The structure of low complexity subshifts

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2. Structures via directive sequences
3. Two structure theorems
4. Discussion

Symbolic dynamics

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  - We write $x = \dots x_{-1}.x_0x_1\dots$ if $x \in \mathcal{A}^{\mathbb{Z}}$.
  - A sequence $(x^k)_k$ converges iff $\forall j$, $(x_j^k)_k$ is eventually constant.

Symbolic dynamics

- Define the **shift map** $S\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ as

$$S\colon \; \ldots x_{-2}x_{-1}\bullet x_0x_1\cdots \mapsto \ldots x_{-1}x_0\bullet x_1x_2\ldots$$

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- The growth of p_X measures how "random" are the orbits of S in X.

Symbolic dynamics: example

- Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $u \in \{0, 1, \ldots, 9\}^{\mathbb{N}}$ be its decimal expansion.

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- The frequency f_0 of the digit 0 in u is equal to:

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{0 \leq n < N} \chi_U \circ T^n(u),$$

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- If $\liminf_{n \to +\infty} p_X(n)/n < +\infty$, then x is transcendental (Adamczewski and Bugeaud, Ann. Math. 2007.).

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- Classic examples: substitutions, Sturmians, IETs, linearly recurrent, “almost every” finite top. rank system, etc.
- X is **transitive** if there is $x \in X$ s.t. $\{S^n(x) : n \in \mathbb{Z}\}$ is dense in X.

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- X is **transitive** if there is $x \in X$ s.t. $\{S^n(x) : n \in \mathbb{Z}\}$ is dense in X.
- We will only consider **minimal** subshifts *i.e.* s.t. every orbit is dense in X.

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The $\mathcal{S}$-adic conjecture. Consider the class (L) of **linear-growth complexity subshifts**, *i.e.*, consisting in subshifts X s.t.

$$\limsup_{n \to +\infty} p_X(n)/n < +\infty.$$

Then, there is an $\mathcal{S}$-adic structure theorem for (L).

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$$p_X(1) = 2 \qquad\qquad p_X(2) = 3$$

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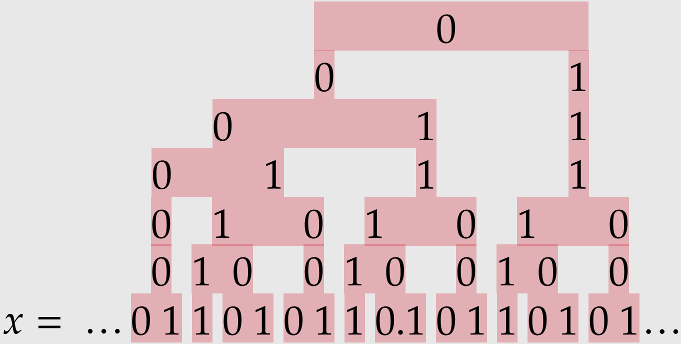

Definitions

- A **substitution** is a map $\tau\colon \mathcal{A}^+ \to \mathcal{B}^+$ s.t.

$$\tau(a_1 \cdots a_k) = \tau(a_1) \cdots \tau(a_k), \ \forall a_1 \cdots a_k \in \mathcal{A}^+.$$

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- A **directive sequence** is a sequence of substitutions $\boldsymbol{\tau} = (\tau_n)_{n \geq 0}$ having the form

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  We write $\tau_{n,m} = \tau_n \ldots \tau_{m-1}$ for $m > n \geq 0$.

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- Words of the form $\tau_{0,n}(a)$, $a \in \mathcal{A}_n$, are *base blocks* for the level n.

Definitions

- Let $\mathcal{L}_{\boldsymbol{\tau}}^{(n)} \subseteq \mathcal{A}_n^+$ be the set of subwords of

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Definition. The ***n*-th $\mathcal{S}$-adic subshift** $X_{\boldsymbol{\tau}}^{(n)}$ is defined as

$$\{x \in \mathcal{A}_n^{\mathbb{Z}} : \text{any finite subword of } x \text{ belongs to } \mathcal{L}_{\boldsymbol{\tau}}^{(n)}.\}$$

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Fact. For every $n \geq 0$, $X_{\boldsymbol{\tau}}^{(n)} = \bigcup_{k \in \mathbb{Z}} S^k \tau_n \left(X_{\boldsymbol{\tau}}^{(n+1)} \right)$.

