String attractors and pseudopalindromic closures

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Program

Program

String attractor

Let $v = v_0 v_1 \cdots v_n$ be a word and let $z = v_i v_{i+1} \cdots v_j$ be its factor. Then $\{i, i+1, \ldots, j\}$ is an *occurrence* of z in v.

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Definition (Kempa & Prezza, 2018)

Consider a finite word $v = v_0 v_1 \cdots v_n$, where v_i are letters. Then $\Gamma \subset \{0, 1, \ldots, n\}$ is a *(string) attractor* of v if each factor of v has an occurrence containing an element of Γ.

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Example

$\Gamma_1 = \{0, 1, 2, 5\}$: $v = \underline{012}01\underline{3}012$.
$\Gamma_2 = \{3, 4, 5, 8\}$: $v = 012\underline{013}01\underline{2}$.

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Remark

The size of minimal attractor is not monotone.
$1\underline{1}0\underline{10}0$ *vs* $1\underline{1}010\underline{0}10$

Attractors in CoW

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In CoW, attractors of minimum size have been determined:

- for particular prefixes:
  - of standard Sturmian sequences by Mantaci, Restivo, Romana, Rosone, Sciortino, 2021
  - of the Thue-Morse sequence by Kutsukake et al., 2020
- for prefixes:
  - of standard Sturmian sequences by Restivo, Romana, Sciortino, 2022
  - of the Tribonacci sequence by Schaeffer & Shallit, 2021
  - of the Thue-Morse sequence by Schaeffer & Shallit, 2021
  - of the period-doubling sequence by Schaeffer & Shallit, 2021
  - of the powers of two sequence by Schaeffer & Shallit, 2021
- for factors:
  - of the Thue-Morse sequence by Dolce, 2023

Attractors in CoW

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- Romana: String Attractor: a Combinatorial Object from Data Compression, October 2022

Program

Reversal and exchange antimorphism

- $R : \{0, 1, \ldots, \mathtt{d} - 1\}^* \to \{0, 1, \ldots, \mathtt{d} - 1\}^*$, called *reversal*, is defined by $R(w_0 w_1 \cdots w_{n-1}) = w_{n-1} \cdots w_1 w_0$.

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- $E : \{0, 1\}^* \rightarrow \{0, 1\}^*$, called *exchange antimorphism*, is defined by $E(w_0 w_1 \cdots w_{n-1}) = \overline{w_{n-1}} \cdots \overline{w_1}\ \overline{w_0}$.

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- A word w is a *palindrome* if $w = R(w)$ and w is an *antipalindrome* if $w = E(w)$. A word is a *pseudopalindrome* if it is a palindrome or an antipalindrome.

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Example

$R(0110110) = 0110110$ and $E(01001101) = 01001101$

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Definition (Droubay, Justin, Pirillo, 2001)

Let $\Delta = \delta_0\delta_1\delta_2\cdots$ be a sequence of letters and define $u_0 = \varepsilon$ and $u_{n+1} = (u_n\delta_n)^R$ for all $n \in \mathbb{N}$. Then we denote $\mathbf{u}(\Delta) = \lim_{n\to\infty} u_n$.

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Example

Let $\Delta = (01)^\omega$. Then $\mathbf{u}(\Delta)$ is the Fibonacci sequence. The first six prefixes of $\mathbf{u}(\Delta)$ read: $u_0 = \varepsilon$, $u_1 = 0$, $u_2 = 010$, $u_3 = 010010$, $u_4 = 01001010010$, $u_5 = 0100101001001010010$.

Episturmian sequences

Definition

Let $\mathbf{u}$ be a sequence whose language is closed under reversal and such that for each length n it contains at most one left special factor. Then $\mathbf{u}$ is called an *episturmian sequence*. An episturmian sequence is *standard* if all left special factors are prefixes.

