Synchronizing automatic sequences along Piatetski-Shapiro sequences

Arithmetic subword complexity of automatic sequences - part I

Clemens Müllner

TU Wien

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Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad t_{22} = 1$$

$$(t(n))_{n \geq 0} = 0110100110010110100101100110 1001 \dots$$

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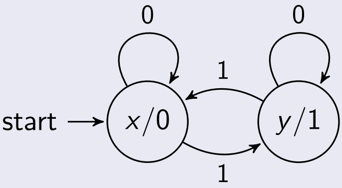

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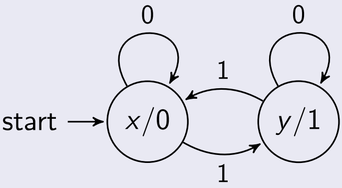

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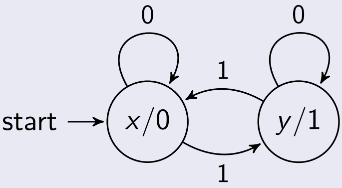

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Different Points of View I

$$(a(n))_{n\geq 0} = 01101001100101101001011001101001\ldots$$

Automaton (Computer Science)

Substitution (Dynamics)

Coding of the fixpoint of a substitution:

$$x \to xy \qquad x \mapsto 0$$
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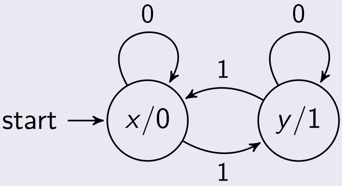

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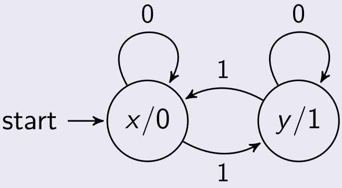

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Formal Power Series (Algebra)

Algebraicity over $\mathbf{F}_q(X)$.

$$t(X) := \sum_{n\geq 0} a(n)X^n$$

$$X + (1+X)^2 t(X) + (1+X)^3 t(X)^2 = 0$$

Finite Kernel

The *k-kernel* of a sequence $a(n)$ is defined as

$$\{(a(nk^\lambda + r))_{n\geq 0} : \lambda \geq 0, 0 \leq r < k^\lambda\}.$$

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- Relatively easy to define (structured).
- Complex enough that interesting phenomena appear.
- Every subsequence $(a(xn + y))_{n \geq 0}$ along an arithmetic progression of an automatic sequence $\mathbf{a}$ is again automatic.

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Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = x \quad \forall q.$

Example

$\mathbf{w}_0 = 010.$

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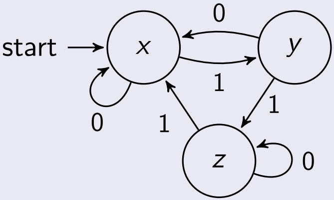

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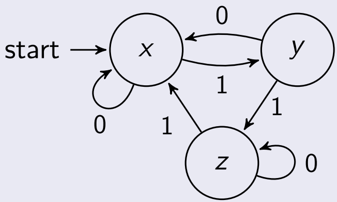

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Synchronizing Automata

Key Property

- Any word $\mathbf{w}$ containing $\mathbf{w}_0$ is also synchronizing.
- Most words are synchronizing.
- $\mathbf{a}$ can be approximated by periodic sequences:
  Let λ be large. Most words of length λ are synchronizing.
  $a(n) = a(n \bmod k^\lambda)$ if $n \bmod k^\lambda$ is synchronizing.

"Usual" Strategy

- Understand the problem for periodic sequences.
- Transfer the result to synchronizing automatic sequences.

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Subword Complexity

Let $\mathcal{A}$ be a finite alphabet and $\mathbf{u} = (u(n))_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$.

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The subword complexity of a sequence $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ is defined by

$$p_{\mathbf{u}}(L) := \#\{\mathbf{w} \in \mathcal{A}^L : \exists k, (u(k), \ldots, u(k+L-1)) = \mathbf{w}\}.$$

$$p_{\mathbf{u}}(L) \leq |\mathcal{A}|^L$$

Subword complexity of automatic sequences

Let $\mathbf{a}$ be an automatic sequence. Then there exists $C > 0$ such that for all $L \in \mathbb{N}$

$$p_{\mathbf{a}}(L) \leq C \cdot L.$$

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Theorem (Avgustinovich, Fon-Der-Flaass and Frid; 2003)

- A certain class of invertible automatic sequences has maximal arithmetic subword complexity. (E.g. Thue-Morse sequence)
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Main Result

Definition (polynomial subword complexity)

Let $\mathbf{u}$ be a sequence over a finite alphabet $\mathcal{A}$.

