# Orbits of N-expansions with a finite set of digits

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# Joint work with ...

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and

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# The regular continued fraction expansions

It is well known that every real number $x$ can be written as a finite (in case $x \in \mathbb{Q}$) or infinite (regular) continued fraction of the form:

$$x = r_0 + \cfrac{1}{r_1 + \cfrac{1}{n_2 + \ddots + \cfrac{1}{r_n + \ddots}}} = [r_0; r_1, r_2, \dots, r_n, \dots], \qquad (1)$$

where $r_0 \in \mathbb{Z}$ such that $x - r_0 \in [0, 1)$, and $r_n \in \mathbb{N}$ for $n \geq 1$.

Such a *regular continued fraction expansion* (RCF) of $x$ is unique if and only if $x$ is irrational; in case $x \in \mathbb{Q}$ one has two expansions of the form (1).

# The regular continued fraction expansions

Apart from this, very many things are known about the RCF ... too many to mention, but some are:

- There is an underlying dynamical system, which is ergodic (actually, it has much stronger mixing properties). The invariant measure was found by Gauss on October 25, 1800 (mentioned in his mathematical diary). Later, in 1812, Gauss asked Lagrange about the speed of convergence of the Lebesgue measure to this invariant (*Gauss*) measure. This problem was solved more than a century later independently by Kuzmin and Lévy.

  Lévy and in the 1930ies Khintchine used this *Gauss-Kuzmin-Lévy* result to get very many properties of the RCF for almost all $x$ (a.a. with respect to Lebesgue measure).

  After the Second World War Ryll-Nardzewski showed that these results can also be obtained using the ergodic theorem. In fact, Doeblin also knew this (before WWII !).

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The natural extension of the RCF (and many related continued fraction expansions, in particular Nakada's $\alpha$-expansions) was obtained by Shigeru Tanaka, Shunji Ito and Hitoshi Nakada.

These natural extensions were fundamental in solving the so-called *Doeblin-Lenstra* conjecture, by Wieb Bosma, Henk Jager and Freek Wiedijk.

Apart from this, these natural extensions were also fundamental in obtaining many arithmetic results for the RCF and variants (like Nakada's $\alpha$-expansions).

An important arithmetic properties of the RCF (and many of it's siblings) is:

*A real number $x$ is a quadratic irrational if and only if the RCF-expansion of $x$ is eventually periodic.*

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# $N$-expansions

In 2008, Ed Burger and his co-authors introduced a new continued fraction expansions, which are a nice variation on the RCF-expansion.

Let $N \in \mathbb{N}$ be a fixed positive integer, and define the map $T_N : [0,1) \to [0,1)$ by:

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \ x \neq 0; \quad T_N(0) = 0.$$

Of course, choosing $N = 1$ yields the Gauss-map and the RCF.

Setting $d_1 = d_1(x) = \lfloor N/x \rfloor$, and $d_n = d_n(x) = d_1\left(T_N^{n-1}(x)\right)$, whenever $T_N^{n-1}(x) \neq 0$, we find:

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \ddots + \cfrac{N}{d_n + T_N^n(x)}}}. \tag{2}$$

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Taking finite truncations yield the convergents, which converge to $x$.

Following Burger *et al.*, we abbreviate

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \ddots + \cfrac{N}{d_n + \ddots}}}$$

by $x = [0; d_1, d_2, \ldots, d_n, \ldots]_N$.

# $N$-expansions

In fact, Burger *et al.* call any continued fraction of $x \in \mathbb{R}$ of the form:

$$x = n_0 + \cfrac{N}{n_1 + \cfrac{N}{n_2 + \ddots + \cfrac{N}{n_k + \ddots}}} = [n_0; n_1, n_2, \dots, n_k, \cdots]_N, \qquad (3)$$

where $N \in \mathbb{N}$ and $(n_i)_{i \geq 0} \in \mathbb{N}^{\mathbb{N}}$, an $N$-expansion of $x$.

Note that if the RCF-expansion of $x$ is given by $x = [r_0; r_1, r_2, \dots]$, with $r_0 \in \mathbb{Z}$ such that $x - r_0 \in [0, 1)$, and $r_n \in \mathbb{N}$ for $n \geq 1$, the continued fraction expansion:

$$x = [r_0; Nr_1, r_2, Nr_3, r_4, Nr_5, r_6, \dots]_N$$

is an continued fraction expansion of $x$ of the form (3) which is usually different from the $N$-expansion yielded by the map $T_N$.

