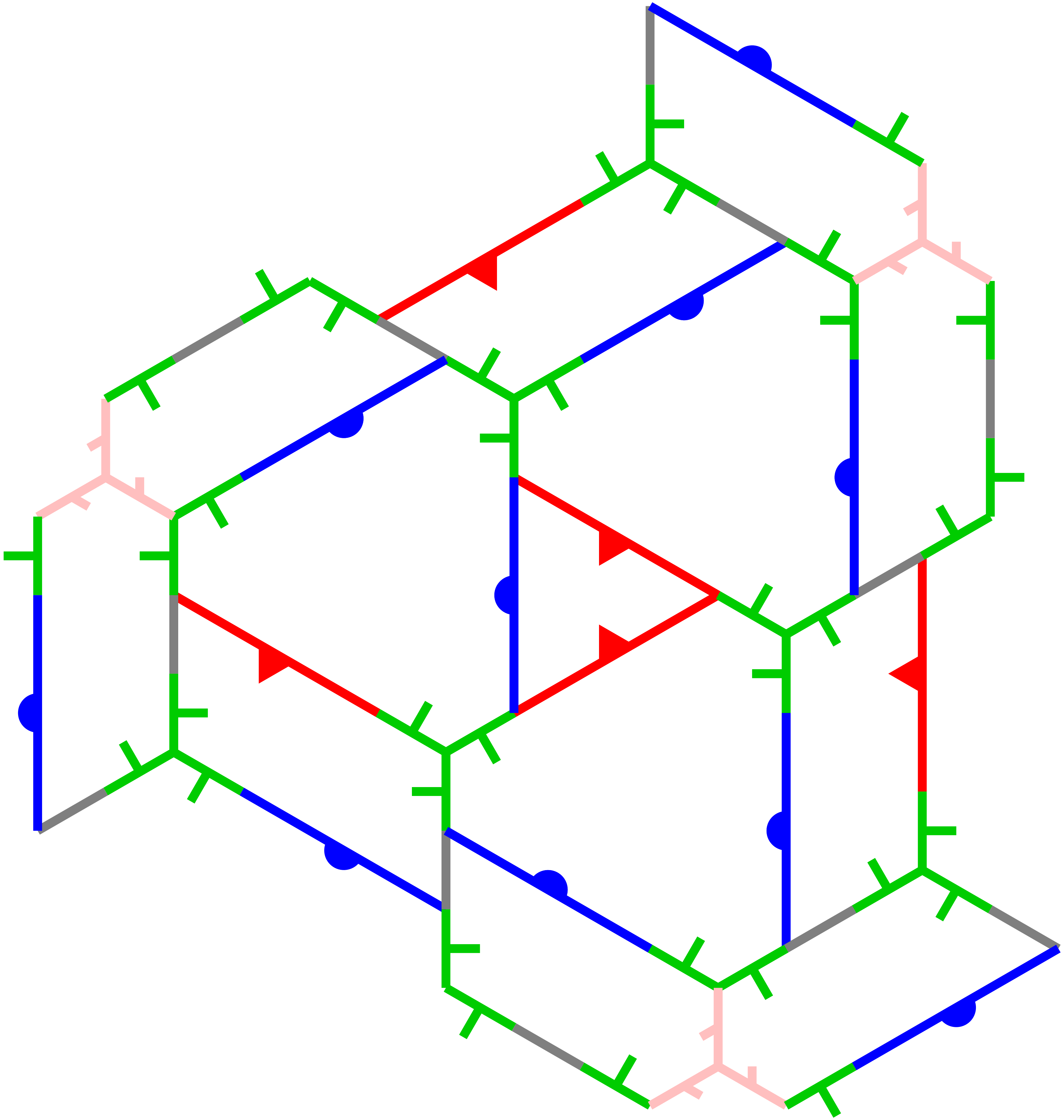

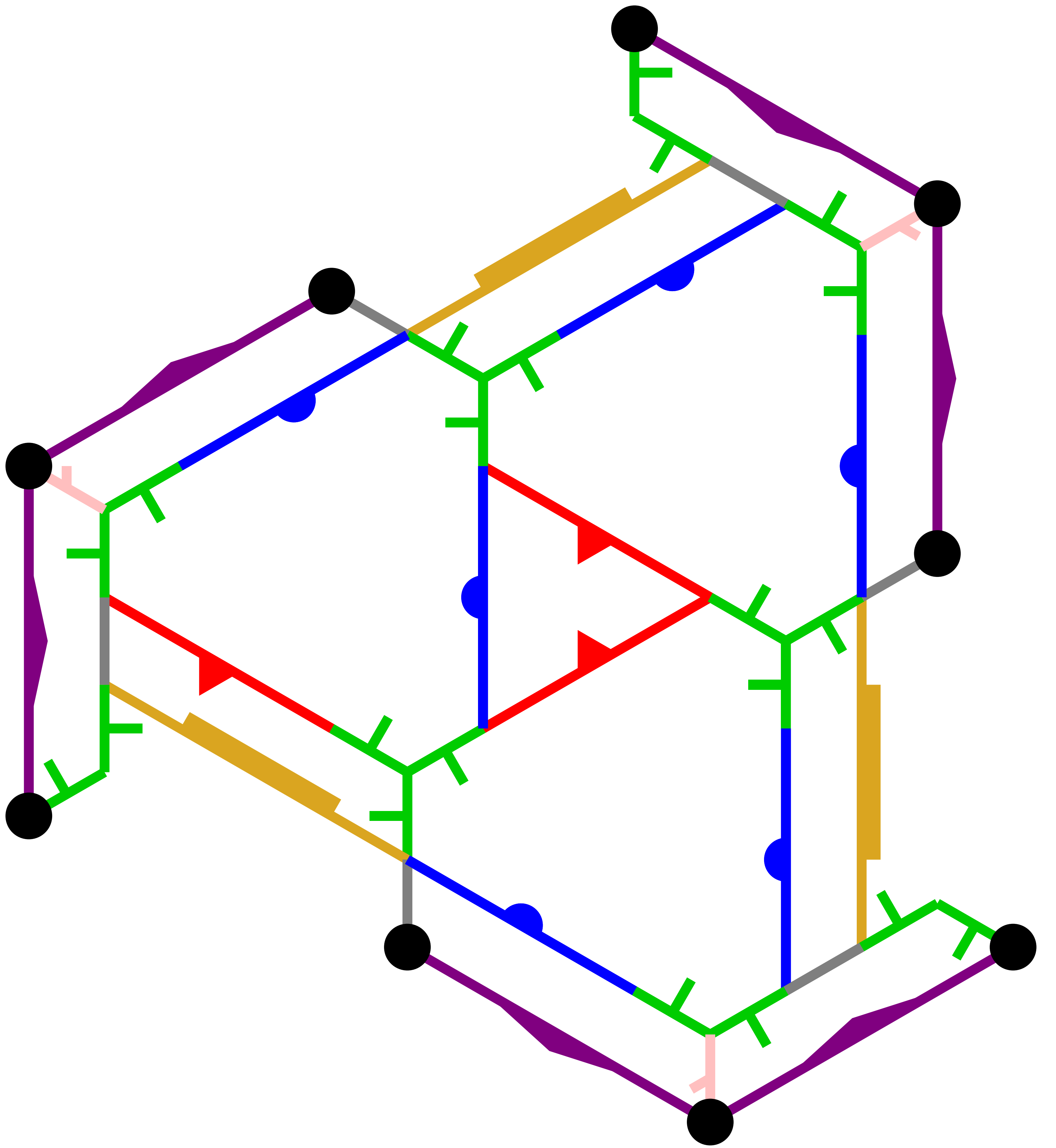

