Analogs of overlap-freeness

James D. Currie

University of Winnipeg

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Patterns

Word w *encounters pattern* p if $h(p)$ is a factor of w for some non-erasing morphism h. Word *illegible* encounters xyx. (Let $h(x) = le, h(y) = gib$, for example.)

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If w is a finite overlap-free binary word, then $w = x\mu(y)z$ where y is an overlap-free binary word, and $|x|, |z| \leq 2$.

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If $\boldsymbol{u}$ is an infinite binary overlap-free word, then $\boldsymbol{u} = x\mu(\boldsymbol{w})$, where $\boldsymbol{w}$ is overlap-free, and $x \in \{\epsilon, 0, 1\}$.

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- Fife's Theorem (a characterization of all infinite overlap-free binary words);
- the only patterns encountered by $\boldsymbol{t}$ which are not factors of $\boldsymbol{t}$ are 00100 and 11011 (Shur).

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- The word 01001101 is not ϕ-good. (It contains factor 11.)
- The word 01010100 is not ϕ-good. (It contains factor $XXXX^-$, where $X = 01$.)

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Proof.

We show $\boldsymbol{\ell} = \phi(\boldsymbol{m})$. Since $\boldsymbol{f}$ has final segments beginning 00, word $\boldsymbol{\ell}$ begins 00. By our structure theorems on ϕ-good words, $\boldsymbol{\ell} = \phi(\boldsymbol{u})$, some ϕ-good $\boldsymbol{u}$. However, we see that ϕ is order-reversing on infinite words. It follows that $\boldsymbol{u} = \boldsymbol{m}$. □

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Corollary

We have $\boldsymbol{\ell} = 0\phi^2(\boldsymbol{\ell})$, *and* $\boldsymbol{m} = 1\phi^2(\boldsymbol{m})$.

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This allows us to calculate arbitrarily long prefixes of $\boldsymbol{\ell}$ and $\boldsymbol{m}$. For example, $\boldsymbol{\ell}$ begins with 0, hence with $0\phi^2(0) = 0010$, hence with $0\phi^2(0010) = 001001001010$.

Corollary

Every factor of $\mathbf{f}$ is a factor of $\boldsymbol{\ell}$, but there are infinitely many factors of $\boldsymbol{\ell}$ which are not factors of $\mathbf{f}$. Word $\boldsymbol{\ell}$ is not a fixed point of a binary morphism.

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Theorem

If $\mathbf{w}$ is an infinite δ-good word, then $\mathbf{w} = a\delta(\mathbf{u})$, for some δ-good word $\mathbf{u}$ where $a \in \{\epsilon, 0, 1\}$.

Fife's Theorem for δ-good words

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Fife's Theorem for δ-good words

Suppose $w \in \{0,1\}^*$ has a suffix $\delta^n(01)$, $n \geq 0$, where n is as large as possible. Write $w = y\delta^n(01)$. Define mappings α, β and γ on w by

$$\alpha(w) = y\delta^{n+1}(01)$$
$$\beta(w) = y\delta^{n+1}(001)$$
$$\gamma(w) = y\delta^{n+1}(0001).$$

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Each of these has a w as a prefix, and a suffix $\delta^{n+1}(01)$.

The δ-good words

An example

Let $w = 00\ 0100\ 0101$. Here $y = 00$, $n = 2$, $\delta^n(0) = 0100$, $\delta^n(1) = 0101$, so that

$$\alpha(w) = 00\ 0100\ 0101\ 0100\ 0100$$
$$\beta(w) = 00\ 0100\ 0101\ 0100\ 0101\ 0100\ 0100$$
$$\gamma(w) = 00\ 0100\ 0101\ 0100\ 0101\ 0100\ 0101\ 0100\ 0100.$$

Fife's Theorem for δ-good words

Suppose $u \in \{\alpha, \beta, \gamma\}^*$, $u = u_1 u_2 \cdots u_n$, $u_i \in \{\alpha, \beta, \gamma\}$. We define

$$01 \bullet u = u_n(u_{n-1}(\cdots(u_2(u_1(01))\cdots)).$$

For an infinite sequence $\boldsymbol{u}$ over $\{\alpha, \beta, \gamma\}$, $\boldsymbol{u} = u_1 u_2 \cdots$, we define $01 \bullet \boldsymbol{u}$ to be the binary sequence having each $01 \bullet u_1 u_2 \cdots u_n$ as a prefix.

