Large normalizers of odometers and $\mathbb{Z}^d$-automatic sequences

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Basic dynamical notions

Definition

Let $(X, T, \mathbb{Z}^d)$ be a topological dynamical system, X a compact metric space. An isomorphism $\phi\colon X \to X$ is an homeomorphism s.t. for some $M_\phi \in \mathrm{GL}(d, \mathbb{Z})$

$$\phi \circ T^{\boldsymbol{n}} = T^{M\boldsymbol{n}} \circ \phi, \quad \forall \boldsymbol{n} \in \mathbb{Z}^d.$$

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When $M = \mathrm{Id}$

- an isomorphism is an automorphism.
- $\mathrm{Aut}(X) = \{\phi\colon \forall \boldsymbol{n} \in \mathbb{Z}^d, \phi \circ T^{\boldsymbol{n}} = T^{\boldsymbol{n}} \circ \phi\}$.

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For $d = 1, \quad T : X \to X$
An isomorphism is a flip conjugacy

$$\phi \circ T = T^{\pm} \circ \phi.$$

Isomorphisms are also called extended symmetries.

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$$\mathrm{N}(X) = \{\phi \text{ isomorphism of } (X, T, \mathbb{Z}^d)\}$$

$$\{T^{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^d} = \langle T \rangle \lhd \mathrm{Aut}(X) \lhd \mathrm{N}(X).$$

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$$\mathrm{Aut}(X) = \mathrm{Cent}_{\mathrm{Homeo}(X)}(\langle T\rangle)$$

$$\begin{aligned}\mathrm{N}(X) &= \mathrm{Norm}_{\mathrm{Homeo}(X)}(\langle T\rangle)\\ &= \{\phi \in \mathrm{Homeo}(X); \quad \phi\langle T\rangle\phi^{-1} = \langle T\rangle\}.\end{aligned}$$

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Set the linear representation group or linear symmetries

$$\vec{\mathrm{N}}(X) := \{M \in \mathrm{GL}(d,\mathbb{Z});\ \exists \phi \in \mathrm{N}(X) \text{ with } M_\phi = M\}.$$

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For aperiodic systems the following exact sequence hold:

$$1 \to \quad \mathrm{Aut}(X) \quad \stackrel{i}{\to} \quad \mathrm{N}(X) \quad \stackrel{j}{\to} \quad \vec{\mathrm{N}}(X) \quad \to \quad 1,$$

where

i is the natural injection

$j(\phi) = M$ whenever $M_\phi = M$.

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- Q: What can we say on $\mathrm{N}(X)$, $\vec{\mathrm{N}}(X)$ as a group? Commutative? Amenable? What are the subgroups? the quotients?. When $\mathrm{N}(X) \equiv \mathrm{Aut}(X) \rtimes \vec{\mathrm{N}}(X)$?...
- Q: What do dynamical properties of $(X, T, \mathbb{Z}^d)$ say about properties of $\mathrm{N}(X)$ and vice versa ?

Constant base $\mathbb{Z}^d$-odometer

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$$\mathbb{Z}^d_{(L^n)} = \left\{ (x_n)_{n\geq 0} \in \prod_{n=0}^{\infty} \mathbb{Z}^d / L^n(\mathbb{Z}^d) : x_{n+1} \equiv x_n \mod L^n(\mathbb{Z}^d)\ \forall n \right\}.$$

It is an abelian group: $(x_n)_n + (y_n)_n = (x_n + y_n \mod L^n(\mathbb{Z}^d))_n$

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The action T is aperiodic, minimal and equicontinuous.

Q: What is $\vec{\mathrm{N}}(\mathbb{Z}^d_{(L^n)})$?