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Theorem (Droubay, Justin, Pirillo, 2001)

Let $\mathbf{u}$ *be a standard episturmian sequence over* $\mathcal{A}$. *Then* $\mathbf{u} = \mathbf{u}(\Delta)$ *for a unique sequence* $\Delta = \delta_0\delta_1\delta_2\cdots$ *with* $\delta_i \in \mathcal{A}$.

Attractors of episturmian sequences

Theorem (D., 2022)

Let v be a non-empty palindromic prefix of a standard episturmian sequence. For every letter a occurring in v, denote

$$m_a = \max\{|p| : p \textit{ is a palindrome and } pa \textit{ is a prefix of } v\}.$$

Then $\Gamma = \{m_a : a \text{ occurs in } v\}$ is a minimal attractor of v.

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$$\begin{aligned}
u_1 &= \underline{0} \\
u_2 &= \underline{0}\underline{1}0 \\
u_3 &= \underline{0}\underline{1}0\underline{2}010 \\
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\end{aligned}$$

Attractors of Sturmian sequences

For a standard Sturmian sequence $\mathbf{u}(\Delta)$, a minimal attractor of u_n:
$\Gamma = \cup_{a \text{ letter in } u_n} \max\{|p| : p \text{ palindrome}, \ pa \text{ prefix of } u_n\}$.

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Proof.

by mathematical induction on n, WLOG $\Delta = 0\cdots$, $u_1 = \underline{0}$ and for $n \geq 2$, $u_n = (u_{n-1}a)^R$ has three possible forms:

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Attractors of episturmian sequences

Theorem (D., 2022)

Let $\mathbf{u}$ *be an episturmian sequence. Each factor of* $\mathbf{u}$ *containing* d *distinct letters has an attractor of size* d*.*

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Definition (de Luca, De Luca, 2006)

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Attractors of pseudostandard sequences

Theorem (D., Hendychová, 2023)

Let v be a non-empty antipalindromic prefix of a pseudostandard sequence starting with the letter 0*. Denote*

$$m_a = \max\{|q| : q \text{ is an antipalindrome and } qa \text{ is a prefix of } v\}.$$

Then $\Gamma = \{m_0, m_1, |v| - m_1 - 1\}$ is an attractor of v.
Γ is minimal (with only minor exceptions).

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Example

Consider $\Delta = 01001\cdots$.

$$\begin{aligned}
u_1 &= \underline{01} \\
u_2 &= \underline{0}1\underline{1}\underline{0}01 \\
u_3 &= 01\underline{1}001\underline{0}11\underline{0}01 \\
u_4 &= 01\underline{1}00101100\underline{1}0\underline{1}1001 \\
u_5 &= 011001011001\underline{0}11\underline{0}01\underline{1}001011001011001\,.
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Example

Let $\Delta = 01^{\omega}$ and $\Theta = (RE)^{\omega}$, then $\mathbf{u} = \mathbf{u}(\Delta, \Theta)$ is the Thue-Morse sequence. The first six prefixes of $\mathbf{u}$ read:

$u_0 = \varepsilon$, $u_1 = 0$, $u_2 = 01$, $u_3 = 0110$,

$u_4 = 01101001$, $u_5 = 0110100110010110$.

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- factors of the Thue-Morse sequence have minimal attractors of size ≤ 5 (Dolce, 2023)
- pseudopalindromic prefixes of complementary-symmetric Rote sequences have minimal attractors of size 2 (D., Hendrychová, 2023)

Rote sequences

Definition

Complementary-symmetric (CS) Rote sequences are binary sequences having complexity $2n$ and such that their language is closed under the letter exchange.

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Let $u = u_0 u_1 \cdots u_{n-1}$, $u_i \in \{0, 1\}$. Then $S(u)$ is defined by

$$S(u)_i = (u_{i+1} + u_i) \bmod 2 \quad \text{for } i = 0, 1, \ldots, n-2.$$

For example, if $u = 0011010$, then $S(u) = 010111$.