$$p_{\mathbf{u}}^{\leq d}(L) := \#\{\mathbf{w} \in \mathcal{A}^L : \exists P \in \mathbb{Z}[x], P(\mathbb{N}) \subseteq \mathbb{N}, \deg P \leq d : u(P(i)) = \mathbf{w}(i) \text{ for } i = 0, \ldots, L-1\}.$$

Theorem 1 (Deshouillers, Drmota, M., Shubin, Spiegelhofer; 2024+)

Let $a(n)$ be a synchronizing automatic sequence. Then for any $d \geq 1$

$$p_{\mathbf{a}}^{\leq d}(L) \leq \exp(o(L)).$$

Basically the same proof: there exist $c > 0, \eta > 0$ such that

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Consequences of Theorem 1

Theorem 2 (Deshouillers, Drmota, M., Shubin, Spiegelhofer; 2024+)

Let $a(n)$ be a synchronizing automatic sequence. Then for any $c > 0$ the subword complexity of $a(\lfloor n^c \rfloor)$ grows sub-exponentially ($\exp(o(L))$).

Remark: The same result holds for any function f with "nice" derivatives.

Theorem 3 (Konieczny, M., 2024+)

Theorem 1 can be used to give rather sharp upper bounds for $p_{\mathbf{a}}^{\leq d}$ for general automatic sequences $\mathbf{a}$.

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Background to Theorem 1 and 2

Theorem (Drmota, Mauduit, Rivat; 2019)

Let **t** denote the Thue-Morse sequence. Then

$$p_{t(n^2)}(L) = 2^L.$$

Actually, the Thue-Morse sequence is normal along the squares.

Theorem (M., 2018)

The same also holds for block-additive functions modulo m instead of the Thue-Morse sequence.

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Let **t** denote the Thue-Morse sequence and let $1 < c < \frac{3}{2}$. Then $(t(\lfloor n^c \rfloor))_{n\in\mathbb{N}}$ is normal.

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Proof of Theorem 1

Naive approach

- Let f be a m-periodic function. Then $p_f^{\leq d}(L) \leq m^{d+1}$.
- Approximate $a(n)$ by a k^λ-periodic function $f(n)$.
- $a(n)$ and $f(n)$ agree on most residue classes modulo k^λ.
- Problem: P can hit the "bad" residue classes very often. (Trivial example: $P(x) = k^\lambda x + r$.)

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Reductions

We study $(a(P(n)), a(P(n+1)), \ldots, a(P(n+L-1)))$.

- Let $Q(\ell) = P(n+\ell)$. Study $(a(Q(0)), a(Q(1)), \ldots, a(Q(L-1)))$.
- Avoid trivial problems: $Q(\ell) = k^{\lambda_0}\left(z'_d\ell^d + \ldots + z'_1\ell + z'_0\right) + r$.
  Using the kernel: $\exists b_i \in Ker_k(a)$ with $b_i(n) = a(nk^{\lambda_0} + r)$.
  $a(Q(\ell)) = b_i(z'_d\ell^d + \ldots + z'_1\ell + z'_0) = b_i(Q'(\ell))$.
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  $a(Q(\ell)) = b_i(z_d'\ell^d + \ldots + z_1'\ell + z_0') = b_i(Q'(\ell))$.
- Remains to study $b_i(Q'(\ell))$ where some z_i' $(i \geq 1)$ is not divisible by k.

Example

Consider $Q'(\ell) = 5 \cdot 3^4 \cdot \ell$ modulo 6^4.

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(Q'(0))_6 &= 00000 & (Q'(8))_6 &= 23000\\
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(Q'(4))_6 &= 11300 & (Q'(12))_6 &= 34300\\
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- Problem: We still only hit few residue classes modulo k^λ. (E.g. $Q'(\ell) = 5 \cdot 3^\lambda \cdot \ell \bmod 6^\lambda$.)
- "low" digits: $Q'(\ell)$ might still not equidistribute $\bmod k^\lambda$.
- "high" digits work: $\exists \varepsilon(k) > 0$ such that for any $\mathbf{w} \in \mathcal{A}^{\varepsilon\lambda}$ we have $\#\{\ell < k^\lambda : (Q'(\ell) \bmod k^\lambda)_k \text{ starts with } \mathbf{w}\} \approx k^{\lambda(1-\varepsilon)}$.