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Lemma

$M \in \vec{N}(\mathbb{Z}^d_{(L^n)}) \iff$

$$\forall n > 0, \exists m > 0, \quad L^{-n} M L^m (\mathbb{Z}^d) \subset \mathbb{Z}^d.$$

In particular

$$\bigcup_{k>0} \mathrm{Cent}_{\mathrm{GL}(d,\mathbb{Z})}(L^k) \subset \vec{\mathrm{N}}(\mathbb{Z}^d_{(L^n)}).$$

Similar characterization by Giordano-Putnam-Skau

Fact:

$$\mathrm{Aut}(\overleftarrow{\mathbb{Z}^d}_{(L^n(\mathbb{Z}^d))}) \cong \overleftarrow{\mathbb{Z}^d}_{(L^n(\mathbb{Z}^d))}.$$

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Proposition (C., Petite)

- *The symmetry semigroup* $\vec{N}(\overleftarrow{\mathbb{Z}^d}_{(L^n(\mathbb{Z}^d))})$ *is a group.*
- $N(\overleftarrow{\mathbb{Z}^d}_{(L^n(\mathbb{Z}^d))}) \cong \mathrm{Aut}(\overleftarrow{\mathbb{Z}^d}_{(L^n(\mathbb{Z}^d))}) \rtimes \vec{N}(\overleftarrow{\mathbb{Z}^d}_{(L^n(\mathbb{Z}^d))})$.

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Hence:

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Classification of linear symmetry group of $\mathbb{Z}^2_{(L^n)}$

Examples:

- If $L_1 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$, then $\vec{N}(\overleftarrow{\mathbb{Z}^2}_{(L_1^n(\mathbb{Z}^2))}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

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- If $L_3 = \left(\begin{smallmatrix} 6 & 4 \\ 0 & 2 \end{smallmatrix}\right)$, then

$$\vec{N}(\overleftarrow{\mathbb{Z}^2}_{(L_3^n(\mathbb{Z}^2))}) = \left\{ \left(\begin{smallmatrix} 1-m & -m \\ m & 1+m \end{smallmatrix}\right), \left(\begin{smallmatrix} 1-m & 2-m \\ m & m-1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1-m & -2-m \\ m & m-1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1-m & -m \\ m & m-1 \end{smallmatrix}\right) : m \in \mathbb{Z} \right\}.$$

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In the two-dimensional case, for a constant-base odometer system $\overleftarrow{\mathbb{Z}^2}_{(L^n(\mathbb{Z}^2))}$, the matrices in its symmetry group $\vec{N}(\mathbb{Z}^2_{(L^n(\mathbb{Z}^2))})$ satisfy explicit linear relations with respect to the determinant and the trace of L.

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Open problem

Problem

For higher dimensional constant-base odometer systems $d \geq 3$.

Are the elements of $\vec{\mathrm{N}}(\mathbb{Z}^d_{(L^n)})$ computable?

Is its group structure computable?

Subshift

Let $\mathcal{A}$ be a finite alphabet and consider in $\mathcal{A}^{\mathbb{Z}^d}$ the product topology.
The shift action is defined on $\mathcal{A}^{\mathbb{Z}^d}$ as

$$\forall x \in \mathcal{A}^{\mathbb{Z}^d}, \forall \boldsymbol{n}, \boldsymbol{z} \in \mathbb{Z}^{\boldsymbol{d}}, \quad (S^{\boldsymbol{n}} x)_{\boldsymbol{z}} = x_{\boldsymbol{z}+\boldsymbol{n}}.$$

$X \subset \mathcal{A}^{\mathbb{Z}^d}$, is a subshift if it is closed and S-invariant.

Multidimensionnal constant-shape substitutive subshift

- Let $L \in \mathcal{M}(d, \mathbb{Z})$ be an expansion matrix,
  i.e. L is invertible, $\|L\| > 1$, $\|L^{-1}\| < 1$.
- Let $F \subset \mathbb{Z}^d$ be a fundamental domain of $L(\mathbb{Z}^d)$ in $\mathbb{Z}^d$.
- A substitution map $\xi \colon \mathcal{A} \to \mathcal{A}^F$

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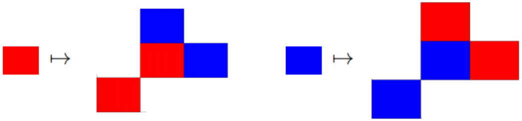
Ex.: $L = 2\mathrm{Id}_{\mathbb{R}^2}$, $F = \{(0,0),(1,0),(0,1),(-1,-1)\}$.



