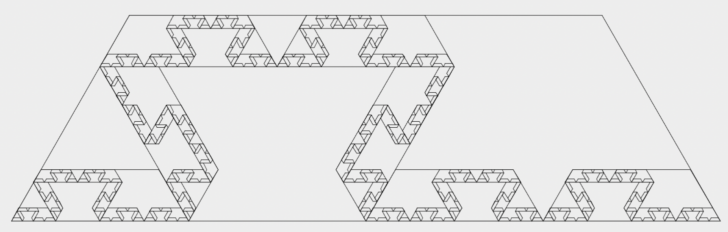
Improved bound for the Gerver-Ramsey collinearity problem

Finn Lidbetter

Independent Researcher
Software Developer at RideCo, Waterloo, Canada



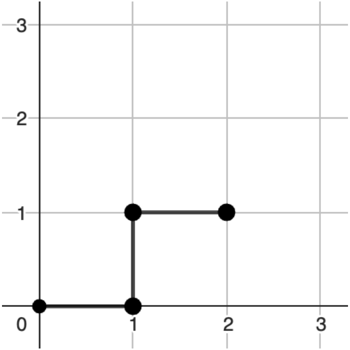
Gerver-Ramsey collinearity problem

- Let S be a finite subset of $\mathbb{Z}^n$
- A vector sequence $(\mathbf{z}_i)$ is an S-walk if and only if $\mathbf{z}_{i+1} - \mathbf{z}_i$ is an element of S for all i

For a given set, S, what is the longest S-walk we can construct that avoids having m collinear points?

Gerver-Ramsey collinearity problem: example

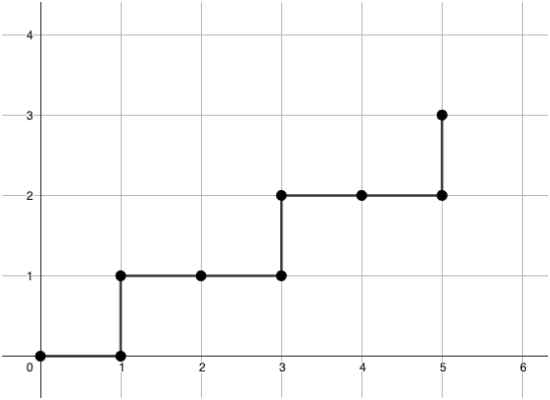
$S = \{\mathbf{i}, \mathbf{j}\}$



A longest S-walk avoiding 3 collinear points.

Gerver-Ramsey collinearity problem: example

$S = \{\mathbf{i}, \mathbf{j}\}$



A longest S-walk avoiding 4 collinear points.

The two dimensional case

Theorem (Ramsey; 1977)

Let $S \subset \mathbb{Z}^2$ and let K be any positive integer. There exists $N(K)$ such that any S-walk of length at $N(K)$ must have K collinear points.

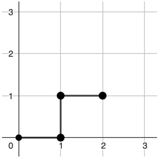
There is no infinite S-walk for $S \subset \mathbb{Z}^2$ avoiding K collinear points.

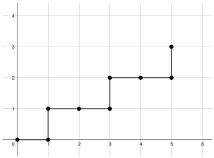
The Two dimensional case

For $S = \{\mathbf{i}, \mathbf{j}\}$, define $a(n)$ as the smallest integer t such that every length-t S-walk is guaranteed to have at least n collinear points.

The sequence $a(n)$ begins $0, 1, 4, 9, 29, 97$.

See `https://oeis.org/A231255`.





Gerver; 1979: $a(n)$ grows faster than any polynomial function of n.

The three dimensional case

Is there an infinite S-walk for $S \subset \mathbb{Z}^3$ that can avoid K collinear points for some K?

The three dimensional case

Is there an infinite S-walk for $S \subset \mathbb{Z}^3$ that can avoid K collinear points for some K?

Yes!

Theorem (Gerver, Ramsey; 1979)

For $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, there exists an infinite S-walk for which no $5^{11} + 1 = 48{,}828{,}126$ points are collinear.

The construction (Gerver, Ramsey; 1979)

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ be sequences of vectors. Define:

- $RA = (a_n, \ldots, a_1)$
- $(A, B) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$
- $\beta A = (\beta a_1, \ldots, \beta a_n)$ for vector operator β

The construction (Gerver, Ramsey; 1979)

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ be sequences of vectors. Define:

- $RA = (a_n, \ldots, a_1)$
- $(A, B) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$
- $\beta A = (\beta a_1, \ldots, \beta a_n)$ for vector operator β

Define vector operators α and β that operate on $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as

$$\alpha\mathbf{i} = \mathbf{j}, \quad \alpha\mathbf{j} = \mathbf{i}, \quad \alpha\mathbf{k} = \mathbf{k}, \qquad \beta\mathbf{i} = \mathbf{i}, \quad \beta\mathbf{j} = \mathbf{k}, \quad \beta\mathbf{k} = \mathbf{j}.$$

The construction (Gerver, Ramsey; 1979)

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ be sequences of vectors. Define:

- $RA = (a_n, \ldots, a_1)$
- $(A, B) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$
- $\beta A = (\beta a_1, \ldots, \beta a_n)$ for vector operator β

Define vector operators α and β that operate on $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as

$$\alpha\mathbf{i} = \mathbf{j}, \quad \alpha\mathbf{j} = \mathbf{i}, \quad \alpha\mathbf{k} = \mathbf{k}, \qquad \beta\mathbf{i} = \mathbf{i}, \quad \beta\mathbf{j} = \mathbf{k}, \quad \beta\mathbf{k} = \mathbf{j}.$$

Let $A_0 = (\mathbf{i})$, and $A_{n+1} = (A_n, \alpha A_n, R\beta A_n, A_n, R\beta\alpha A_n, R\beta A_n, A_n)$.