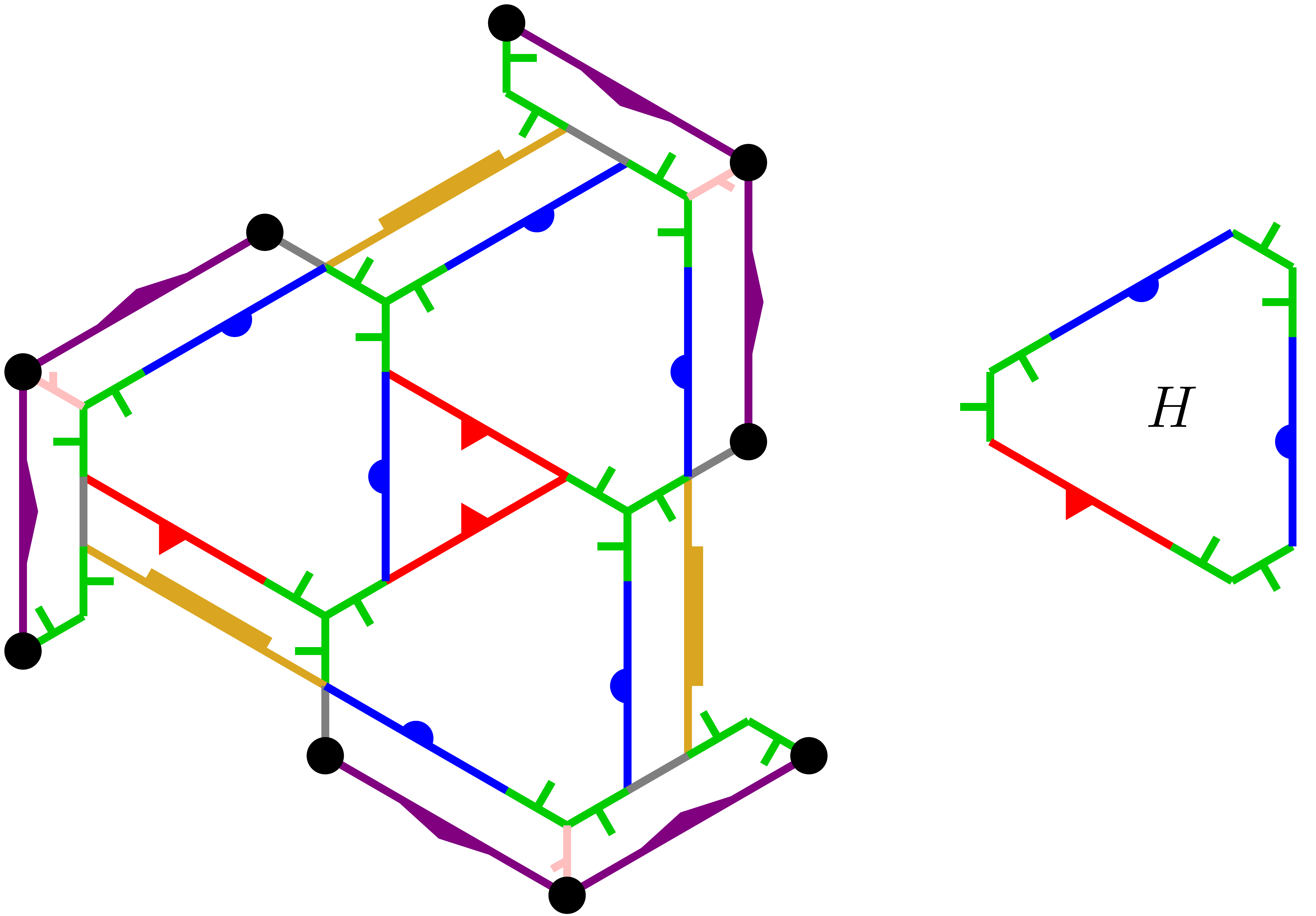
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