On the evaluation of infinite products involving automatic sequences

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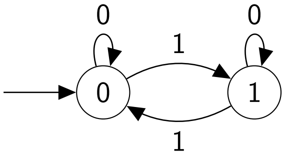
J.O. Shallit, On infinite products associated with sums of digits, J. Number Theory 21 (1985) 128–134.

$$u_0 = 0 \qquad u_{i+1} = \begin{cases} 0 & \text{if } \prod_{0 \le j \le i} \left(\frac{2j+1}{2j+2}\right)^{(-1)^{u_j}} > \frac{1}{\sqrt{2}}, \\ 1 & \text{otherwise.} \end{cases}$$

Shallit conjectured that u_i equals the i^{th} entry t_i of the **Thue–Morse sequence**.

$$t_i = \# \text{ of ones in the binary expansion of } i \bmod 2.$$



Infinite products and automatic sequences

$$u_0 = 0$$

$$0.5 = \prod_{0 \le j \le 0} \left(\frac{2j+1}{2j+2} \right)^{(-1)^{u_j}} < \frac{1}{\sqrt{2}}$$

$$\therefore u_1 = 1$$

$$0.666... = \prod_{0 \le j \le 1} \left(\frac{2j+1}{2j+2} \right)^{(-1)^{u_j}} < \frac{1}{\sqrt{2}}$$

$$\therefore u_2 = 1$$

$$0.8 = \prod_{0 \le j \le 2} \left(\frac{2j+1}{2j+2} \right)^{(-1)^{u_j}} > \frac{1}{\sqrt{2}}$$

Infinite products and automatic sequences

J.-P. Allouche, H. Cohen, Dirichlet series and curious infinite products, Bull. London Math. Soc. 17 (1985) 531–538.

$q \in \mathbb{N}_{\geq 2}$

ζ: an r^{th} root of unity such that $\zeta \neq 1$ and $r|q$

$s_q(n)$: the sum of the digits of n in the q-ary expansion

$$f(\mathcal{S}) = \sum_{n=0}^{\infty} \frac{\zeta^{s_q(n)}}{(n+1)^{\mathcal{S}}}$$

$$g(\mathcal{S}) = \sum_{n=1}^{\infty} \frac{\zeta^{s_q(n)}}{n^{\mathcal{S}}}$$

Infinite products and automatic sequences

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$$
\begin{aligned}
f(\mathcal{S}) &= \sum_{a=0}^{q-1} \sum_{n=0}^{\infty} \zeta^{s_q(qn+a)} (qn+a+1)^{-\mathcal{S}} \\
&= \sum_{a=0}^{q-1} \zeta^{a} \sum_{n=0}^{\infty} \zeta^{s_q(n)} \left(q(n+1) - (q-a-1)\right)^{-\mathcal{S}} \\
&= \sum_{a=0}^{q-1} \zeta^{q} q^{-\mathcal{S}} \sum_{n=0}^{\infty} \zeta^{s_q(n)} (n+1)^{-\mathcal{S}} \left(1 - \frac{q-a-1}{q(n+1)}\right)^{-\mathcal{S}} \\
&= \frac{1}{q^{\mathcal{S}}} \sum_{b=0}^{q-1} \zeta^{-b-1} \sum_{n=0}^{\infty} \zeta^{s_q(n)} (n+1)^{-\mathcal{S}} \sum_{k=0}^{\infty} \binom{\mathcal{S}+k-1}{k} \frac{b^k}{(n+1)^k q^k},
\end{aligned}
$$

where $b = q - a - 1$. (The expression involving "0^0" is omitted by convention).

Observe that $\sum_{b=0}^{q-1} \zeta^{-b-1} = 0$.

Theorem (Allouche & Cohen, 1985)

The Dirichlet series $f(\mathcal{S})$ can be analytically continued to an entire function which vanishes at all non-positive integers and satisfies

$$f(\mathcal{S}) = \sum_{k=1}^{\infty} \binom{\mathcal{S}+k-1}{k} \frac{f(\mathcal{S}+k)}{q^{\mathcal{S}}+k} \left(\sum_{b=0}^{q-1} \zeta^{-b-1} b^k \right).$$

This can be applied to obtain

$$\zeta f(\mathcal{S})(1-\zeta^{q-1}q^{-\mathcal{S}}) = g(\mathcal{S})(1-q^{-\mathcal{S}}),$$

$$f'(0) = -\log q/(\zeta-1).$$

J.-P. Allouche, H. Cohen, Dirichlet series and curious infinite products, Bull. London Math. Soc. 17 (1985) 531–538.