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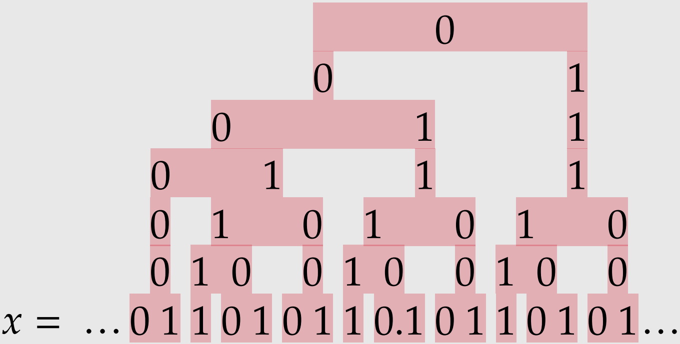

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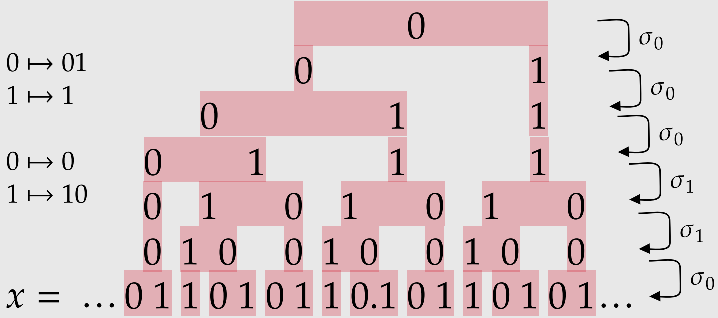

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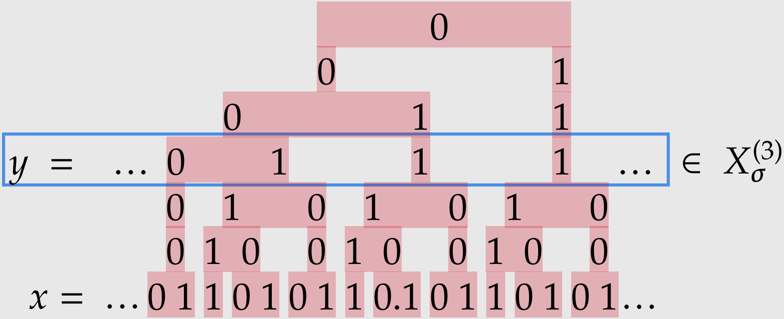

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The main question

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- **Intuition:** There is a *hidden structure* explaining this rigidity.

The $\mathcal{S}$-adic conjecture. Consider the class (L) of **linear-growth complexity subshifts**, *i.e.*, consisting in subshifts X s.t. Then, there is an $\mathcal{S}$-adic structure theorem for (L).

- The class (L) has connections with many other areas.

Past work

- Sturmian subshifts ($p_X(n) = n + 1$) have an $\mathcal{S}$-adic structure that uses just 2 substitutions (Coven, Hedlund '73).

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> **Theorem** (Cassaigne '95). A transitive subshift in (L) is such that $p_X(n+1) - p_X(n)$ is bounded.

- We say that $\boldsymbol{\tau}$ is **finitary** if $\{\tau_n \colon n \geq 0\}$ is finite.

> **Theorem** (Ferenczi '96). Any transitive subshift in (L) is generated by a finitary directive sequence.

Past work

- In (Leroy '13), a finitary $\mathcal{S}$-adic structure is described for the case $p_X(n+1) - p_X(n) \leq 2$.

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- Other works try to narrow down the type of searched $\mathcal{S}$-adic structure, but none of the proposed conditions is considered satisfactory.

Main result

- If w is a word, then $\mathrm{root}(w)$ defined as its shortest prefix v s.t. $w = v^k$ for some $k \geq 1$.
- Example: $\mathrm{root}(ababab) = ab$ and $\mathrm{root}(abababa) = abababa$.

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Theorem. A minimal subshift X has linear-growth complexity if and only if there exist $d \geq 1$ and an directive sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n \geq 0}$ generating X such that for every $n \geq 1$:

($\mathcal{C}_1$) $\mathrm{root}(\tau_{0,n}(\mathcal{A}_n)) := \{\mathrm{root}(\tau_{0,n}(a)) : a \in \mathcal{A}_n\}$ has at most d elements.

($\mathcal{C}_2$) $|\tau_{0,n}(a)| \leq d \cdot |\tau_{0,n}(b)|$ for every $a, b \in \mathcal{A}_n$.

($\mathcal{C}_3$) $|\tau_{n-1}(a)| \leq d$ for every $a \in \mathcal{A}_n$.