Multidimensionnal constant-shape substitutive subshift

Iterating the substitution map, for $n > 0$

$$\xi^n \colon \mathcal{A} \to \mathcal{A}^{F_n}, \qquad \text{with } F_{n+1} = L(F_n) + F.$$

hypothesis: $(F_n)_n$ is a Følner sequence, i.e. $\forall \boldsymbol{z} \in \mathbb{Z}^d$

$$\lim_{n \to +\infty} \frac{|F_n \Delta (\boldsymbol{z} + F_n)|}{|F_n|} = 0.$$

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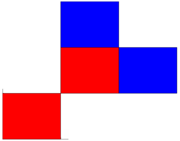

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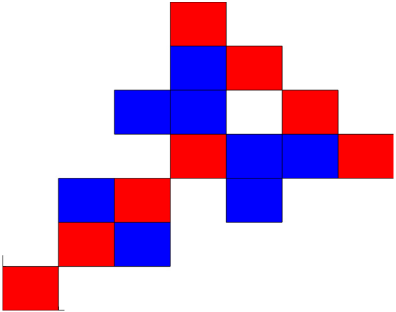

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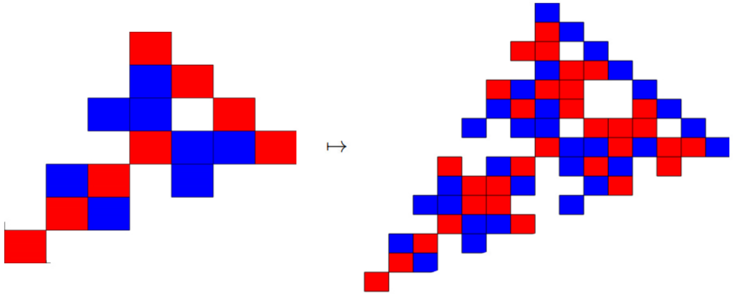

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The substitutive subshift associated to ξ is X_ξ

$$\{x \in \mathcal{A}^{\mathbb{Z}^d}; \forall R > 0, \boldsymbol{z} \in \mathbb{Z}^d,\ x|_{B_R(\boldsymbol{z})} \text{ occurs in some } \xi^n(a), n > 0, a \in \mathcal{A}\}$$

with primitivity assumption:

$$\exists n > 0, \quad \forall a, b \in \mathcal{A},\ b \text{ occurs in } \xi^n(a).$$

The substitution is aperiodic if any point $x \in X_\zeta$ is aperiodic.

Basic topological notions

Theorem (Curtis-Hedlund-Lyndon)

Let ϕ be an isomorphism of a subshift $(X, S, \mathbb{Z}^d)$.
There exists a local map $\hat{\phi}\colon \mathcal{A}^{B_r(\mathbf{0})} \to \mathcal{A}$ *s.t.*

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$\mathrm{N}(X)$ is countable.

Sufficient conditions on substitutions ξ to get $\vec{N}(X_\xi)$ finite:

M. Baake, J. A. G. Roberts and R. Yassawi (2018): Normalizer group of the chair tiling (and Ledrappier's shift).

A. Bustos (2019): Normalizer group of the Robinson tiling.

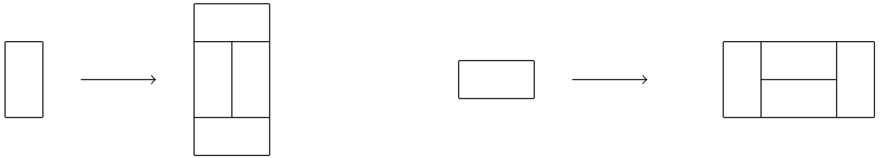
A. Bustos, D. Luz, N. Mañibo (2021): Normalizer of block substitutions.