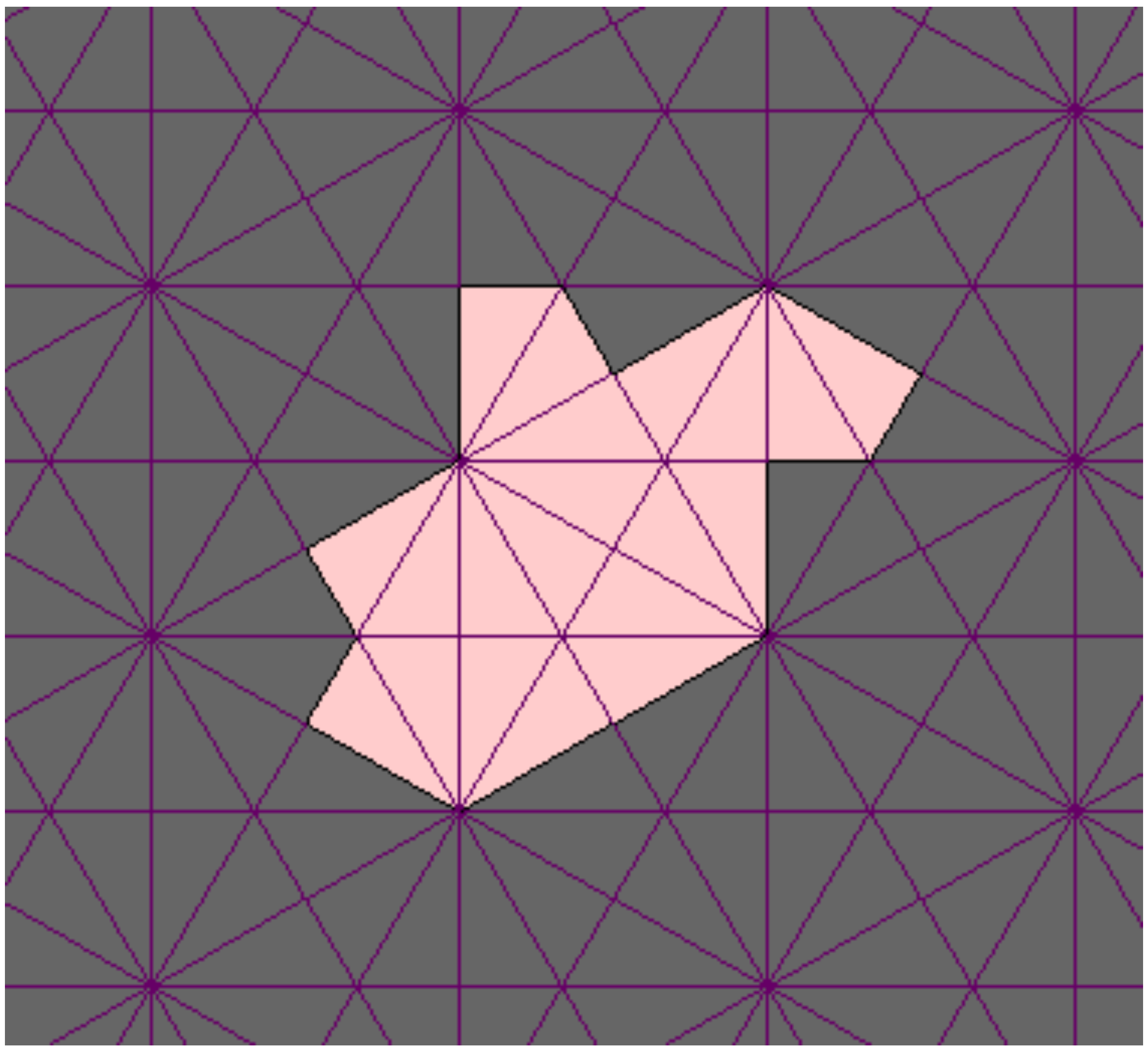
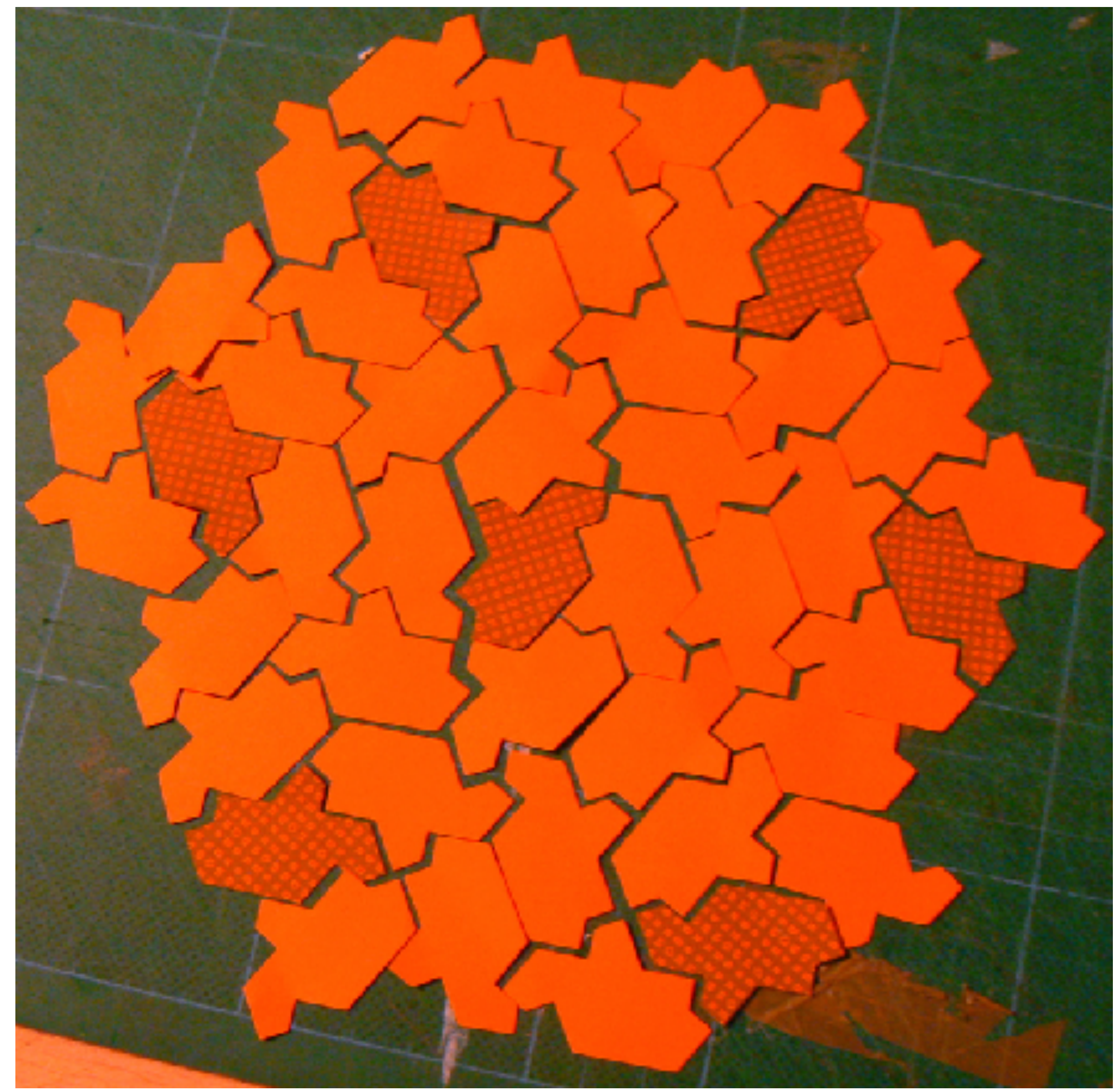
A second einstein!