The construction (Gerver, Ramsey; 1979)

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ be sequences of vectors. Define:

- $RA = (a_n, \ldots, a_1)$
- $(A, B) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$
- $\beta A = (\beta a_1, \ldots, \beta a_n)$ for vector operator β

Define vector operators α and β that operate on $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as

$$\alpha \mathbf{i} = \mathbf{j}, \quad \alpha \mathbf{j} = \mathbf{i}, \quad \alpha \mathbf{k} = \mathbf{k}, \qquad \beta \mathbf{i} = \mathbf{i}, \quad \beta \mathbf{j} = \mathbf{k}, \quad \beta \mathbf{k} = \mathbf{j}.$$

Let $A_0 = (\mathbf{i})$, and $A_{n+1} = (A_n, \alpha A_n, R\beta A_n, A_n, R\beta\alpha A_n, R\beta A_n, A_n)$.

Take $(\mathbf{v}_1, \ldots, \mathbf{v}_{7^n}) = A_n$.

Define $\mathbf{z}_p = \Sigma_{q=1}^{p} \mathbf{v}_q$ for integers $p > 0$ and $\mathbf{z}_0 = \mathbf{0}$.

The S-walk is $W = (\mathbf{z}_i)_{i \geq 0}$.

The construction (Gerver, Ramsey; 1979)

Vector operators:

$$\alpha\mathbf{i} = \mathbf{j}, \quad \alpha\mathbf{j} = \mathbf{i}, \quad \alpha\mathbf{k} = \mathbf{k}, \qquad \beta\mathbf{i} = \mathbf{i}, \quad \beta\mathbf{j} = \mathbf{k}, \quad \beta\mathbf{k} = \mathbf{j}.$$

Recurrence rule: $A_{n+1} = (A_n, \alpha A_n, R\beta A_n, A_n, R\beta\alpha A_n, R\beta A_n, A_n)$

First 35 terms:

$$\mathbf{i},\mathbf{j},\mathbf{i},\mathbf{i},\mathbf{k},\mathbf{i},\mathbf{i},\ \mathbf{j},\mathbf{i},\mathbf{j},\mathbf{j},\mathbf{k},\mathbf{j},\mathbf{j},\ \mathbf{i},\mathbf{i},\mathbf{j},\mathbf{i},\mathbf{i},\mathbf{k},\mathbf{i},\ \mathbf{i},\mathbf{j},\mathbf{i},\mathbf{i},\mathbf{k},\mathbf{i},\mathbf{i},\ \mathbf{k},\mathbf{k},\mathbf{j},\mathbf{k},\mathbf{k},\mathbf{i},\mathbf{k}.$$

`https://www.geogebra.org/classic/tnetaqjn`

Improving the bound

Previous bound on number of collinear points: $5^{11} + 1 = 48{,}828{,}126$.

This work:

Theorem (L. 2024)

There exists an infinite $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$-walk for which no 189 points are collinear.

Improving the bound

Previous bound on number of collinear points: $5^{11} + 1 = 48{,}828{,}126$.

This work:

Theorem (L. 2024)

There exists an infinite $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$-walk for which no 189 points are collinear.

- Same construction
- Re-state the walk as the fixed point of iterating a morphism
- More compute power!

Proof approach (Gerver and Ramsey; 1979)

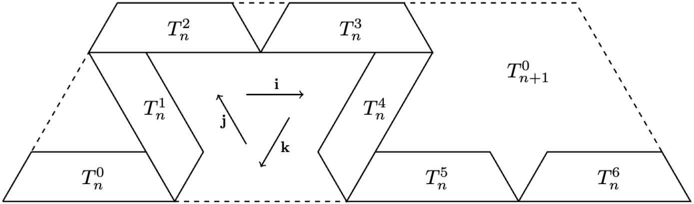
Goal: show that a pair of points that are separated by a relatively small number of indices cannot be collinear with another pair of points that are separated by a relatively large number of indices.

Project the points of W onto a plane perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Projection in the plane

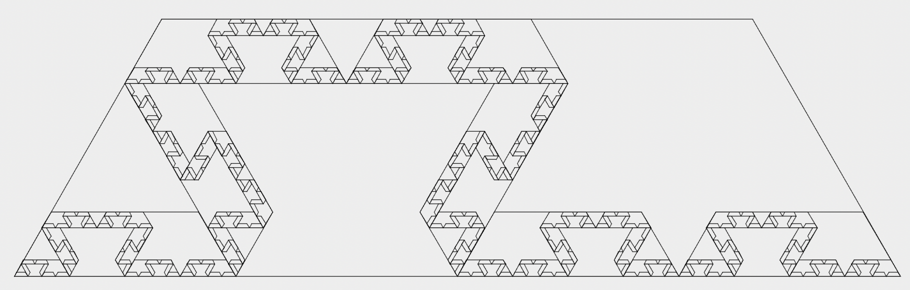
Let γ be the length of the component of $\mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$. That is, $\gamma = (2/3)^{1/2}$

The points $(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{7^n})$ lie in a trapezoid of base length $4^n\gamma$, base angles 60°, and adjacent sides $4^n\gamma/3$, with $\mathbf{z}_0$ and $\mathbf{z}_{7^n}$ at opposite ends of the long edge.



https://www.geogebra.org/classic/tnetaqjn

Trapezoid configurations



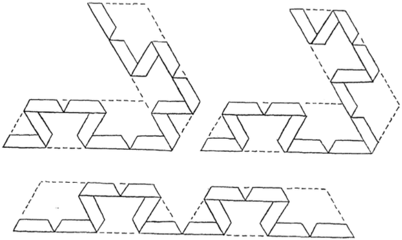


FIGURE 2

Figure 2 from Gerver, Ramsey; 1979.

The only three possible ways for T^k_{n+1} and T^{k+1}_{n+1} to be arranged.