Main Result

- Suppose that $\boldsymbol{\tau}$ satisfies $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$.

Corollary. There exists a constant d s.t. for every $x \in X_{\boldsymbol{\tau}}$ and $\ell \geq 1$, we can find at most d words $\{w_a\}_a$ decomposing x as

$$x = \dots w_{a_0}^{p_0} w_{a_1}^{p_1} w_{a_2}^{p_2} w_{a_3}^{p_3} \dots,$$

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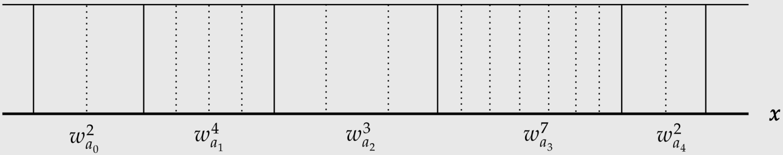

Main Result: variation

- We have an analogous theorem for the class (NSL) of **nonsuperlinear-growth complexity subshifts**.

Theorem. A minimal subshift X is in (NSL), *i.e.*,

$$\liminf_{n\to+\infty} p_X(n)/n < +\infty,$$

if and only if there exist $d \geq 1$ and an directive sequence $\boldsymbol{\tau} = (\tau_n \colon \mathcal{A}_{n+1} \to \mathcal{A}_n^+)_{n\geq 0}$ generating X such that for every $n \geq 1$:

$(\mathcal{C}_1)$ $\mathrm{root}(\tau_{0,n}(\mathcal{A}_n)) := \{\mathrm{root}(\tau_{0,n}(a)) : a \in \mathcal{A}_n\}$ has at most d elements.

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Finitary structures

- Our main theorem is intrinsically non-finitary.

Theorem. There is a minimal subshift X in (L) s.t. any $\mathcal{S}$-adic structure satisfying $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ is not finitary.

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Conjecture. There is no finitary $\mathcal{S}$-adic structure theorem for (L).

- This is not a formal statement.

Applications

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- Our main theorem allowed us to give a new proof of Cassaigne's Theorem using known $\mathcal{S}$-adic techniques.

Applications

The known $\mathcal{S}$-adic tools permit to give new proofs for:

- that subshifts in (NSL) are partially rigid (Creutz '23).
- that subshifts in (NSL) have finite top. rank (DDMP, '21).
- the characterization of (L) from (Cassaigne, Frid, Puzynina and Zamboni, '19).

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Our conclusion. The classes (NSL) and (L) gain effective access to the $\mathcal{S}$-adic tool set, and thus our theorems provide a **unified framework** for these results.

Thank you!

Proof idea

- A *coding* of $Y \subseteq \mathcal{B}^{\mathbb{Z}}$ is a pair $(X \subseteq \mathcal{A}^{\mathbb{Z}}, \tau\colon \mathcal{A}^+ \to \mathcal{B}^+)$ s.t. $Y = \bigcup_{k\in\mathbb{Z}} S^k\tau(X)$.

- Ex. If $\boldsymbol{\tau}$ is an directive sequence, then $(X_{\boldsymbol{\tau}}^{(n+1)}, \tau_n)$ is a coding of $X_{\boldsymbol{\tau}}^{(n)}$.

Proof Idea.

- Step 1: build appropriate codings $(X_n \subseteq \mathcal{A}_n^{\mathbb{Z}}, \sigma_n\colon \mathcal{A}_n^+ \to \mathcal{A}^+)$ of $X \subseteq \mathcal{A}^{\mathbb{Z}}$, where $|\sigma_{n+1}| \gg |\sigma_n|$.
- Step 2: define substitutions $\gamma_n\colon \mathcal{A}_{n+1}^+ \to \mathcal{A}_n^+$ s.t. σ_{n+1} is equal to $\sigma_n\gamma_n$ (up to a shift).
- Then, $\boldsymbol{\tau} = (\sigma_0, \gamma_0, \gamma_1, \dots)$ generates X and inherits properties from the σ_n.

Proof idea

Idea for building the codings:

- Consider the "returns to right-special words" coding (X_n, σ_n) of X.
- There are two types of behaviors: a periodic one and an aperiodic one.
- The periodic parts occur when there are too many consecutive short return words; we control them using tricks from combinatorics on words.
- The aperiodic parts greatly contribute to the complexity, so they are controlled by p_X.
- Several technical conditions are needed in the interface of the words $\sigma_n(a)$ for defining the connecting substitutions γ_n.