In December, David had found a second unusual shape (the "turtle").



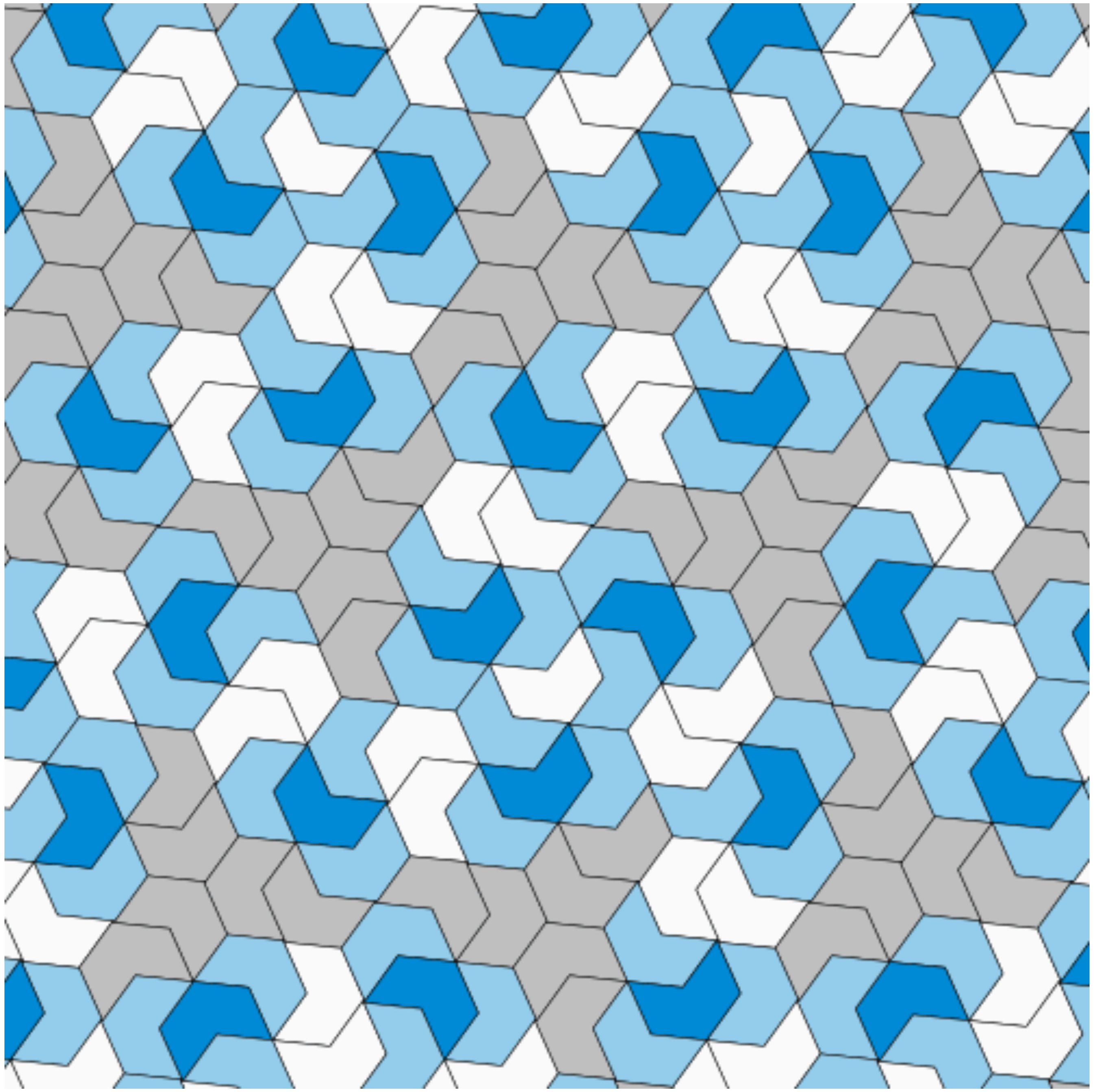


Joseph: "Tile B is also aperiodic. In fact, we have an infinite family of aperiodic 13-gon tiles, determined by a parameter that can be any positive real except maybe 1…"