C. (2023): Normalizer group for some extremally permutative substitutions.

Sufficient conditions on substitutions ξ to get $\vec{N}(X_\xi)$ finite:
M. Baake, J. A. G. Roberts and R. Yassawi (2018): Normalizer group of the chair tiling (and Ledrappier's shift).
A. Bustos (2019): Normalizer group of the Robinson tiling.
A. Bustos, D. Luz, N. Mañibo (2021): Normalizer of block substitutions.
C. (2023): Normalizer group for some extremally permutative substitutions.

Some subshifts with $\vec{N}(X)$ infinite:
M. Baake, J. A. G. Roberts and R. Yassawi (2018): Normalizer group of the full-shift.
M. Baake, A. Bustos, C. Huck, M. Lemanczyk, and A. Nickel (2021): Normalizer group of some number-theoretic positive entropy shifts.

The table tiling:



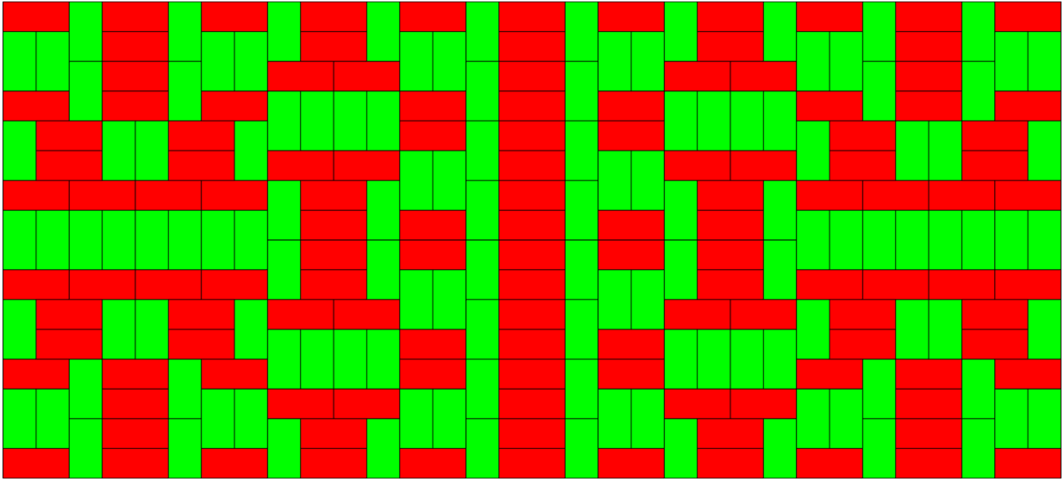


Figure: A pattern of the table tiling

The table tiling: $L_t = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, $F_1^t = [0,1]^2 \cap \mathbb{Z}^2$

$$0 \mapsto \begin{smallmatrix} 3 & 0 \\ 1 & 0 \end{smallmatrix}, \qquad 1 \mapsto \begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}, \qquad 2 \mapsto \begin{smallmatrix} 2 & 3 \\ 2 & 1 \end{smallmatrix}, \qquad 3 \mapsto \begin{smallmatrix} 0 & 2 \\ 3 & 3 \end{smallmatrix}.$$

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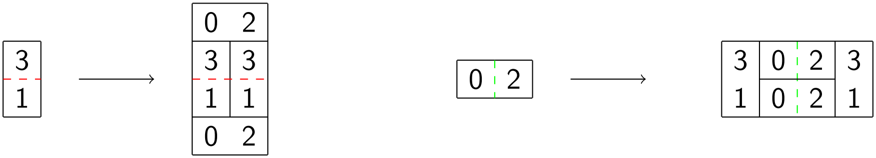

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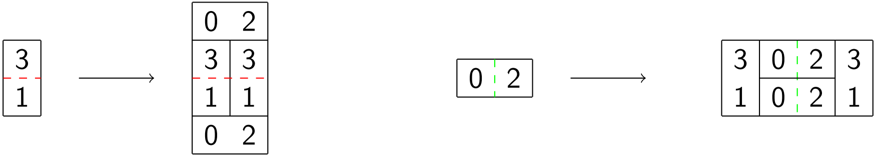

Proposition (C. (2023))

For the table substitution, we have $N(X_t, S, \mathbb{Z}^2) \cong \mathbb{Z}^2 \rtimes D_4$.