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# $N$-expansions

In 2011, Anselm and Weintraub further studied $N$-expansions. They showed that every positive real number $x$ always has an $N$-expansion, and for $N \geq 2$ even infinitely many, and that rationals always have finite and infinite expansions.

Furthermore, in case $N \geq 2$ every quadratic irrational has both periodic and non-periodic expansions. In their algorithm to find an $N$-expansion of a real number $x$ there is a *best choice* for the partial quotient (i.e., digit), and if one always makes this best choice for the partial quotients one finds what they call the *best expansion* of $x$.

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# $N$-expansions

Neither Burger and his co-authors, nor Anselm and Weintraub use the ‘dynamic approach’ to $N$-expansions (i.e., use ‘Gauss’-maps). If one does so, it is trivial to see that every $x \in (0, N)$ has infinitely many expansions.

# $N$-expansions

If we 'stick' to the *greedy* expansion, many classical properties can be obtained:

- There is an underlying ergodic dynamical system, with an invariant measure which has a density reminiscent of the Gauss-density in case $N = 1$.
- Burger and his co-authors studied $N$-expansions because for every quadratic irrational number $x$ there exists an $N \in \mathbb{N}$ such that the $N$-expansion of $x$ is periodic with period-length 1 (and from their proof it then follows there are infinitely many such $N$).

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# $N$-expansions: a conjecture of Anselm and Weintraub

## Conjecture Anselm and Weintraub

There are quadratic irrational numbers $x$ and integers $N \geq 2$ such that the greedy $N$-expansion of $x$ is **not** periodic.

To illustrate this we will look at the number $[0; \overline{1,2,3}]_1$ and try to convert this into a 2-expansion:

$$x = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\ddots}}}}} = \cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{6+2\cfrac{1}{1+\cfrac{1}{2+\ddots}}}}} \tag{4}$$

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# $N$-expansions: a conjecture of Anselm and Weintraub

Here we see that the first time our multiplication trick works, but the second time we are stuck with the problem of finding the RCF expansion of $2 \cdot [0; \overline{1, 2, 3}]_1$.

To do this you can use a so called transducer. Transducers were introduced by Raney, and using the right one it has as it's input the RCF expansion $y = [r_0; r_1, r_2, r_3, r_4, \ldots]_1$ and gives $2y = [\tilde{r}_0; \tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \ldots]_1$ as it's output.

Because our input is a periodic RCF-expansion we know that our input is a quadratic irrational $y$. This implies that $2y$ is also a quadratic irrational as well, so we know that the expression we receive from Raney's transducer is again a periodic RCF-expansion aswell. When we apply Raney's transducer we go on untill we have determined the periodicity of $2y$.

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# $N$-expansions: a conjecture of Anselm and Weintraub

In our example we find that $2 \cdot [0; \overline{1, 2, 3}]_1 = [1; \overline{2, 1, 1, 2, 1, 7}]_1$. This means that we can write the continued fraction from equation (4) as:

$$x = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{6 + 2\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \ddots}}}}}} = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{6 + 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}$$

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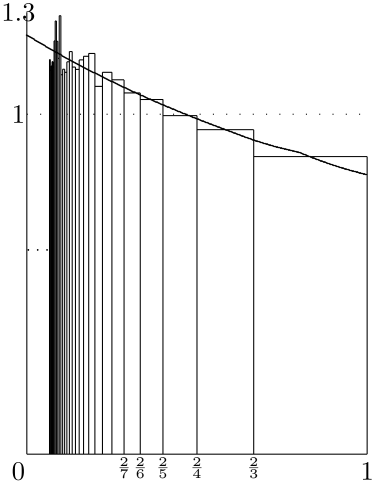


Figure: Using 70825 digits on the 2-CF of $[0; \overline{1,2,3}]_1$

# $N$-expansions with finitely many digits

In his MSc-thesis from 2015, Niels Langeveld studied an nice subclass of $N$-expansions: the $N$-expansions with finitely many digits.

For $N \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq \sqrt{N} - 1$, we define the continued fraction map $T_\alpha : [\alpha, \alpha + 1] \to [\alpha, \alpha + 1)$ as:

$$T_\alpha(x) := \frac{N}{x} - d(x),$$

where $d : I_\alpha \to \mathbb{N}$ is defined by:

$$d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor.$$

In the following we will write $I_\alpha$ for $[\alpha, \alpha + 1]$ and $I_\alpha^-$ for $[\alpha, \alpha + 1)$.