Perpendicular and parallel distances

For $\mathbf{z} = z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$, define:

$$\|\mathbf{z}\|^{\parallel} = z_1 + z_2 + z_3$$

$$\|\mathbf{z}\|^{\perp} = \sqrt{z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1}$$

The component of $\mathbf{z}$ perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$ has length $\gamma\|\mathbf{z}\|^{\perp}$

Fact 1:
For $\mathbf{z}_p, \mathbf{z}_q$ in W, we have $\|\mathbf{z}_p - \mathbf{z}_q\|^{\parallel} = |p - q|$

Fact 2:
If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}$, $\mathbf{s}$ are collinear, then $\dfrac{\|\mathbf{u}-\mathbf{v}\|^{\perp}}{\|\mathbf{u}-\mathbf{v}\|^{\parallel}} = \dfrac{\|\mathbf{r}-\mathbf{s}\|^{\perp}}{\|\mathbf{r}-\mathbf{s}\|^{\parallel}}$

Bounding $\|\mathbf{z}_p - \mathbf{z}_q\|^{\perp} / \|\mathbf{z}_p - \mathbf{z}_q\|^{\parallel}$

Consider integers p, q with $7^n \leq |p - q| < 7^{n+1}$.

Projections of $\mathbf{z}_p$ and $\mathbf{z}_q$ are not in adjacent trapezoids of order $n - 1$.

Projections of $\mathbf{z}_p$ and $\mathbf{z}_q$ are in the same trapezoid or adjacent trapezoids of order $n + 1$.

Use the geometry of the trapezoids to bound $\|\mathbf{z}_p - \mathbf{z}_q\|^{\perp}$

Bounding $\|\mathbf{z}_p - \mathbf{z}_q\|^\perp / \|\mathbf{z}_p - \mathbf{z}_q\|^\parallel$

Consider integers p, q with $7^n \le |p - q| < 7^{n+1}$.

Projections of $\mathbf{z}_p$ and $\mathbf{z}_q$ are not in adjacent trapezoids of order $n - 1$.

Projections of $\mathbf{z}_p$ and $\mathbf{z}_q$ are in the same trapezoid or adjacent trapezoids of order $n + 1$.

Use the geometry of the trapezoids to bound $\|\mathbf{z}_p - \mathbf{z}_q\|^\perp$

$$\frac{4^{n-1}}{\sqrt{3}} \le \|\mathbf{z}_p - \mathbf{z}_q\|^\perp$$

$$\|\mathbf{z}_p - \mathbf{z}_q\|^\perp \le 2 \cdot 4^{n+1}$$

$$\|\mathbf{z}_p - \mathbf{z}_q\|^\parallel = |p - q|$$

$$\frac{4^{n-1}}{7^{n+1}\sqrt{3}} \le \frac{\|\mathbf{z}_p - \mathbf{z}_q\|^\perp}{\|\mathbf{z}_p - \mathbf{z}_q\|^\parallel} \le \frac{2 \cdot 4^{n+1}}{7^n}$$

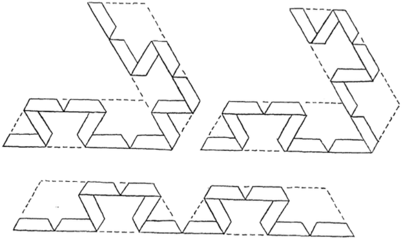


FIGURE 2

Bounding the relative index separation of collinear points

Consider integers r, s with $7^m \le |r - s| < 7^{m+1}$.

Assume $\mathbf{z}_p, \mathbf{z}_q, \mathbf{z}_r, \mathbf{z}_s$ are collinear.

Then $\dfrac{\|\mathbf{z}_p - \mathbf{z}_q\|^{\perp}}{\|\mathbf{z}_p - \mathbf{z}_q\|^{\parallel}} = \dfrac{\|\mathbf{z}_r - \mathbf{z}_s\|^{\perp}}{\|\mathbf{z}_r - \mathbf{z}_s\|^{\parallel}}$

Manipulate inequalities to get $\left(\dfrac{7}{4}\right)^{|m-n|} < 2 \cdot 4^2 \cdot 7\sqrt{3} = 224\sqrt{3}$

Integrality of m, n implies $|m - n| \le 10$.

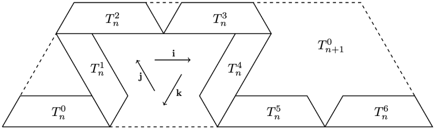
So $\dfrac{|r - s|}{|p - q|} < 7^{11}$ and we get at most 7^{11} collinear points in W.

Lines through trapezoids

Suppose X is a set of collinear points in W with all points in X in a trapezoid of order n, but not within a trapezoid of $n-1$.

Then no two points in X can be in the same trapezoid of order $n-11$.

No line can intersect more than five trapezoids of order $n-1$ within a trapezoid of order n.



Therefore there are at most 5^{11} collinear points in W.

Improving the bound

1. No need to lump together all values of $|p - q|$ between 7^n and 7^{n+1}.

   Using a finer partition of $[7^n, 7^{n+1}]$ for each given $|p - q|$ we could further constrain the values that $\frac{\|\mathbf{z}_p - \mathbf{z}_q\|^{\perp}}{\|\mathbf{z}_p - \mathbf{z}_q\|^{\parallel}}$ range over.

Improving the bound

1. No need to lump together all values of $|p - q|$ between 7^n and 7^{n+1}.

   Using a finer partition of $[7^n, 7^{n+1}]$ for each given $|p - q|$ we could further constrain the values that $\frac{\|\mathbf{z}_p - \mathbf{z}_q\|^{\perp}}{\|\mathbf{z}_p - \mathbf{z}_q\|^{\|}}$ range over.

2. Instead of the largest number of trapezoids of order $n - 1$ inside a trapezoid of order n intersected by a single straight line, why not $n - 2$? Or $n - 3$? Etc.

