Université Paris Cité

Density of Rational Languages Under Invariant Measures

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One-World Combinatorics on Words Seminar

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FONDAMENTALE

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1. Densities

2. Skew Product

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3. Coloring

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Part 1

Densities

Notation

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(v) Measures μ on $A^{\mathbb{Z}}$ are viewed as maps on words,

$$\mu(u) = \mu(\{x \in A^{\mathbb{Z}} \mid x_{[0,|u|)} = u\}),$$

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(vi) A *shift space* is a closed, shift invariant subset of $A^{\mathbb{Z}}$.

(vii) For shift spaces, *minimal* means *minimal for inclusion*.

Density

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$$\delta_\mu(L) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(L \cap A^i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(\{x \in A^{\mathbb{Z}} \mid x_{[0,i)} \in L\}).$$

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(ii) $\delta_\mu(L \cup K) = \delta_\mu(L) + \delta_\mu(K)$ when $L \cap K = \emptyset$.

On 1 letter

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In other words *rational subsets of* $\mathbb{N}$ *are evenly distributed.*

When is a rational language evenly distributed in a minimal shift space?

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When does the density depend *only* on the syntactic monoid?

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Theorem (Michel, 1974)

Every primitive substitution has *exactly one* invariant probability measure.

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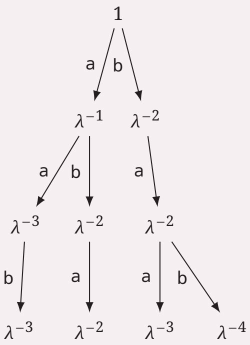
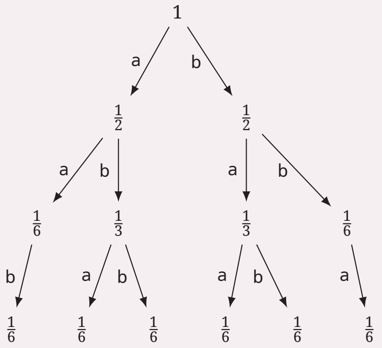
By Michel's theorem, shift spaces defined by primitive substitutions are uniquely ergodic.

Invariant measures



Fibonacci ($\lambda^2 = \lambda + 1$)



Thue–Morse

Values of ergodic measures on cylinders.

Examples

$$\varphi\colon \{\mathsf{a},\mathsf{b}\}^* \to \mathbb{Z}/2\mathbb{Z},\ \mathsf{a} \mapsto 1,\ \mathsf{b} \mapsto 0, \quad L = \varphi^{-1}(0).$$

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For the invariant measure of X, $\delta_\mu(L) = 5/9$.
The parity of occurrences of $\mathsf{a} + \mathsf{c}$ *is not evenly distributed.*

Part 2

Skew Product

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Let $\varphi\colon A^* \to G$ be a morphism onto a finite group and X be a minimal shift space.

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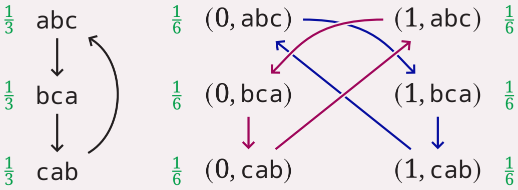

(X,S) $\qquad$ $(\mathbb{Z}/2\mathbb{Z}\times_\varphi X,T_\varphi)$

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Theorem 3

When X is dendric all skew products are ergodic.
This includes in particular all Sturmian shifts.

Ergodic sums in skew product

$$\varphi(x_{[0,3)}) = g$$

$$\cdots \quad x_{-1} \quad \boxed{x_0 \quad x_1 \quad x_2} \quad x_3 \quad \cdots$$

A cylinder in $\varphi^{-1}(g) \cap A^3$.

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Non-minimal example

$$\sigma\colon \mathsf{a} \mapsto \mathsf{ab},\ \mathsf{b} \mapsto \mathsf{ba}, \quad \varphi\colon \{\mathsf{a},\mathsf{b}\}^* \to \mathbb{Z}/2\mathbb{Z},\ \mathsf{a} \mapsto 1,\ \mathsf{b} \mapsto 0.$$

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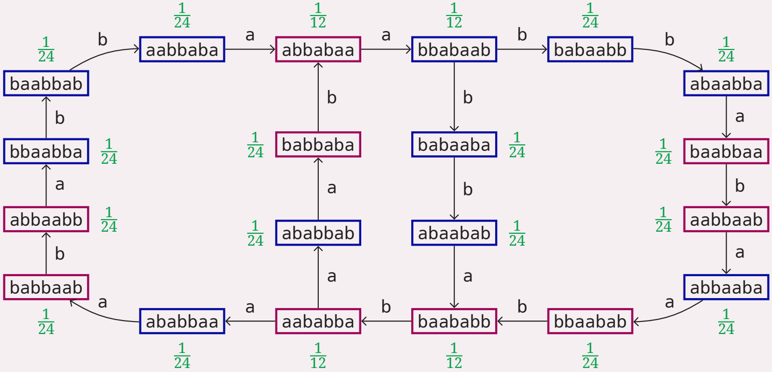

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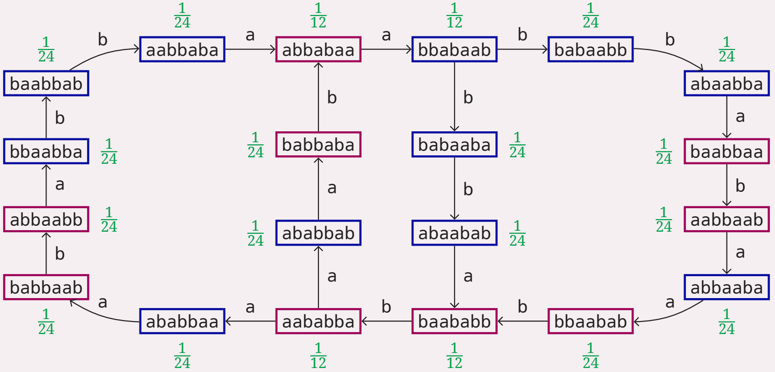

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$\{(g,x) \mid \alpha(x) = g\}$ is a closed invariant subspace of the skew product.