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Theorem (Rote, 1994)

A binary sequence $\mathbf{u}$ *is a CS Rote sequence if and only if the sequence* $S(\mathbf{u})$ *is a Sturmian sequence.*

Rote and Sturmian sequences

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Remark

There is no straightforward relation with known atractors in Sturmian sequences. Let $\Delta = 0100\cdots$. *Then* $u = 010010010$ *is a prefix of* $\mathbf{u}(\Delta)$ *and* $w = 0011100011$ *is the prefix of the corresponding Rote sequence, i.e.,* $S(w) = u$.

"Prague attractor": $u = 0\underline{1}0010\underline{0}10$

"Palermo attractor": $u = 010010\underline{01}0$.

The factor 10 *has a unique occurrence in* w*, therefore each attractor of* w *has to contain either the position* 4 *or* 5.

Rote and generalized pseudostandard sequences

Each CS Rote sequence is a generalized pseudostandard sequence.

Theorem (Blondin-Massé et al., 2013)

Let (Δ, Θ) *be a directive bi-sequence. Then* $\mathbf{u} = \mathbf{u}(\Delta, \Theta)$ *is a standard CS Rote sequence if and only if* $\mathbf{u}$ *is aperiodic and no factor of length two of the directive bi-sequence is in the following sets:*

$$\{(ab, EE) : a, b \in \{0,1\}\},$$
$$\{(a\bar{a}, RR) : a \in \{0,1\}\},$$
$$\{(aa, RE) : a \in \{0,1\}\}.$$

Attractors of Rote sequences

Theorem (D., Hendrychová, 2023)

Assume (Δ, Θ) is the directive bi-sequence of a standard CS Rote sequence $\mathbf{u}$ and u_n contains both letters.

1. *If $u_n = E(u_n)$, $\delta_{n-1} = a$, and u_i is the longest antipalindromic prefix of u_n followed by $\overline{a}$, then $\Gamma = \{|u_i|, |u_{n-1}|\}$ is an attractor of u_n.*
2. *If $u_n = R(u_n)$, $\delta_{n-1} = a$, $u_{n-1} = E(u_{n-1})$, and u_j is the longest palindromic prefix of u_n followed by $\overline{a}$, then $\Gamma = \{|u_j|, |u_{n-1}|\}$ is an attractor of u_n.*
3. *If $u_n = R(u_n)$, $u_{n-1} = R(u_{n-1})$, then the attractor of u_n equals the attractor of u_{n-1}.*

Attractors of Rote sequences

Example

Let (Δ, Θ) start in $(00111, RRERR)$

$$u_1 = 0$$
$$u_2 = 00$$
$$u_3 = \underline{0}0\underline{1}1$$

If $u_n = E(u_n)$, $\delta_{n-1} = a$, and u_i is the longest antipalindromic prefix of u_n followed by $\overline{a}$, then $\Gamma = \{|u_i|, |u_{n-1}|\}$. Here $n = 3$, $a = 1$ and $u_i = \varepsilon$.

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If $u_n = R(u_n)$, $\delta_{n-1} = a$, $u_{n-1} = E(u_{n-1})$, and u_j is the longest palindromic prefix of u_n followed by $\overline{a}$, then $\Gamma = \{|u_j|, |u_{n-1}|\}$ is an attractor of u_n. Here $n = 4$, $a = 1$ and $u_j = 0 = u_1$.

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u_3 &= \underline{0}0\underline{1}1 \\
u_4 &= 0\underline{0}11\underline{1}00 \\
u_5 &= 0\underline{0}11\underline{1}0011100
\end{aligned}
$$

If $u_n = R(u_n)$, $u_{n-1} = R(u_{n-1})$, then the attractor of u_n is the same as of u_{n-1}.

Program

- Find minimal attractors of pseudopalindromic prefixes of all generalized pseudostandard sequences. We believe that attractors of size 4 suffice for pseupalindromic prefixes.

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- What about generalized pseudostandard sequences over larger alphabets?

Thank you for attention!