Improving the bound

1. No need to lump together all values of $|p - q|$ between 7^n and 7^{n+1}.

   Using a finer partition of $[7^n, 7^{n+1}]$ for each given $|p - q|$ we could further constrain the values that $\frac{\|\mathbf{z}_p - \mathbf{z}_q\|^{\perp}}{\|\mathbf{z}_p - \mathbf{z}_q\|^{\parallel}}$ range over.

2. Instead of the largest number of trapezoids of order $n - 1$ inside a trapezoid of order n intersected by a single straight line, why not $n - 2$? Or $n - 3$? Etc.
3. Can we work with trapezoidal prisms in 3 dimensions, instead of the projection of points onto a plane?

Improving the bound

1. No need to lump together all values of $|p-q|$ between 7^n and 7^{n+1}.

   Using a finer partition of $[7^n, 7^{n+1}]$ for each given $|p-q|$ we could further constrain the values that $\frac{\|\mathbf{z}_p-\mathbf{z}_q\|^{\perp}}{\|\mathbf{z}_p-\mathbf{z}_q\|^{\|}}$ range over.

2. Instead of the largest number of trapezoids of order $n-1$ inside a trapezoid of order n intersected by a single straight line, why not $n-2$? Or $n-3$? Etc.
3. Can we work with trapezoidal prisms in 3 dimensions, instead of the projection of points onto a plane?

Note: these were all suggested by Gerver, Ramsey; 1979.

Improving the bound

1. No need to lump together all values of $|p-q|$ between 7^n and 7^{n+1}.

   Using a finer partition of $[7^n, 7^{n+1}]$ for each given $|p-q|$ we could further constrain the values that $\frac{\|\mathbf{z}_p-\mathbf{z}_q\|^{\perp}}{\|\mathbf{z}_p-\mathbf{z}_q\|^{\parallel}}$ range over.

2. Instead of the largest number of trapezoids of order $n-1$ inside a trapezoid of order n intersected by a single straight line, why not $n-2$? Or $n-3$? Etc.

3. Can we work with trapezoidal prisms in 3 dimensions, instead of the projection of points onto a plane?

Note: these were all suggested by Gerver, Ramsey; 1979.

Let's start by re-stating the construction of W as the fixed point of iterating a morphism to aid in some proofs!

Morphism construction of W

Due to Luke Schaeffer.

Vector operators:

$$\alpha\mathbf{i} = \mathbf{j}, \quad \alpha\mathbf{j} = \mathbf{i}, \quad \alpha\mathbf{k} = \mathbf{k}, \qquad \beta\mathbf{i} = \mathbf{i}, \quad \beta\mathbf{j} = \mathbf{k}, \quad \beta\mathbf{k} = \mathbf{j}.$$

Recurrence rule: $A_{n+1} = (A_n, \alpha A_n, R\beta A_n, A_n, R\beta\alpha A_n, R\beta A_n, A_n)$

- Vector operators α and β actions on $\mathbf{i}, \mathbf{j}, \mathbf{k}$ correspond to S_3
- Reversal corresponds to cyclic group of order 2
- Product of these groups is the dihedral group of order 12

Morphism construction of W

$$A_{n+1} = (A_n, \alpha A_n, R\beta A_n, A_n, R\beta\alpha A_n, R\beta A_n, A_n)$$

For the alphabet $\Sigma = \{i, j, k, i', j', k', i_b, j_b, k_b, i'_b, j'_b, k'_b\}$ define the morphism $\mu : \Sigma^* \to \Sigma^*$ as:

$$\mu(i) = i\ j'\ i'_b\ i\ k_b\ i'_b\ i \qquad \mu(i_b) = i_b\ i'\ k\ i_b\ i'\ j'_b\ i_b$$

$$\mu(j) = j\ k'\ j'_b\ j\ i_b\ j'_b\ j \qquad \mu(j_b) = j_b\ j'\ i\ j_b\ j'\ k'_b\ j_b$$

$$\mu(k) = k\ i'\ k'_b\ k\ j_b\ k'_b\ k \qquad \mu(k_b) = k_b\ k'\ j\ k_b\ k'\ i'_b\ k_b$$

$$\mu(i') = i'\ k\ i_b\ i'\ j'_b\ i_b\ i' \qquad \mu(i'_b) = i'_b\ i\ j'\ i'_b\ i\ k_b\ i'_b$$

$$\mu(j') = j'\ i\ j_b\ j'\ k'_b\ j_b\ j' \qquad \mu(j'_b) = j'_b\ j\ k'\ j'_b\ j\ i_b\ j'_b$$

$$\mu(k') = k'\ j\ k_b\ k'\ i'_b\ k_b\ k' \qquad \mu(k'_b) = k'_b\ k\ i'\ k'_b\ k\ j_b\ k'_b$$

Morphism construction of W

Define the output map $\phi : \Sigma^* \to \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}^*$ with

$$\begin{aligned}
\phi(\varepsilon) &= \varepsilon \\
\phi(i) = \phi(i_b) = \phi(i') = \phi(i'_b) &= \mathbf{i} \\
\phi(j) = \phi(j_b) = \phi(j') = \phi(j'_b) &= \mathbf{j} \\
\phi(k) = \phi(k_b) = \phi(k') = \phi(k'_b) &= \mathbf{k}
\end{aligned}$$

and $\phi(uv) = \phi(u)\phi(v)$ for $u, v \in \Sigma^*$.