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Equidistribution of high digits

- Detection of digits: The digits of ℓ in base k between positions μ and λ coincide with the digits of $m < k^{\lambda-\mu}$ iff

$$\left\{\frac{\ell}{k^\lambda}\right\} \in \left[\frac{m}{k^{\lambda-\mu}}, \frac{m+1}{k^{\lambda-\mu}}\right).$$

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$$\sum_{\ell<k^\lambda} \mathbf{1}_{[\{\frac{Q'(\ell)}{k^\lambda}\}\in[\frac{m}{k^{\lambda-\mu}},\frac{m+1}{k^{\lambda-\mu}})]} \approx \sum_{|h|<H} c_h \sum_{\ell<k^\lambda} \mathrm{e}\left(\frac{h\cdot Q'(\ell)}{k^\lambda}\right).$$

- Use classical estimates for

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- Approximate $b_i(n)$ with a k^λ-periodic function $f(n)$.
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Proof of Theorem 2

Theorem 2 (Deshouillers, Drmota, M., Shubin, Spiegelhofer; 2024+)

Let $a(n)$ be a synchronizing automatic sequence. Then for any $c > 0$ the subword complexity of $a(\lfloor n^c \rfloor)$ grows sub-exponentially $(\exp(o(L)))$.

Connection to Theorem 1

We use Taylor expansion to write

$$(n+\ell)^c = \underbrace{\sum_{t=0}^{d} A_t^{(n)} \ell^t}_{P^{(n)}(\ell)} + g_\ell^{(n)}.$$

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The gray area corresponds to the intersection of the strips $0.5 < x_0 < 1$, $0.5 < x_0 + x_1 < 1$ and $1 < x_0 + 2x_1 < 1.5$.

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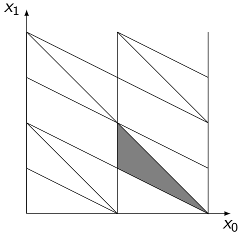

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- There are at most $\sum_{i=0}^{d+1} \binom{mL^{d+2}}{i} \ll_d m^{d+1} L^{(d+1)(d+2)}$ regions.
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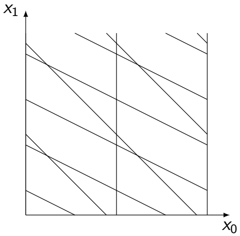

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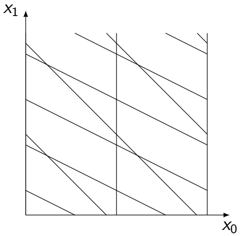

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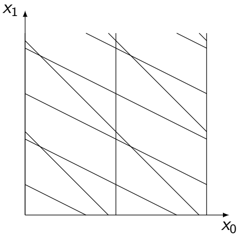

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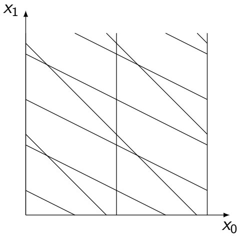

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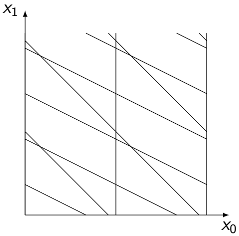

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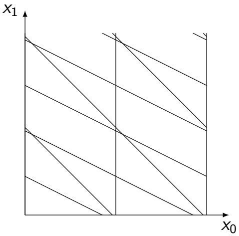

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Shifted hyperplanes

- The picture does not change qualitatively!
- Proof needs some special properties of the error term $g_\ell^{(n)}$

Proposition

There exists c_d such that for any m-periodic function f we have $p_{f(\lfloor n^c \rfloor)}(L) \ll_d m^{d+1} L^{c_d}$.

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Lemma (original idea due to Weyl)

Suppose $h : \mathbb{Z} \to \mathbb{R}$ is a polynomial

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2. There exists $1 \leq s \ll \delta^{-O_d(1)}$

$$\sup_{1 \leq j \leq d} L^j \|s\beta_j\| \ll \delta^{-O_d(1)}.$$

Proof of Theorem 2 (general case)

$a(\lfloor (n+\ell)^c \rfloor) = a(\lfloor P^{(n)}(\ell) + g_\ell^{(n)} \rfloor)$
Strategy: Approximate $a(n)$ by a k^λ-periodic function $f(n)$.

Lemma (original idea due to Weyl)

Suppose $h : \mathbb{Z} \to \mathbb{R}$ is a polynomial

$$h(\ell) = \beta_0 + \ell\beta_1 + \ldots + \ell^d \beta_d.$$

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At least one of the following holds

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- $g_\ell^{(n)}$ is small.
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- This happens rarely.
- $p_{f(\lfloor (n+\ell)^c \rfloor)}(L) \ll k^{(d+1)\lambda} L^{c_d}$.
- (Optional: optimize λ as a function of L.)

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- Along arithmetic progressions (with step size s) we approximate $(n+\ell)^c$ with $Q \in \mathbb{Z}[x]$ with small error term.
- Recall: the polynomial subword-complexity grows sub-exponentially.
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- Synchronizing automatic sequences are easier to treat than general (invertible) automatic sequences. (We can treat higher degrees.)
- However, questions about subword complexity are still difficult!

Open Problem

We know $p_a^{\leq d}(L) \ll \exp(c \cdot L^{1-\eta})$.
Is there a better upper bound for $p_a^{\leq d}(L)$ (maybe even polynomial)?

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