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For $\alpha \in (0, \sqrt{N} - 1]$ we define: $d_n = d_n(x) := d\left(T_\alpha^{n-1}(x)\right)$.

Applying $T_N$ on $I_\alpha$, we get:

$$x = T_\alpha^0(x) = \frac{N}{d_1 + T_\alpha(x)} = \frac{N}{d_1 + \frac{N}{d_2 + T_\alpha^2(x)}} = \cdots = \frac{N}{d_1 + \frac{N}{d_2 + \frac{N}{d_3 + \ddots}}} \quad ,$$

which is abbreviated by $x = [0; d_1, d_2, d_3, \ldots]_{N,\alpha}$. The numbers $d_i$, $i \geq 1$ are called the partial quotients of the $N$-continued fraction expansion of $x$.

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Given $N$, the sequence $T_\alpha^k(x), k = 0, 1, 2, \ldots$, is called the orbit of $x$ under $T_\alpha$.

It is obvious that $T_\alpha$ has one fixed point $f_d$ in each cylinder set $\Delta_d := \{x \in I_\alpha; \lfloor N/x - \alpha \rfloor = d\}$; it is easy to find that:

$$f_d = \frac{\sqrt{4N + d^2} - d}{2}.$$

Moreover, with $p_d$ we denote the discontinuity point between two cylinder sets $\Delta_{d-1}$ and $\Delta_d$; it is not hard to see that

$$p_d = \frac{N}{\alpha + d}.$$

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Since $T'_\alpha(x) = -N/x^2$ and because $0 < \alpha \leq \sqrt{N} - 1$, we have $|T'_\alpha(x)| > 1$ on $I^-_\alpha$. So the fixed points act as repellers and the maps $T_\alpha$ are expanding.

A cylinder set $\Delta_d$ is called full if $T_\alpha(\Delta_d) = I^-_\alpha$. When a cylinder set is not full, it contains either $\alpha$ (in which case $T_\alpha(\alpha) - d_{\max} < \alpha + 1$) or $\alpha + 1$ (in which case $T_\alpha(\alpha + 1) - d_{\min} > \alpha$), and is called incomplete.

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# A 2-expansion with $\alpha = \sqrt{2} - 1$

Much of what has been done, not just in the paper by Hitoshi and Jaap, but also in other subsequent papers, can be seen as a consequence of Niels Langeveld's thesis work, and the paper Niels and I wrote afterwards.

Niels made extensive simulation of the entropy in case $N = 2$, and I noticed there is an "entropy plateau", and we were able to show that for the values of $\alpha$ in the "plateau" the natural extension of the $N - \alpha$ expansions could be determined explicitly.

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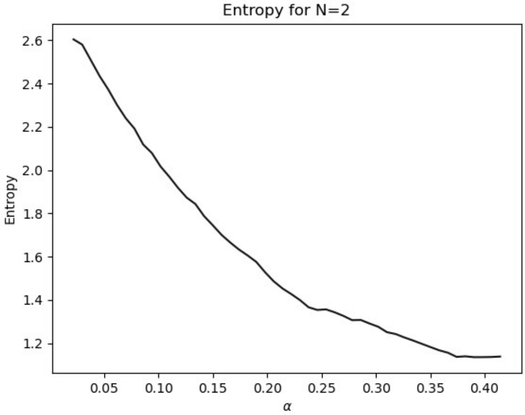
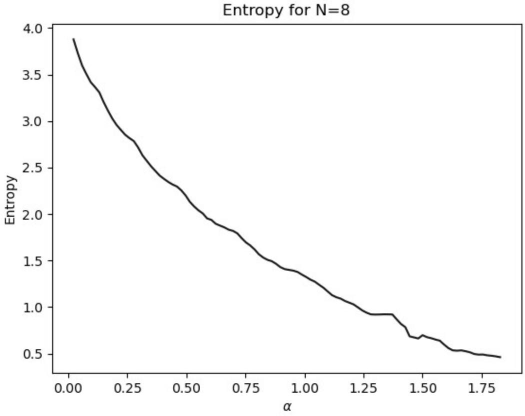
Here is a simulation of the case $N = 2$ by Yufei Chen, which s exactly the same as Niels' simulation:



# A 2-expansion with $\alpha = \sqrt{2} - 1$

Here is a simulation of the case $N = 8$ by Yufei Chen, where a similar phenomenon can be observed (although less pronounced) as in the case $N = 2$:



# A 2-expansion with $\alpha = \sqrt{2} - 1$

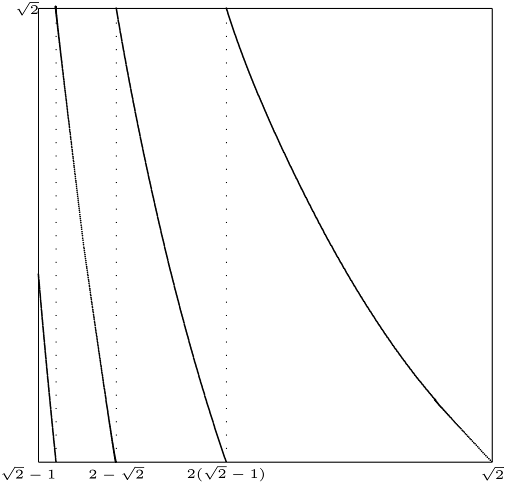
As an example we take $N = 2$ and $\alpha = \sqrt{2} - 1$.

Let $T_\alpha(x) : [\sqrt{2} - 1, \sqrt{2}] \to [\sqrt{2} - 1, \sqrt{2}]$ be defined by:

$$T_\alpha(x) = \begin{cases} \dfrac{2}{x} - 1 & \text{for} \quad 2(\sqrt{2}-1) < x \leq \sqrt{2} \\ \dfrac{2}{x} - 2 & \text{for} \quad 2 - \sqrt{2} < x \leq 2(\sqrt{2}-1) \\ \dfrac{2}{x} - 3 & \text{for} \quad \frac{1}{7}(6 - 2\sqrt{2}) < x \leq 2 - \sqrt{2} \\ \dfrac{2}{x} - 4 & \text{for} \quad \sqrt{2} - 1 \leq x \leq \frac{1}{7}(6 - 2\sqrt{2}) . \end{cases}$$

# A 2-expansion with $\alpha = \sqrt{2} - 1$

The map $T_\alpha$ looks like this:



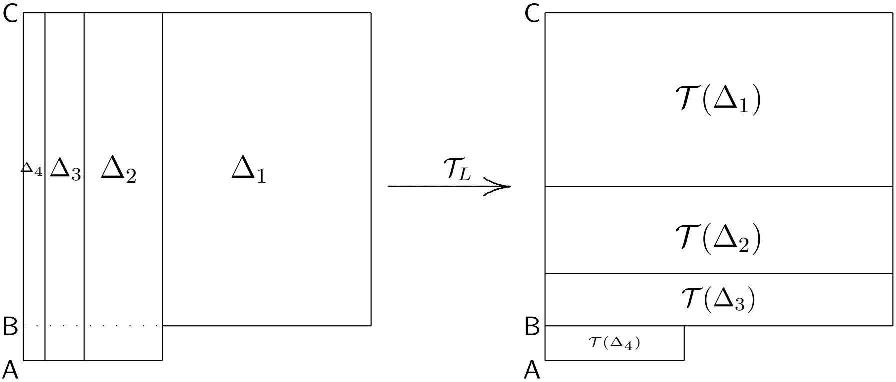
# A 2-expansion with $\alpha = \sqrt{2} - 1$

We can find the invariant measure for $T_\alpha$ (for $\alpha = \sqrt{2} - 1$ !!) by constructing the natural extension for $T_\alpha$. So we construct a domain $\Omega_\alpha \subset \mathbb{R}^2$ from which the map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$, given by

$$\mathcal{T}(x, y) = \left( T(x), \frac{N}{d(x) + y} \right) .$$

is a.e. bijective (and this domain is the 'smallest' with such property). This works well, since we know in advance the density of the invariant measure of $\mathcal{T}_\alpha$

# A 2-expansion with $\alpha = \sqrt{2} - 1$




We get the following equations for the heights $A, B$ and $C$:

$$A = \frac{2}{4+C}\,, \quad B = \frac{2}{3+C} \quad \text{and } C = \frac{2}{1+B}\,.$$

# A 2-expansion with $\alpha = \sqrt{2} - 1$

This results in $A = \frac{1}{2}(\sqrt{33} - 5)$, $B = \frac{1}{6}(\sqrt{33} - 3)$ and $C = \frac{1}{2}(\sqrt{33} - 3)$.