Claim $\phi(\mu^\omega(i)) = W$

Define $\lambda = \mu^\omega(i)$

Bound the relative index separation of collinear points

Previously:

$$7^n \leq |p-q| < 7^{n+1}$$

$$\frac{4^{n-1}}{7^{n+1}\sqrt{3}} \leq \frac{\|\mathbf{z}_p - \mathbf{z}_q\|^{\perp}}{\|\mathbf{z}_p - \mathbf{z}_q\|^{\parallel}} \leq \frac{2 \cdot 4^{n+1}}{7^n}$$

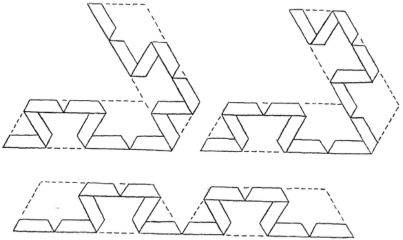


FIGURE 2

Improvement:
Sharpen the bounds by considering each $c \in \{7, 8, \ldots, 48\}$ and looking at

$$\frac{c7^n}{7} \leq |p-q| < \frac{(c+1)7^n}{7}$$

If $p < q$ and the projection of $\mathbf{z}_p$ lies in T_{n-1}^{k} then the projection of $\mathbf{z}_q$ lies in either T_{n-1}^{k+c} or T_{n-1}^{k+c+1}

Minimum and maximum distances between trapezoids

We need to consider trapezoids separated by up to 49 indices and all possible relative configurations.

How can we do this?

1. Establish a correspondence between the symbols in λ and trapezoid orientations.

Minimum and maximum distances between trapezoids

We need to consider trapezoids separated by up to 49 indices and all possible relative configurations.

How can we do this?

1. Establish a correspondence between the symbols in λ and trapezoid orientations.
2. Find all distinct subwords of length up to 49 in λ.

Minimum and maximum distances between trapezoids

We need to consider trapezoids separated by up to 49 indices and all possible relative configurations.

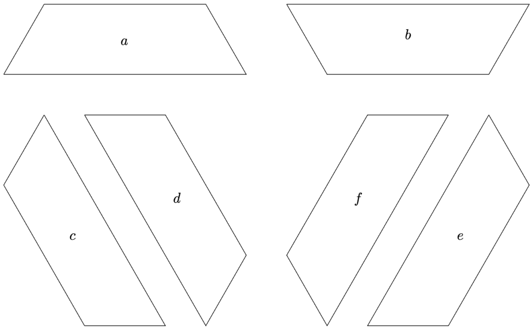
How can we do this?

1. Establish a correspondence between the symbols in λ and trapezoid orientations.
2. Find all distinct subwords of length up to 49 in λ.
3. Use computational geometry on all corresponding sequences of trapezoids to get upper and lower bounds on distances between points in the trapezoids.

1. Correspondence between λ and trapezoid orientations

Map symbols in Σ to orientations a, b, c, d, e, f.

$$\psi(i) = \psi(i'_b) = a$$
$$\psi(i') = \psi(i_b) = b$$
$$\psi(j) = \psi(j'_b) = c$$
$$\psi(j') = \psi(j_b) = d$$
$$\psi(k) = \psi(k'_b) = e$$
$$\psi(k') = \psi(k_b) = f$$



Lemma 6 (L. 2024)

The orientation of trapezoid T_n^m is given by $\psi(\lambda[m])$.

2. Finding all distinct subwords of length n

Lemma 3 (L. 2024)

Let $\mathcal{I}(n)$ be the index of the last new subword of length n in λ.
Then $\mathcal{I}(1) = 215$ and $\mathcal{I}(2) = 558$ and $\mathcal{I}(n) \leq 7 \cdot \mathcal{I}(\lceil n/7 \rceil + 1)$

2. Finding all distinct subwords of length n

Lemma 3 (L. 2024)

Let $\mathcal{I}(n)$ be the index of the last new subword of length n in λ.
Then $\mathcal{I}(1) = 215$ and $\mathcal{I}(2) = 558$ and $\mathcal{I}(n) \leq 7 \cdot \mathcal{I}(\lceil n/7 \rceil + 1)$

Use the upper bound to brute force a search for distinct subwords of length n in λ.

`https://github.com/FinnLidbetter/avoiding-collinearity`

2. Finding all distinct subwords of length n

Lemma 3 (L. 2024)

Let $\mathcal{I}(n)$ be the index of the last new subword of length n in λ.
Then $\mathcal{I}(1) = 215$ and $\mathcal{I}(2) = 558$ and $\mathcal{I}(n) \leq 7 \cdot \mathcal{I}(\lceil n/7 \rceil + 1)$

Use the upper bound to brute force a search for distinct subwords of length n in λ.

`https://github.com/FinnLidbetter/avoiding-collinearity`

Command:

```
IndexOfLastNewSubword 49
```

Output:

```
The (0-based) index of the last new subword of length 49 is 27334.
The subword at this index is gkbfkbgkgdcgdkgdcgdkgdcdlchlcgdcgdkgdcgdkgdkbfkbg.
```

3. Upper and lower bound on distances

$c, d \in \{7, 8, \ldots, 48\}$

$$\frac{c7^n}{7} \leq |p - q| < \frac{(c+1)7^n}{7} \quad \text{and} \quad \frac{d7^m}{7} \leq |r - s| < \frac{(d+1)7^m}{7}$$

Define:

$$d(T_n^a, T_n^b) = \min\{d(p_a, p_b) \;:\; p_a \in T_n^a, p_b \in T_n^b\}$$
$$D(T_n^a, T_n^b) = \max\{d(p_a, p_b) \;:\; p_a \in T_n^a, p_b \in T_n^b\}$$

3. Upper and lower bound on distances

$c, d \in \{7, 8, \ldots, 48\}$

$$\frac{c7^n}{7} \leq |p - q| < \frac{(c+1)7^n}{7} \quad \text{and} \quad \frac{d7^m}{7} \leq |r - s| < \frac{(d+1)7^m}{7}$$