Q (BBHLN (2021)): Is $\vec{\mathrm{N}}(X_\xi)$ finite for any substitution ξ ?

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NO

Theorem (C., Petite)

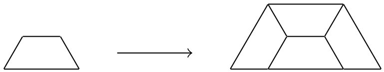
For any $d \geq 2$, there exists an aperiodic primitive constant-shape substitution ξ such that

$$\vec{N}(X_\xi) = GL(d, \mathbb{Z}).$$

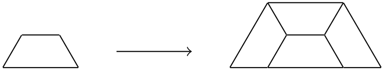
First examples known of zero-entropy minimal subshifts!

$\mathrm{N}(X_\xi)$ is not amenable, even for low complexity subshift

Half-hex tiling



Half-hex tiling



Discrete half-hex tiling: $L_{hh} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, $F_1^{hh} = \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \right\}$

$$0 \mapsto \begin{array}{cc} 0 & \\ 0 & 2 \\ & 1 \end{array}, \qquad 1 \mapsto \begin{array}{cc} 0 & \\ 1 & 2 \\ & 1 \end{array}, \qquad 2 \mapsto \begin{array}{cc} 0 & \\ 2 & 2 \\ & 1 \end{array}.$$

Proposition

There is an almost 1-1 factor map $\pi : X_{hh} \to \overleftarrow{\mathbb{Z}^2}_{(L^n_{hh})}$.

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But $\vec{\mathrm{N}}(\overleftarrow{\mathbb{Z}^2}_{(L^n_{hh})}) = \mathrm{GL}(2, \mathbb{Z})$.

The half-hex is a 2-dimensional automatic sequence: There is a map

$$\tau : \mathbb{Z}^2 \setminus \{\mathbf{0}\} \to F_1^{hh} \setminus \{\mathbf{0}\}$$

where $\tau(\boldsymbol{n}) = \boldsymbol{f} \in F_1^{hh} \setminus \{\mathbf{0}\}$, if $\boldsymbol{n} = L^p(\boldsymbol{f}) + L^{p+1}(\boldsymbol{m})$ for some $\boldsymbol{m} \in \mathbb{Z}^2$ and $p > 0$.

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Theorem (C., Petite (2023))

$\mathrm{Aut}(X_{hh}) = \langle S \rangle$, $\quad \vec{N}(X_{hh}) = GL(2, \mathbb{Z})$, $\quad N(X_{hh}) = \mathbb{Z}^2 \rtimes GL(2, \mathbb{Z})$.

Theorem (C., Petite (2023))

For any expansion matrix $L \in \mathcal{M}(d, \mathbb{Z})$ *with* $|\det L| > 3$*, there is an aperiodic minimal substitutive* $\mathbb{Z}^d$*-subshift* X *with expansion matrix* L *such that*

- *it is coalescent.*
- $\mathrm{Aut}(X) = \langle S \rangle$.
- $\vec{N}(X)$ *is*

$$\Big\{ M \in \bigcup_{k \geq 0} \bigcap_{n \geq k} L^n GL(d, \mathbb{Z}) L^{-n}; \\ \exists n_0, L^{-n} M L^n = L^{-p} M L^p \ (mod\ L(\mathbb{Z}^d)), \forall n, p \geq n_0 \Big\}$$

- $N(X) \simeq \mathbb{Z}^d \rtimes \vec{N}(X)$.

The isomorphisms are explicit.

Open problem

Problem

For general substitution ξ

Are the elements of $\vec{\mathrm{N}}(X_\xi)$ computable?

Is its group structure computable?

Same question for $\mathrm{N}(X_\xi)$.

THANKS