We find the following invariant density up to a normalizing constant:

$$f(x) = \begin{cases} \frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-5}{4+(\sqrt{33}-5)x} & \text{for } \sqrt{2}-1 < x \leq 2(\sqrt{2}-1) \\ \frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-3}{12+(\sqrt{33}-3)x} & \text{for } 2(\sqrt{2}-1) < x \leq \sqrt{2}\ . \end{cases}$$

At first we thought we were lucky to find such a nice natural extension. However, for $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$ by simulation we found that all these continued fraction expansions have the same entropy. We have the following theorem.

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**Theorem 3.1**

Let $N = 2$, and let $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2} - 1\right)$. Then $T^3(\alpha) = T^3(\alpha + 1)$.

And immediate consequence of this result is the following theorem.

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And immediate consequence of this result is the following theorem.

# A 2-expansion with $\alpha = \sqrt{2} - 1$

## Theorem 3.2

For $\alpha \in \left(\frac{\sqrt{33}-5}{2}, \sqrt{2}-1\right)$ the natural extension can be build explicitly as in the case $\alpha = \sqrt{2} - 1$. Moreover the invariant density is given by:

$$\begin{aligned} f(x) &= H\Big(\frac{D}{2+Dx}\mathbf{1}_{(\alpha,T(\alpha+1))} + \frac{E}{2+Ex}\mathbf{1}_{(T(\alpha+1),T^2(\alpha))} \\ &+ \frac{F}{2+Fx}\mathbf{1}_{(T^2(\alpha),\alpha+1)} - \frac{A}{2+Ax}\mathbf{1}_{(\alpha,T^2(\alpha+1))} \\ &- \frac{B}{2+Bx}\mathbf{1}_{(T^2(\alpha+1),T(\alpha))} - \frac{C}{2+Cx}\mathbf{1}_{(T(\alpha),\alpha+1)}\Big) \end{aligned}$$

with $A = \frac{\sqrt{33}-5}{2}, B = \sqrt{2}-1, C = \frac{\sqrt{33}-3}{6}, D = 2\sqrt{2}-2, E = \frac{\sqrt{33}-3}{2}, F = \sqrt{2}$ and $H^{-1} = \log\left(\frac{1}{32}(3+2\sqrt{2})(7+\sqrt{33})(\sqrt{33}-5)^2\right) \approx 0.25$ the normalizing constant.

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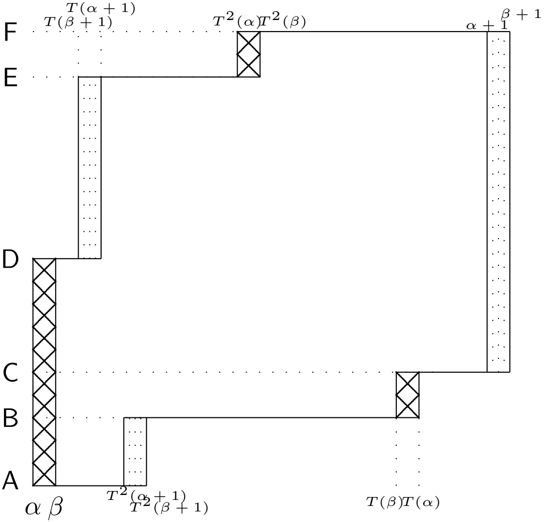

# $N$-expansions

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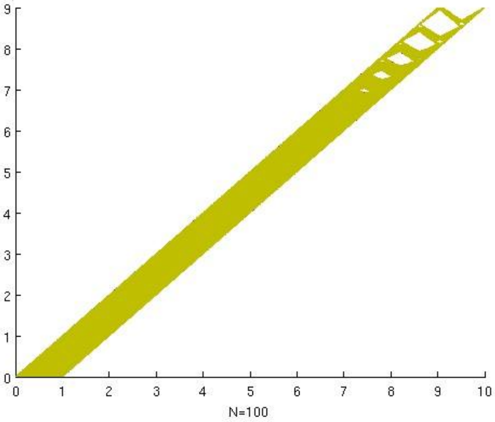