Define:

$$d(T_n^a, T_n^b) = \min\{d(p_a, p_b) \ : \ p_a \in T_n^a, p_b \in T_n^b\}$$

$$D(T_n^a, T_n^b) = \max\{d(p_a, p_b) \ : \ p_a \in T_n^a, p_b \in T_n^b\}$$

$$\ell(n, c) = \min_{k \in \mathbb{N}}\{d(T_{n-1}^k, T_{n-1}^{k+c}), d(T_{n-1}^k, T_{n-1}^{k+c+1})\}$$

$$h(n, c) = \max_{k \in \mathbb{N}}\{D(T_{n-1}^k, T_{n-1}^{k+c}), D(T_{n-1}^k, T_{n-1}^{k+c+1})\}$$

3. Upper and lower bound on distances

$$\ell(n,c) = \min_{k\in\mathbb{N}}\{d(T_{n-1}^{k}, T_{n-1}^{k+c}), d(T_{n-1}^{k}, T_{n-1}^{k+c+1})\}$$

$$h(n,c) = \max_{k\in\mathbb{N}}\{D(T_{n-1}^{k}, T_{n-1}^{k+c}), D(T_{n-1}^{k}, T_{n-1}^{k+c+1})\}$$

1. trapezoids of order n are congruent to trapezoids of order $n-1$.

$$\ell(n,c) = 4^{n-1}\cdot \ell(1,c)$$

$$h(n,c) = 4^{n-1}\cdot h(1,c)$$

2. Use the correspondence between symbols in λ and trapezoid orientations

$$\ell(n,c) = 4^{n-1}\min_{0\le k\le \mathcal{I}(c+2)}\{d(T_0^{k}, T_0^{k+c}), d(T_0^{k}, T_0^{k+c+1})\}$$

$$h(n,c) = 4^{n-1}\max_{0\le k\le \mathcal{I}(c+2)}\{D(T_0^{k}, T_0^{k+c}), D(T_0^{k}, T_0^{k+c+1})\}$$

3. Upper and lower bound on distances

Command:

```
AssertBoundedDistanceRatio 7 48 wholeAndRt3 9 0
```

Output:

```
maxLoDistanceRatio:  10 / sqrt((28 + 0 * sqrt(3)))
maxHiDistanceRatio:  sqrt((964 + 0 * sqrt(3))) / 7
SUCCESS
The ratio of the largest distance to the smallest distance between
trapezoids separated by at least 7 indices and at most 48 indices is
less than (9 + 0 * sqrt(3))
```

$$\left(\frac{7}{4}\right)^{m-n} \leq \frac{10}{\sqrt{28}} \cdot \frac{\sqrt{964}}{7} < 9 < \left(\frac{7}{4}\right)^{4} = 9.37890625$$

$m - n \leq 3$

Lines through trapezoids

The $m - n \leq 3$ bound already implies at most $7^4 = 2401$ collinear points!

Lines through trapezoids

The $m - n \leq 3$ bound already implies at most $7^4 = 2401$ collinear points!

Assume $7^n \leq |p - q| < 7^{n+1}$.

Claim 1:
No two points collinear with both $\mathbf{z}_p$ and $\mathbf{z}_q$ can lie in the same trapezoid of order $n - 4$.

Lines through trapezoids

The $m - n \leq 3$ bound already implies at most $7^4 = 2401$ collinear points!

Assume $7^n \leq |p - q| < 7^{n+1}$.

Claim 1:
No two points collinear with both $\mathbf{z}_p$ and $\mathbf{z}_q$ can lie in the same trapezoid of order $n - 4$.

Claim 2:
No two points collinear with both $\mathbf{z}_p$ and $\mathbf{z}_q$ can lie in two trapezoids of order $n - 4$ separated by more than $7^4 = 2401$ indices.

Lines through trapezoids

The $m - n \leq 3$ bound already implies at most $7^4 = 2401$ collinear points!

Assume $7^n \leq |p - q| < 7^{n+1}$.

Claim 1:
No two points collinear with both $\mathbf{z}_p$ and $\mathbf{z}_q$ can lie in the same trapezoid of order $n - 4$.

Claim 2:
No two points collinear with both $\mathbf{z}_p$ and $\mathbf{z}_q$ can lie in two trapezoids of order $n - 4$ separated by more than $7^4 = 2401$ indices.

Goal:
Find all the ways that 2401 consecutive trapezoids of order $n - 4$ can be arranged and look for the largest number of them that can be intersected by a single straight line.

Lines through trapezoids

How far do we have to look for new sequences of 2401 trapezoids?

Lines through trapezoids

How far do we have to look for new sequences of 2401 trapezoids?

Command:

```
IndexOfLastNewSubword 2401
```

Output:

```
The (0-based) index of the last new subword of length 2401 is 1339414.
```

Lines through trapezoids

How far do we have to look for new sequences of 2401 trapezoids?

Command:

```
IndexOfLastNewSubword 2401
```

Output:

```
The (0-based) index of the last new subword of length 2401 is 1339414.
```

Where are the distinct subwords found?

Lines through trapezoids

How far do we have to look for new sequences of 2401 trapezoids?

Command:

```
IndexOfLastNewSubword 2401
```

Output:

```
The (0-based) index of the last new subword of length 2401 is 1339414.
```

Where are the distinct subwords found?