5 February

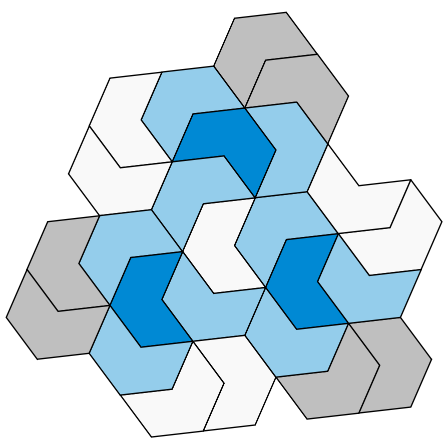
Joseph: "I also now have an outline that *might* work for showing nonexistence of a (strongly) periodic tiling based on the coupling between two polyiamond tilings…"

21 February

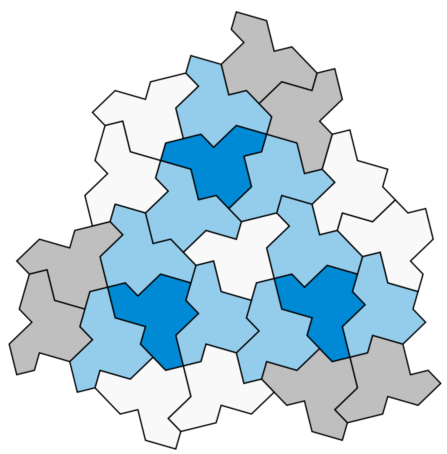


The continuum

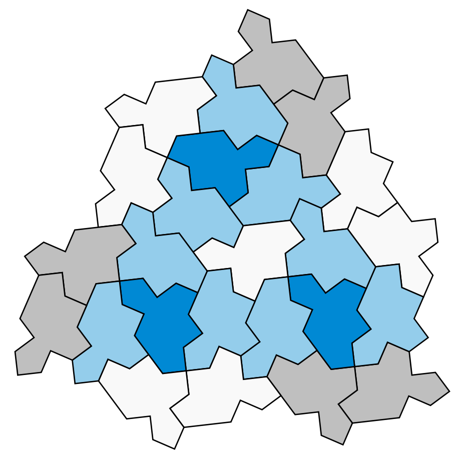
Hat edges come in two lengths in $\sqrt{3}$ proportion. Every edge has a parallel partner. So we can adjust the two lengths independently to produce a continuum of tile shapes denoted $\mathrm{Tile}(a, b)$ for $a, b \geq 0$.