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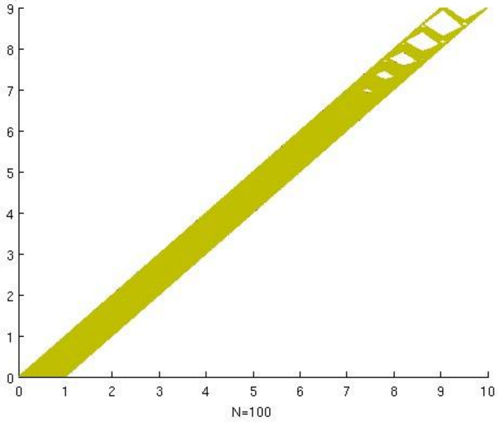

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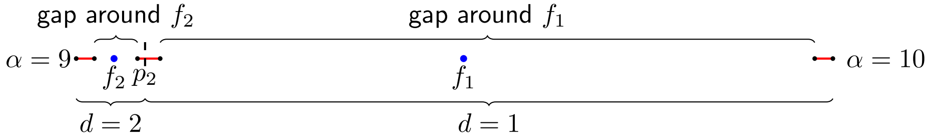

# $N$-expansions with finitely many digits

## Definition

An open interval $(a,b) \subset I_\alpha$ is called a *gap* of $I_\alpha$ if for almost every[a] $x \in I_\alpha$ there is an $n_0 \in \mathbb{N}$ for which $T_\alpha^n(x) \notin (a,b)$ for all $n \geq n_0$.

[a] excluding fixed points and pre-images of fixed points.

# Sufficient conditions for gaplessness

Depending on $N$ and $\alpha$, the number of cylinder sets may vary a lot:

**Lemma**

*Let* $d = \lfloor N/\alpha - \alpha \rfloor$. *Then the following relations hold:*

*if* $\alpha < 1$, *then* $d \geq N - 1$;
*if* $\alpha \geq 1$, *then** $d \leq N - 2$.

*) Ignoring the case that $\alpha$ is a one-point-cylinder set, in case $N/\alpha - d = \alpha + 1$.

Since $\lfloor N/\alpha - \alpha \rfloor$ is a decreasing function of $\alpha$, the number of cylinder sets increases to infinity as $\alpha$ decreases to $0$. Actually, we have infinitely many digits only when $\alpha = 0$.

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Depending on $N$ and $\alpha$ we define the *branch number*[1] $b$ as:

$$b(N,\alpha) := d_{\max} - d_{\min} - 1 \text{ (number of full cylinders)}$$
$$+ \frac{N}{\alpha} - d_{\max} - \alpha \text{ (length of image of leftmost cylinder)}$$
$$+ 1 - \left(\frac{N}{\alpha+1} - d_{\min} - \alpha\right) \text{ (length of image of rightmost cylinder}$$
$$= \frac{N}{\alpha} - \frac{N}{\alpha+1} = \frac{N}{\alpha(\alpha+1)},$$

which is a decreasing function of $\alpha$. As an example we have:

$$b(N,\alpha_{\max}) = \frac{N}{(\sqrt{N}-1)\sqrt{N}} = 1 + \frac{1}{\sqrt{N}-1},$$

so $b(N,\alpha) > 1$ for all $N \geq 2$; the number of cylinder sets is always at least 2.

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# Sufficient conditions for gaplessness

## Theorem

*Let $N \geq 9$ and $b(N, \alpha) \geq 4$. Then $I_\alpha$ contains no gaps.*

To prove this, we apply the following lemma:

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# Sufficient conditions for gaplessness

The next theorem provides a sufficient condition for the absence of gaps in case the branch number is relatively small:

**Theorem**

*The interval $I_\alpha$ is gapless in the following cases:*

(a) $I_\alpha = \Delta_d \cup \Delta_{d-1}$ *such that* $T_\alpha(\alpha) \geq f_{d-1}$ **and** $T_\alpha(\alpha+1) \leq f_d$*;*

(b) $I_\alpha = \Delta_d \cup \ldots \cup \Delta_{d-t}$*, with* $1 < t \leq d-1$*, while* $T_\alpha(\alpha) \geq f_{d-1}$ **or** $T_\alpha(\alpha+1) \leq f_{d-t+1}$.