Command:

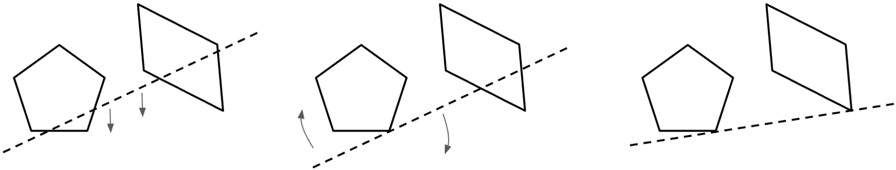
```
DistinctSubwordIntervals 2401
```

Output:

```
[[0,14062], [16808,30869], [67229,76831], [76833,83691], [184878,194480],
[194482,201340], [504211,518272], [1327754,1341815]]
```

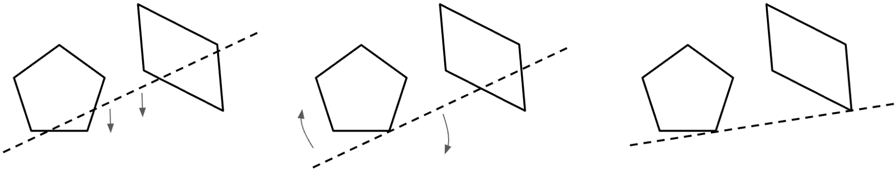
Line through trapezoids algorithm

Observation:
Given a set, P, of polygons in the plane, let L be the largest number of polygons in P intersected by a straight line. Then there is a straight line through a pair of vertices of polygons in P that intersects L polygons in P.



Line through trapezoids algorithm

Observation:
Given a set, P, of polygons in the plane, let L be the largest number of polygons in P intersected by a straight line. Then there is a straight line through a pair of vertices of polygons in P that intersects L polygons in P.



Naive algorithm:

1. For each sequence of 2401 trapezoids, consider each pair of trapezoid vertices to enumerate all candidate lines.
2. For each candidate line, iterate over the 2401 trapezoids to determine the number of trapezoids intersected.

Improved line through trapezoids algorithm

Idea:

- Use a radial line sweep approach
- Consider each vertex as a pivot
- Sort the trapezoid vertices radially, relative to the pivot
- Only consider trapezoids within 2401 indices of the pivot vertex trapezoid
- Rotate a line and keep track of the current count of trapezoid intersections

Efficiently tracking multiple counts

There are 4803 trapezoids in consideration around the pivot.

- Keep one count for each interval of 2401 trapezoids.
- When the line enters a new trapezoid, increment the counts for all intervals that the trapezoid falls within.
- When the line exits a trapezoid, decrement the counts for all intervals that the trapezoid falls within.
- When counts are incremented, check the incremented counts for a new maximum value.

Efficiently tracking multiple counts

There are 4803 trapezoids in consideration around the pivot.

- Keep one count for each interval of 2401 trapezoids.
- When the line enters a new trapezoid, increment the counts for all intervals that the trapezoid falls within.
- When the line exits a trapezoid, decrement the counts for all intervals that the trapezoid falls within.
- When counts are incremented, check the incremented counts for a new maximum value.

Let's use a *Segment Tree* data structure!

- Range updates: $O(\log n)$
- Range queries: $O(\log n)$

Maximum count of 2401 consecutive trapezoids intersected

For each pivot vertex and interval size $n = 2401$:

1. Construct a segment tree with $O(n)$ leaf nodes: $O(n)$.
2. Sort trapezoid vertices relative to the pivot vertex: $O(n \log n)$.
3. Iterate over $O(n)$ sorted vertices and for each vertex perform one range update at cost $O(\log n)$ and at most one range query at cost $O(\log n)$. This step is $O(n \log n)$.

Command:

```
CountCollinearTrapezoids 2401 wholeAndRt3
```

Maximum count of 2401 consecutive trapezoids intersected

For each pivot vertex and interval size $n = 2401$:

1. Construct a segment tree with $O(n)$ leaf nodes: $O(n)$.
2. Sort trapezoid vertices relative to the pivot vertex: $O(n \log n)$.
3. Iterate over $O(n)$ sorted vertices and for each vertex perform one range update at cost $O(\log n)$ and at most one range query at cost $O(\log n)$. This step is $O(n \log n)$.

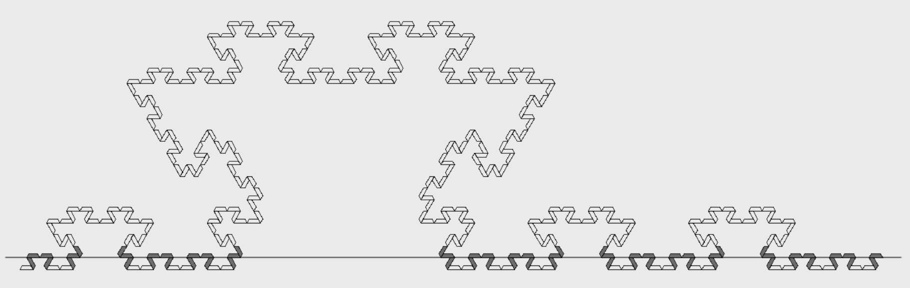
Command:

```
CountCollinearTrapezoids 2401 wholeAndRt3
```

Output:

```
The largest number of trapezoids separated by at most 2401 indices that are
intersected by a single straight line is 188.  The intersection is through
trapezoids 1 and 2402 (0-based indexing) at points ((5 + 0 * sqrt(3)), (0 + 3 *
sqrt(3))) and ((1533 + 0 * sqrt(3)), (0 + 3 * sqrt(3))).
```

Maximum count of 343 consecutive trapezoids intersected



Command:

```
CountCollinearTrapezoids 343 wholeAndRt3
```

Output:

```
The largest number of trapezoids separated by at most 343 indices that are
intersected by a single straight line is 62.  The intersection is through
trapezoids 1 and 344 (0-based indexing) at points ((5 + 0 * sqrt(3)), (0 + 3 *
sqrt(3))) and ((381 + 0 * sqrt(3)), (0 + 3 * sqrt(3))).
```

A bound of 188

Great! We have the bound on the number of 188 collinear points advertised in the theorem. Are we done?

A bound of 188

Great! We have the bound on the number of 188 collinear points advertised in the theorem. Are we done?

Not so fast.