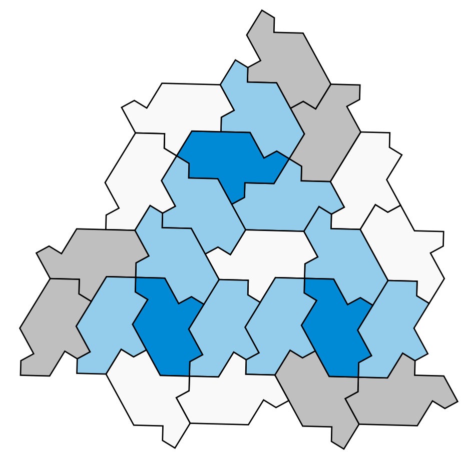
$\mathrm{Tile}(0, 1)$



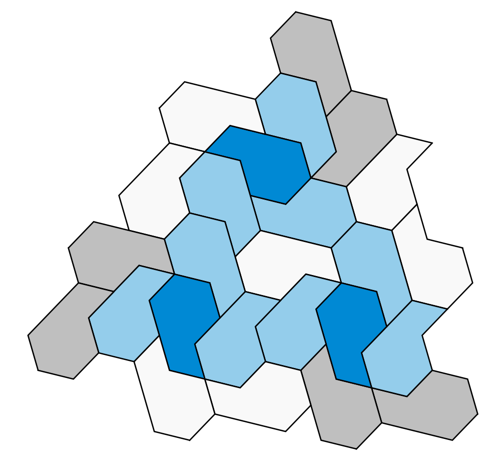
$\mathrm{Tile}(1, \sqrt{3})$



$\mathrm{Tile}(1, 1)$

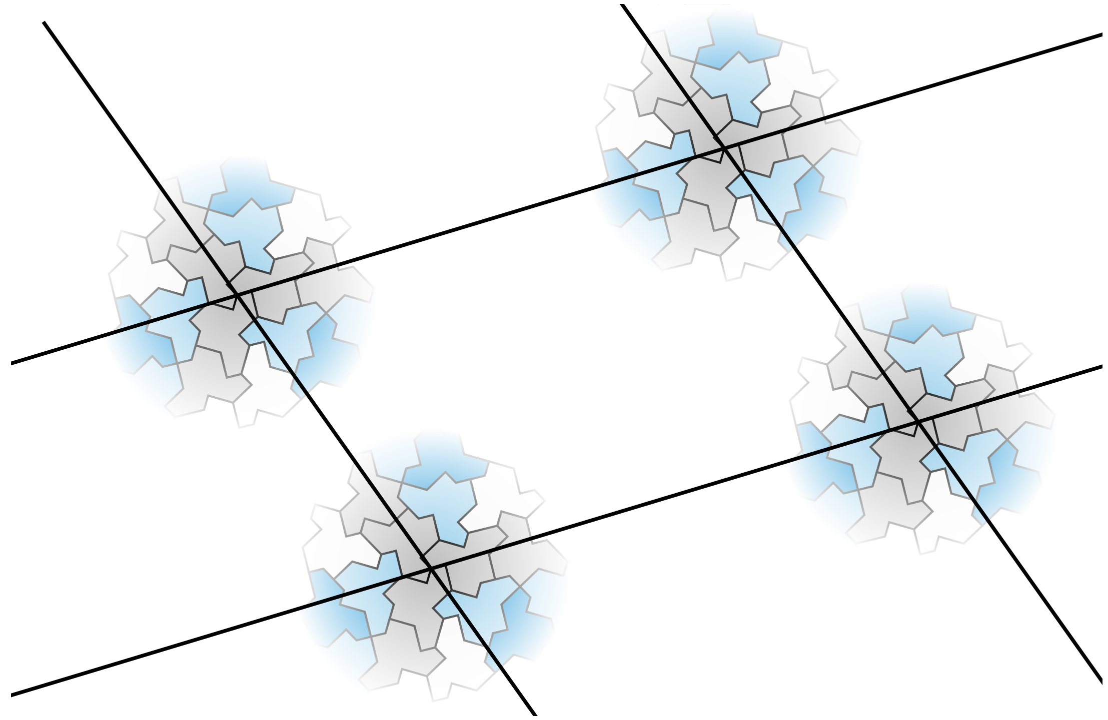


$\mathrm{Tile}(\sqrt{3}, 1)$

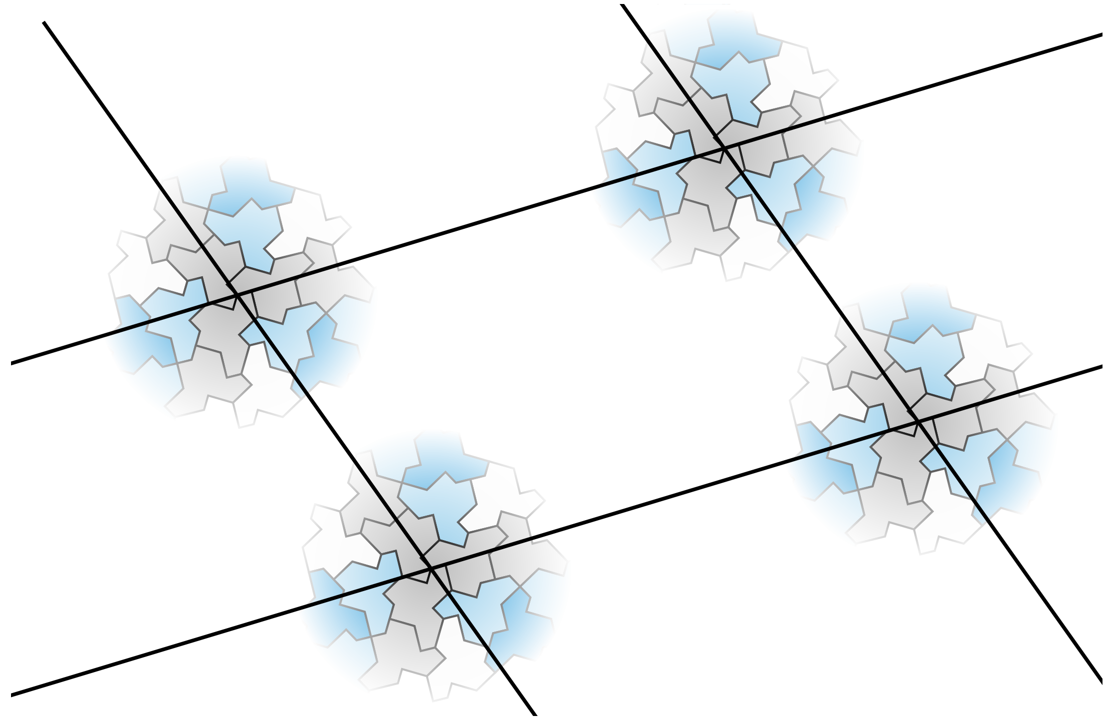


$\mathrm{Tile}(1, 0)$

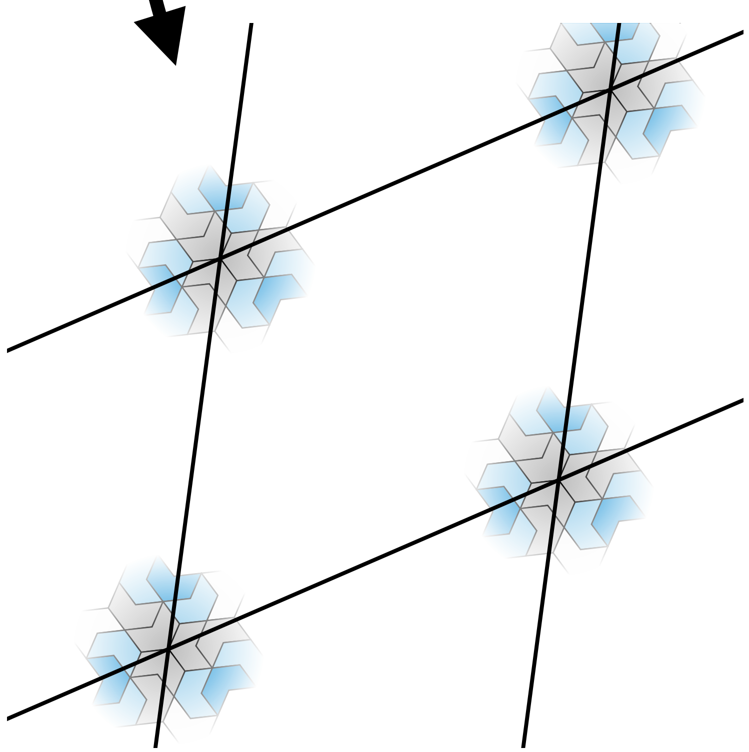
The tiles $\mathrm{Tile}(0,1)$ (the "chevron"), $\mathrm{Tile}(1,1)$, and $\mathrm{Tile}(1,0)$ (the "comet") admit periodic tilings. All others are aperiodic monotiles with combinatorially equivalent tilings.