Part 3

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Lemma

A coloring α defines a closed invariant subspace of $G \times_\varphi X$,

$$Y_\alpha = \{(g, x) \mid \alpha(x) = Hg\}.$$

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Morphisms $\varphi\colon A^* \to G$ define cocycles.

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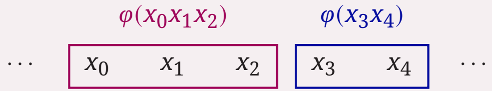

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(iii) Smaller subgroups give stronger coboundary conditions.

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Lemma 3

All *minimal* closed invariant subspaces are of the form Y_α.

Minimality conditions

Let X be a minimal shift space with an invariant measure μ.
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Non-commutative coloring

$$\sigma\colon \mathsf{a} \mapsto \mathsf{aab},\ \mathsf{b} \mapsto \mathsf{acb},\ \mathsf{c} \mapsto \mathsf{ba}, \quad G = \mathrm{Perm}(\{1,2,3\}),$$
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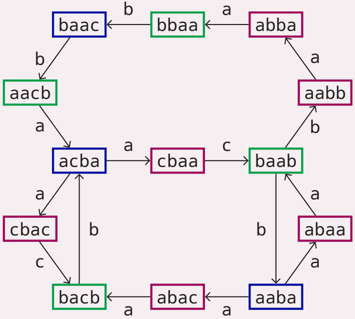
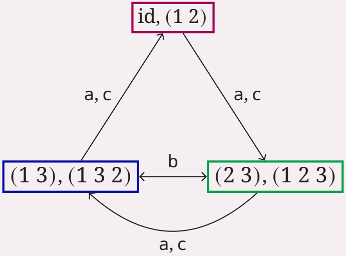


Coloring mod {id, (1 2)}



Right cosets of {id, (1 2)}

Part 4

Return Words

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Example

$$X = X_\sigma, \quad \sigma\colon \mathrm{a} \mapsto \mathrm{ab},\ \mathrm{b} \mapsto \mathrm{ba}.$$

$$\cdots \boxed{\mathrm{a\ b\ b}}\ \boxed{\mathrm{a\ b\ a}}\ \boxed{\mathrm{a\ b\ b\ a}}\ \boxed{\mathrm{a\ b}}\ \boxed{\mathrm{a\ b\ b}}\ \mathrm{a\ b} \cdots$$

$$\mathcal{R}_X(\mathrm{ab}) = \{\mathrm{abb}, \mathrm{aba}, \mathrm{abba}, \mathrm{ab}\}.$$

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Let $\alpha\colon X \to H\backslash G$ be a coloring. There exists $n \in \mathbb{N}$ such that for every $u \in L(X)$ with $|u| \geq n$, $\varphi(\mathcal{R}_X(u)) \subseteq g^{-1}Hg$ for some $g \in G$.

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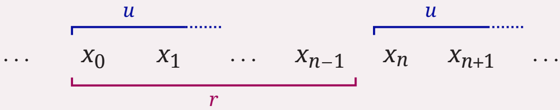

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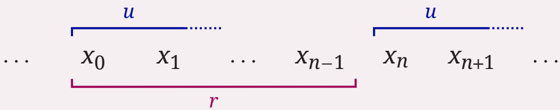

$$\alpha(x) = \alpha(S^n x) = \alpha(x)\varphi^{(n)}(x) = \alpha(x)\varphi(r).$$

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ℹ The *Return Theorem* of Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, & Rindone (2015) states that this is the case for all dendric languages.

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6. What about general formulas for densities when the skew product is *not* ergodic?

References (i)

- Berthé, V. et al. (2015). Acyclic, connected and tree sets. *Monatsh. Math.* 176.4.
- Conze, J. P. (1976). Équirépartition et ergodicité de transformations cylindriques. *Publ. Séminaire Math. Info. Renne* 2.
- Furstenberg, H. (1961). Strict ergodicity and transformation of the torus. *Am. J. Math.* 83.4.
- Krylov, N. & N. Bogolioubov (1937). La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire. *Ann. Math.* 38.1.
- Lemańczyk, M. & M. K. Mentzen (2002). Topological ergodicity of real cocycles over minimal rotations. *Monatsh. Math.* 134.3.

References (ii)

- Mackey, G. W. (1966). Ergodic theory and virtual groups. *Math. Ann.* 166.3.
- Michel, P. (1974). Stricte ergodicité d'ensembles minimaux de substitution. *C. R. Acad. Sci. Paris* 278.
- Schmidt, K. (1977). Cocycles of ergodic transformation groups. MacMillan India.
- Veech, W. A. (1975). Finite group extensions of irrational rotations. *Israel J. Math.* 21.2.
- Zimmer, R. J. (1976). Extensions of ergodic group actions. *Illinois J. Math.* 20.3.