An overlooked detail

The $n = 0$ case: $7^0 \leq |p - q| < 7^1$

We cannot work with trapezoids of order $n - 1 = -1$

An overlooked detail

The $n = 0$ case: $7^0 \leq |p - q| < 7^1$

We cannot work with trapezoids of order $n - 1 = -1$

For collinear $\mathbf{z}_p, \mathbf{z}_q, \mathbf{z}_r, \mathbf{z}_s$ and $7^m \leq |r - s| < 7^{m+1}$ we can bound:

$$\left(\frac{7}{4}\right)^{m-1} \leq \frac{10\sqrt{964}}{21} < 14.89 < \left(\frac{7}{4}\right)^5 = 16.4130859375.$$

In this case we get $m - n \leq 5$.

An overlooked detail

The $n = 0$ case: $7^0 \leq |p - q| < 7^1$

We cannot work with trapezoids of order $n - 1 = -1$

For collinear $\mathbf{z}_p, \mathbf{z}_q, \mathbf{z}_r, \mathbf{z}_s$ and $7^m \leq |r - s| < 7^{m+1}$ we can bound:

$$\left(\frac{7}{4}\right)^{m-1} \leq \frac{10\sqrt{964}}{21} < 14.89 < \left(\frac{7}{4}\right)^5 = 16.4130859375.$$

In this case we get $m - n \leq 5$.

This gives us at most 7^6 collinear points in this case, where previously we had a bound of 7^4.

Collinear points in W

Lemma (L. 2024)

In the first 10 million points of W there are no 7 collinear points.

The first example of 6 collinear points:

$(46, 40, 23)$ at index 109,
$(48, 41, 24)$ at index 113,
$(64, 49, 32)$ at index 145,
$(66, 50, 33)$ at index 149,
$(82, 58, 41)$ at index 181,
$(84, 59, 42)$ at index 185.

Collinear points in W

The index of the last new subword of length 7^5 is 9375904.

Corollary

In every $16807 = 7^5$ consecutive indices of W there are no 7 collinear points.

Collinear points in W

The index of the last new subword of length 7^5 is 9375904.

Corollary

In every $16807 = 7^5$ consecutive indices of W there are no 7 collinear points.

If $1 \leq |p - q| < 7$, and $\mathbf{z}_p, \mathbf{z}_q, \mathbf{z}_r, \mathbf{z}_s$ are collinear and $7^m \leq |r - s| < 7^{m+1}$, then $|r - s| < 7^6$.

Since each consecutive 7^5 indices have at most 6 collinear points, there are at most $7 * 6 = 42$ collinear points in 7^6 consecutive indices.

We have a bound of at most 42 collinear points in the $n = 0$ case!

Collinear points in W

The index of the last new subword of length 7^5 is 9375904.

Corollary

In every $16807 = 7^5$ consecutive indices of W there are no 7 collinear points.

If $1 \leq |p - q| < 7$, and $\mathbf{z}_p, \mathbf{z}_q, \mathbf{z}_r, \mathbf{z}_s$ are collinear and $7^m \leq |r - s| < 7^{m+1}$, then $|r - s| < 7^6$.

Since each consecutive 7^5 indices have at most 6 collinear points, there are at most $7 * 6 = 42$ collinear points in 7^6 consecutive indices.

We have a bound of at most 42 collinear points in the $n = 0$ case!

Together with the 188 bound in the $n > 0$ case, this proves the theorem.

Theorem (L. 2024)

There exists an infinite $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$-walk for which no 189 points are collinear.

Collinear trapezoidal prisms in 3 dimensions?

Can we improve the bound by considering lines through trapezoidal prisms in 3 dimensions?

Collinear trapezoidal prisms in 3 dimensions?

Can we improve the bound by considering lines through trapezoidal prisms in 3 dimensions?

- Stabbing line problem in 3 dimensions.
- A stabbing line for a set of convex polyhedra is an infinite line that intersects at least one facet of each polyhedron in the set.
- There is an $O(n^3 \log n)$ algorithm for finding $O(n^3)$ candidate stabbing lines (Avis, Wenger; 1988).
- We could adapt this to an $O(n^4 \log n)$ algorithm for finding the largest number of trapezoids intersected by a single line.
  - Far too slow

A finer partition of $7^n \leq |p-q| < 7^{n+1}$?

Can we use a finer partition to get $m - n \leq 2$?

It seems unlikely.

We need to get a distance ratio bound less than $(7/4)^3 = 5.359375$.

$c, d \in \{7, \ldots, 48\}$	$\frac{10\sqrt{964}}{7\sqrt{28}}$	8.38227
$c, d \in \{49, \ldots, 342\}$	$\frac{239\sqrt{14400}}{54\sqrt{7168}}$	6.27316
$c, d \in \{343, \ldots, 2400\}$	$\frac{1661\sqrt{236196}}{394\sqrt{115492}}$	6.02884
$c, d \in \{2401, \ldots, 16806\}$	???	???

Open Problems

- Can the bound be improved further? Is 6 the largest number of collinear points in this walk?
- Does there exist an infinite $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$-walk with fewer than 6 collinear points?
- Can we do better in higher dimensions?
- Can we compute more terms of $a(n)$, the smallest integer t such that every $\{\mathbf{i}, \mathbf{j}\}$-walk of length t is guaranteed to have at least n collinear points? Sequence `https://oeis.org/A231255`. Only the first 6 terms are known.

References

- Gerver and Ramsey; 1979. On certain sequences of lattice points.
- Gerver; 1979. Long walks in the plane with few collinear points.
- Ramsey; 1977. Fourier-Stieljes transforms of measures with a certain continuity property.
- Avis and Wenger; 1988. Polyhedral line traversals in space.
- Lidbetter; 2024. Improved bound for the Gerver-Ramsey collinearity problem.
- https://github.com/FinnLidbetter/avoiding-collinearity

Number of distinct subwords of length n

n	Number of distinct subwords of length n in λ
1	12
2	30
7	168
49	1320
343	9384
2401	65832
16807	460968