Fife's Theorem for δ-good words

Example

$$\begin{aligned}
01 \bullet \alpha\beta\gamma =& (01 \bullet \alpha)\beta\gamma \\
=& (\delta^0(01) \bullet \alpha)\beta\gamma \\
=& \delta^1(01) \bullet \beta\gamma \\
=& (\delta^1(01)) \bullet \beta)\gamma \\
=& \delta^2(001) \bullet \gamma \\
=& (\delta^2(0)\delta^2(01)) \bullet \gamma \\
=& (\delta^2(0)\delta^3(0001)) \\
=& \delta(\delta(0\delta(0001))) \\
=& \delta(\delta(001010100)) \\
=& \delta(010100010001000101) \\
=& 0100010001010100010101000101010001000
\end{aligned}$$

Fife's Theorem for δ-good words

Theorem

The infinite δ-good words starting with 01 *are precisely the words* $01 \bullet \boldsymbol{u}$, *where* $\boldsymbol{u}$ *can be walked on this automaton:*

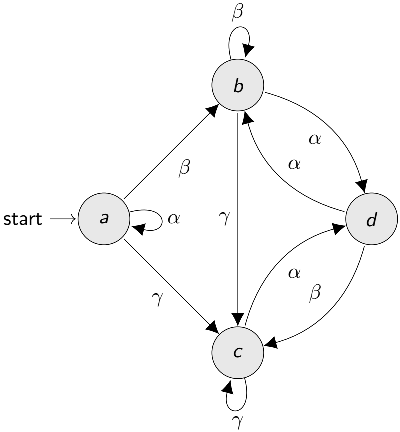

Figure: 'Fife' automaton for δ-good words

Ingredients for Fife's Theorem

Let G be the set of one-sided infinite δ-good words. Let G_u stand for those starting with finite word u.

Lemma

Let $\boldsymbol{w}$ be a one-sided infinite binary word.

(a) $\delta(\boldsymbol{w}) \in G \iff \boldsymbol{w} \in G$;

(b) $1\delta(\boldsymbol{w}) \in G \iff 0\boldsymbol{w} \in G$;

(c) $0\delta(\boldsymbol{w}) \in G \iff (1\boldsymbol{w} \in G)$ *or* $(\boldsymbol{w} \in G_{001})$.

Let $W = \{\boldsymbol{f} \in \{\alpha, \beta, \gamma\}^\omega : 01 \bullet \boldsymbol{f} \in G\}$.

Ingredients for Fife's Theorem

Let $u \in \{\alpha, \beta, \gamma\}^k$ and let $\boldsymbol{f}$ be an infinite word over $\{\alpha, \beta, \gamma\}$ such that $01 \bullet \boldsymbol{f} = \boldsymbol{x}$. Then

$$01 \bullet u\boldsymbol{f} = (01 \bullet u)\delta^k(01)^{-1}\delta^k(\boldsymbol{x}).$$

Ingredients for Fife's Theorem

Getting the automaton in our theorem means proving identities such as $(\beta\gamma)^{-1}W = \gamma^{-1}W$. However,

$$\begin{aligned}
\beta\gamma\boldsymbol{f} \in W &\iff 01 \bullet \beta\gamma\boldsymbol{f} \in G \\
&\iff (01 \bullet \beta\gamma)\delta^2(01)^{-1}\delta^2(\boldsymbol{x}) \in G \\
&\iff 0101000100\ \cancel{01000101(01000101)^{-1}}\delta^2(\boldsymbol{x}) \in G \\
&\iff \delta(0\delta(00\boldsymbol{x})) \in G \\
&\iff 0\delta(00\boldsymbol{x}) \in G \\
&\iff 100\boldsymbol{x} \in G \text{ or } 00\boldsymbol{x} \in G_{001} \\
&\iff 00\boldsymbol{x} \in G.
\end{aligned}$$

Similarly, we calculate that

$$\gamma\boldsymbol{f} \in W \iff 00\boldsymbol{x} \in G.$$

Patterns in $\boldsymbol{d}$

Lemma

Any factor $0u$ *of* $\boldsymbol{d}$ *can be written as* $\phi(p)$ *for some word* p *. Any factor* $u0$ *of* $\boldsymbol{d}$ *can be written as* $\phi^R(p)$ *for some word* p *where* $\phi^R = [10, 0]$*. Word* $\boldsymbol{d}$ *thus has an inverse image under each of* ϕ *and* ϕ^R*.*

Patterns in $\boldsymbol{d}$

Theorem

Word p is a binary pattern encountered by $\boldsymbol{d}$ *if and only if one of the following holds:*

1. *One of p and* $\overline{p}$ *is a factor of* $\boldsymbol{d}$, $\phi_1^{-1}(\boldsymbol{d})$, *or* $(\phi^R)^{-1}(\boldsymbol{d})$.
2. *One of p and* $\overline{p}$ *is among* 0010100, 01001001000, 00100100100, 001001001000, 00010010010, 000100100100, 0010001000100, 00100010001000, 00010001000100, *and* 000100010001000.

The two possibilities are distinct.