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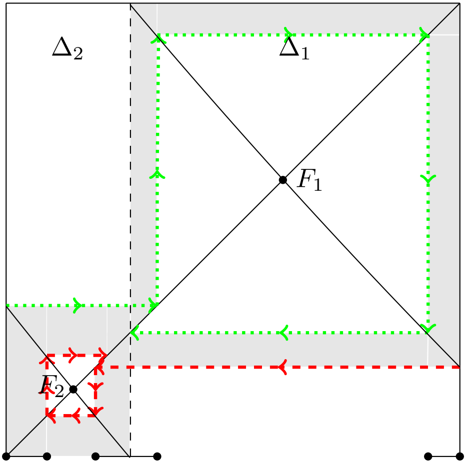
# Gaps in the case of two or three cylinder sets

**Lemma**

*Let* $I_\alpha = \Delta_d \cup \Delta_{d-1}$*. If* $T_\alpha(\alpha) < f_{d-1}$*, there is a gap containing* $f_{d-1}$*; if* $T_\alpha(\alpha+1) > f_d$*, there is a gap containing* $f_d$*.*

The lemma allows for the situation in which both $T_\alpha(\alpha) < f_{d-1}$ and $T_\alpha(\alpha+1) > f_d$, in which case there are gaps around both fixed points; see the next figure.

# Gaps in the case of two or three cylinder sets




$$N = 100, \alpha = \frac{\sqrt{2311076} - 176}{151} = 8.9021\cdots$$

$$T_\alpha^3(\alpha) = p_2$$

# Gaps in the case of two or three cylinder sets

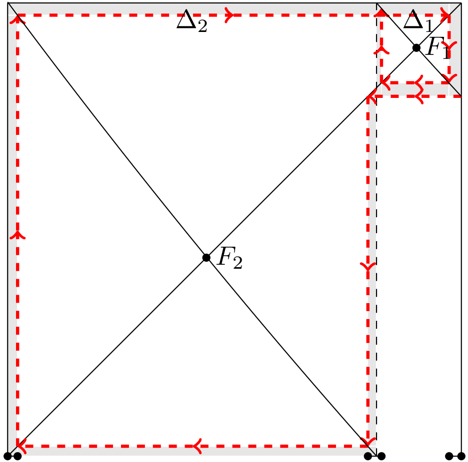
The second case in which gaps occur when $I_\alpha$ exists of two cylinder sets only is given by the following lemma.

## Lemma

*Let* $I_\alpha = \Delta_d \cup \Delta_{d-1}$. *If* $T_\alpha(\alpha) < f_{d-1}$ *and* $T_\alpha^3(\alpha) < f_d$, *or if* $T_\alpha(\alpha+1) > f_d$ *and* $T_\alpha^3(\alpha+1) > f_{d-1}$, *then both* $f_{d-1}$ *and* $f_d$ *are contained in a gap.*

The conditions of the lemma imply that one of the cylinders is almost full if not complete, while the other is very small; see the next figure.

# Gaps in the case of two or three cylinder sets




$$N = 100, \alpha = \frac{\sqrt{409}-3}{2} = 8.61187\cdots$$

$$\alpha = f_3$$

# Gaps in the case of two or three cylinder sets

Now suppose $I_\alpha$ consists of three cylinder sets. Then we have:

**Lemma**

*Let* $I_\alpha = \Delta_d \cup \Delta_{d-1} \cup \Delta_{d-2}$*, and suppose that* $T_\alpha(\alpha) < f_{d-1} < T_\alpha(\alpha+1)$*. Let* $a = \max\{T_\alpha(\alpha), T_\alpha^2(\alpha+1)\}$ *and* $b = \min\{T_\alpha^2(\alpha), T_\alpha(\alpha+1)\}$*. Then* $(a,b)$ *is a gap containing* $f_{d-1}$*.*

Since the branch number is a decreasing function of $\alpha$, there will be a minimal $\alpha$ such that the condition $T_\alpha(\alpha) < f_{d-1} < T_\alpha(\alpha+1)$ of the lemma can be fulfilled in case $I_\alpha$ consists of three cylinder sets. So we have

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*Let* $I_\alpha = \Delta_d \cup \Delta_{d-1} \cup \Delta_{d-2}$*. Then* $I_\alpha$ *is gapless when*

$$\alpha \leq \min_\alpha\{0 < \alpha \leq \sqrt{N} - 1 : T_\alpha(\alpha) = f_{d-1} \text{ and } b(N,\alpha) < 2\}.$$

# Gaps in the case of two or three cylinder sets

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# One huge gap in the case of four cylinder sets

The computer simulations of Kraaikamp and Langeveld suggest that gaps only occur when $I_\alpha$ consists of two or three cylinders sets. We thought that for $\min_\alpha\{0 < \alpha \le \sqrt{N} - 1 : T_\alpha(\alpha) = f_{d-1}$ and $b(N, \alpha) < 2\}$ there would be no gaps, especially when $b(N, \alpha) \ge 2$.