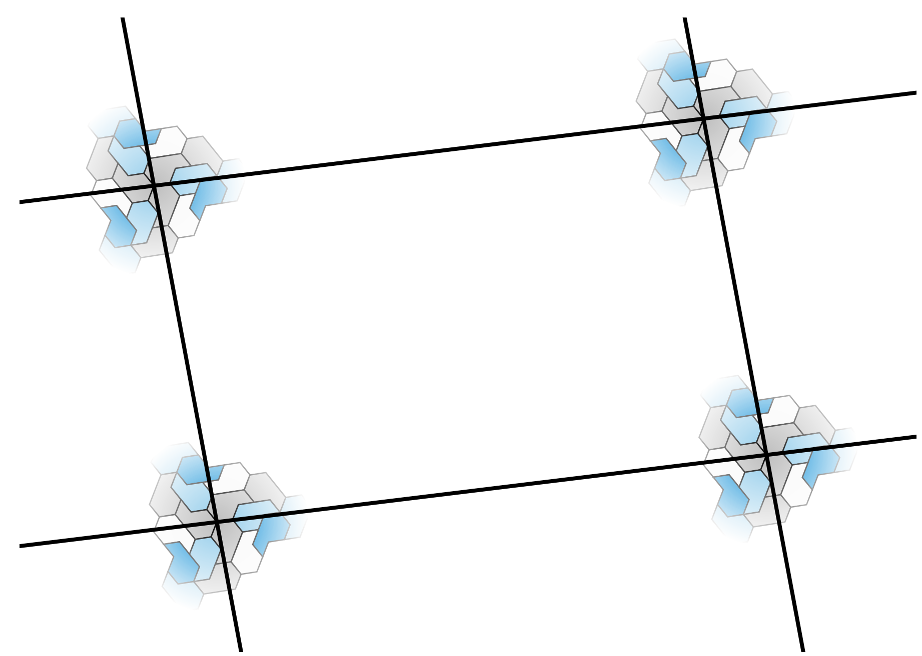


Suppose that a tiling by hats were periodic.

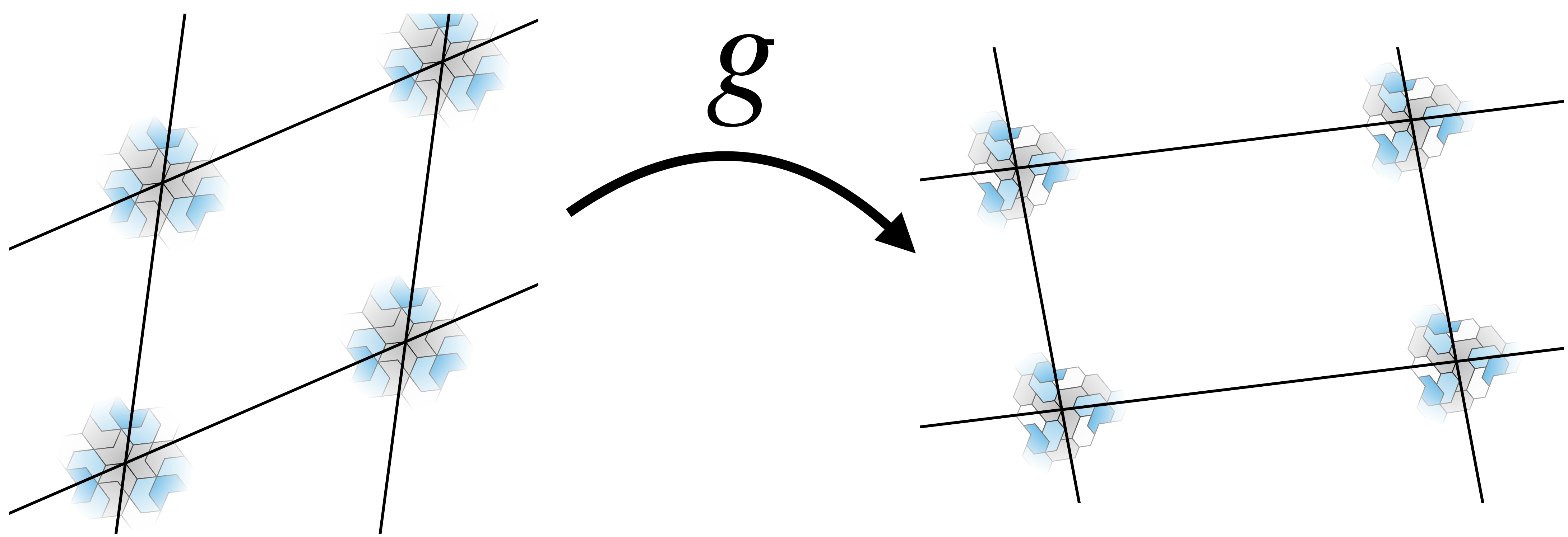


There would be corresponding periodic tilings by chevrons and comets.