But this is not the case. That is, not always. For some $N$ arrangements exist with four cylinders and then one huge gap containing both fixed points of the full cylinders. The ones nearest to $N = 100$ are $N = 83$ and $N = 109$. Since the outer cylinders in these cases are very narrow, we will give two arrangements associated with a small $N$ containing a gap.

The next two figures, where $N = 11$ and $\Upsilon$ consists of four cylinders, mark the $\alpha$-interval on which one huge gap exists. Note that this interval is quite small.

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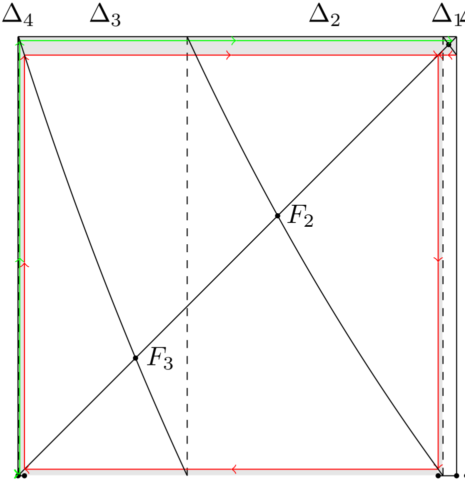
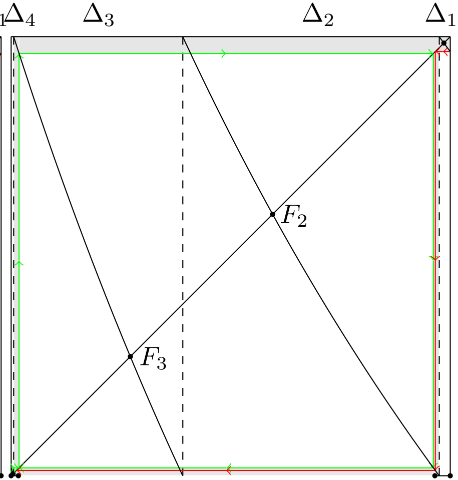



$N = 11, \alpha = 1.8719$

$T_\alpha(\alpha + 1) = T_\alpha^3(\alpha + 1)$




$N = 11, \alpha = 1.8687$

$T_\alpha(\alpha) = T_\alpha^3(\alpha)$

# Concluding Remarks

Up to now, most of this presentation was based on a paper by Jaap, Hitoshi, and myself. Recently, Jaap and I looked at the case that $\alpha = \alpha_{\max} = \sqrt{N} - 1$. In a paper in `Integers` we show that for this "simple case" the number of gaps is a monotonically non-deceasing and unbounded funtion of $N$, and also that the lagest gap tends to "swallow up" most of the space. We even found an explicit expression of the number of gaps as a function of $N$ ... a bit technical, so I decided to skip it. At the moment we are working on the case of general $\alpha$. As you can imagine the situation is even richer and more complicated.

# Concluding Remarks

In Autumn 2022, Yufei Chen and I submitted a paper (which will appear in Tohoku Math. J.) in which we showed that one can always find (so for every $N \in \mathbb{N}$, $N \geq 2$), intervals where for different values of $\alpha$ in such intervals the corresponding natural extensions are isomorphic (and therefore have the same entropy. In this way, Yufei and I generalized what Niels Langeveld and I did. Note that we have matching in these cases, and the idea of "quilting" can be applied here.

At the end of last year, Niels Langeveld and I submitted a paper in which we showed that for every $N \in \mathbb{N}$, $N \geq 2$, there are intervals of $\alpha$ where **no** matching occurs at all. This is remarkable, as matching seemed to be a 'natural phenomena' for various kinds of families of continued fraction transformations. Usually matching occurs almost surely.

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Thank you for your attention!

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