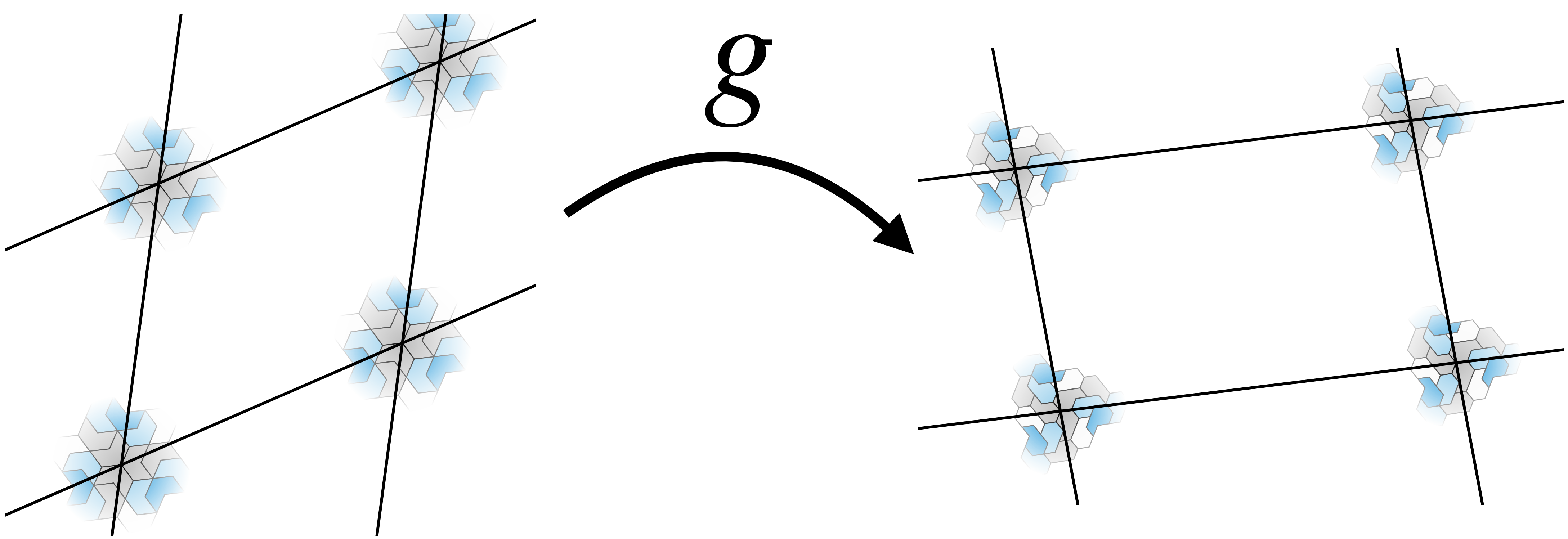


The affine transformation g mapping the lattice of translations of the chevron tiling to that of the comet tiling cannot be a similarity: it must scale areas by $2/3$, which would scale lengths by an impossible amount.

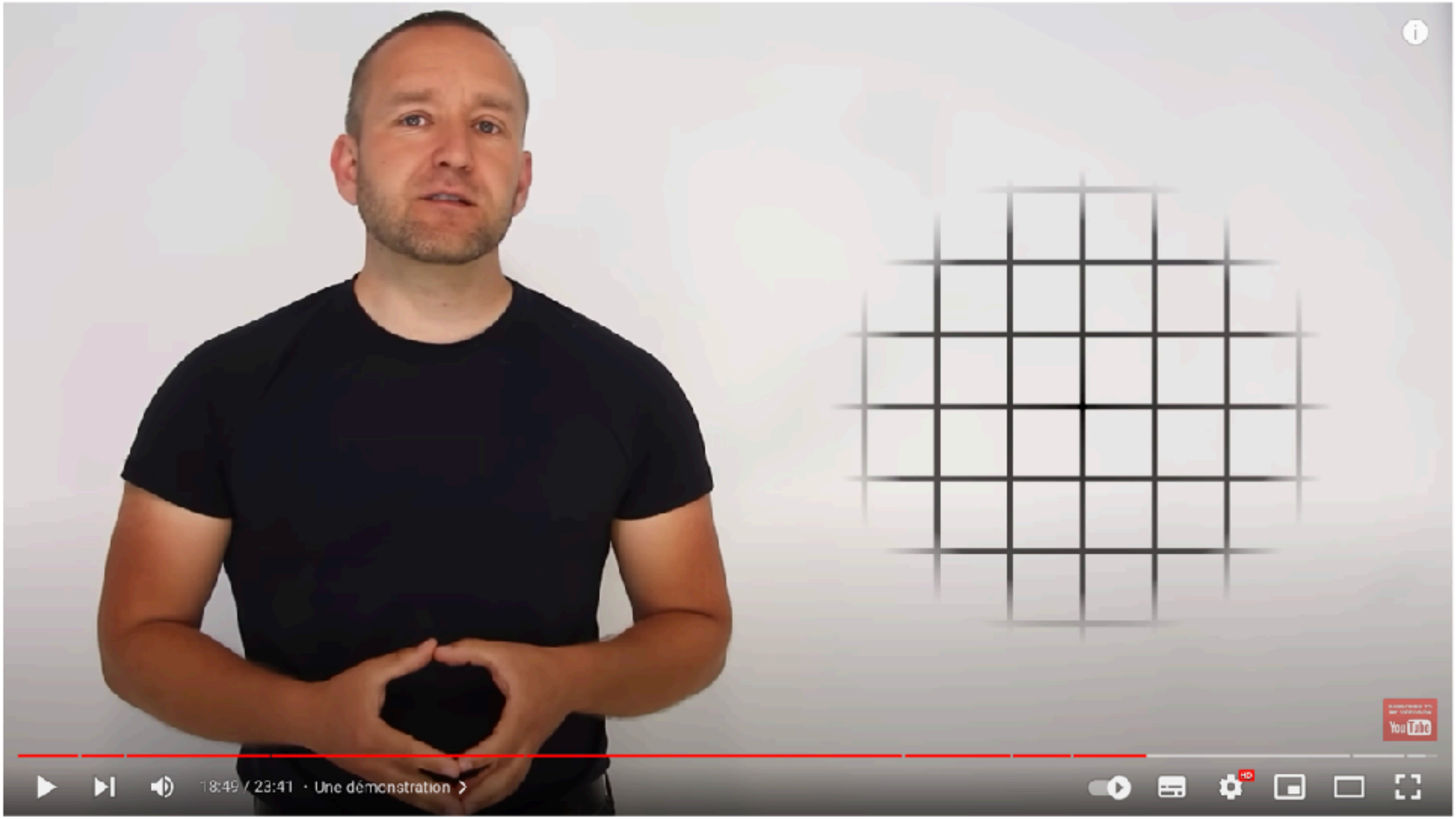


But by taking into account the distribution of tile orientations in all three tilings, we show that g must be a similarity, a contradiction.

Therefore, the original tiling by hats could not have been periodic.



Passe-Science #53 on YouTube



La première tuile apériodique de l'histoire! The Hat - Passe-science #53

Conclusion

The hat is an aperiodic monotile with an unusual origin story.

We provide a "standard" combinatorial proof of aperiodicity, and a new indirect geometric proof.

Related problems for future work:

- Are there "simpler" aperiodic monotiles?
- Is there a chiral aperiodic monotile?
- Is there a 3D aperiodic monotile?
- Are there bounds on isohedral numbers or Heesch numbers?
- Is the tiling problem undecidable for a single tile in the plane?
- Is the periodic tiling problem undecidable for a single tile in the plane?

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Thank you!

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