$$
\begin{aligned}
f'(0) &= \sum_{k=1}^{\infty} q^{-k} f(k) \lim_{\mathcal{S}\to 0} \frac{1}{\mathcal{S}} \binom{\mathcal{S}+k-1}{k} \left(\sum_{b=0}^{q-1} \zeta^{-b-1} b^k \right) \\
&= \sum_{k=1}^{\infty} q^{-k} f(k) k^{-1} \left(\sum_{b=0}^{q-1} \zeta^{-b-1} b^k \right) \\
&= \sum_{b=0}^{q-1} \zeta^{-b-1} \sum_{n=0}^{\infty} \zeta^{s_q(n)} \sum_{k=1}^{\infty} k^{-1} b^k q^{-k} (n+1)^{-k} \\
&= -\sum_{m=0}^{\infty} \zeta^{s_q(m)} \log \left(\frac{m+1}{q\left(\lfloor m/q \rfloor + 1\right)} \right).
\end{aligned}
$$

Theorem (Allouche & Cohen, 1985)

The following equality holds for q and ζ as specified:

$$\sum_{m=0}^{\infty} \zeta^{s_q(m)} \log\left(\frac{m+1}{q(\lfloor m/q \rfloor + 1)}\right) = (\log q)/(\zeta - 1).$$

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In[522]:= q = 8; ξ = i;
          N[ Sum[ξ^s[q, m] Log[(m + 1)/(q (Floor[m / q] + 1))], {m, 0, 2000000}] ]
          N[Log[q] / (ξ - 1)]
Out[523]= -1.03972 - 1.03972 i
Out[524]= -1.03972 - 1.03972 i
```

Future research

$s_q(n)$: the sum of the digits of n in the q-ary expansion

$b(n)$: the n^{th} *evil number* (i.e., with an even number of ones in its base-2 expansion)

$$2^{-\frac{\pi}{2\sqrt{3}}} = \prod_{n=1}^{\infty} \left(\frac{1 + b(n)}{2n} \right)^{\frac{\left(-\frac{1}{3}\right)^{\frac{s_2(b(n))}{2}}}{1+s_2(b(n))}}$$

This recalls a number of product identities due to Gosper, including:

$$2^{\frac{2}{\pi}} = \lim_{n\to\infty} \prod_{m=n}^{2n} \frac{\pi}{2\tan^{-1} m}.$$

Infinite products and the period-doubling sequence

J.M. Campbell, Infinite products involving the period-doubling sequence, Monatsh. Math. 203 (2024) 765–778.

Allouche and Cohen's approach:

Dirichlet series (involving the sum-of-digits function) $\longrightarrow$
summation "tricks" $\longrightarrow$
analytic continuation $\longrightarrow$
a closed form for the derivative at zero

A different but related approach:

Start with recursive properties of an automatic sequence $\longrightarrow$
"translate" these properties into an infinite product identity $\longrightarrow$
search for a reduction to an elementary function $\longrightarrow$
apply integral operators to an equivalent series identity

J.-P. Allouche, S. Riasat, J. Shallit, More infinite products: Thue-Morse and the gamma function, Ramanujan J. 49 (2019) 115–128.

The **Woods–Robbins product** identity:

$$\prod_{n=0}^{\infty}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}} = \frac{1}{\sqrt{2}}.$$

Related evaluations due to Allouche, Riasat, and Shallit:

$$\prod_{n=0}^{\infty}\left(\frac{4n+1}{4n+3}\right)^{(-1)^{t_n}} = \frac{1}{2},$$

$$\prod_{n=0}^{\infty}\left(\frac{(4n+1)(4n+4)}{(4n+2)(4n+3)}\right)^{t_n} = \frac{\pi^{3/4}\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}.$$

J.-P. Allouche, S. Riasat, J. Shallit, More infinite products: Thue-Morse and the gamma function, Ramanujan J. 49 (2019) 115–128.

$$f(a,b) := \prod_{n=1}^{\infty} \left(\frac{n+a}{n+b}\right)^{(-1)^{t_n}}$$

$$g(x) := \frac{f\left(\frac{x}{2}, \frac{x+1}{2}\right)}{x+1}$$

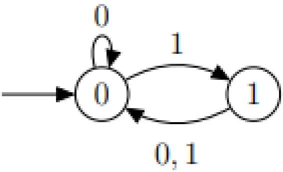
$$f(a,b) = \frac{g(a)}{g(b)}$$

"Translate" the recursive properties of t_n into an infinite product identity

$$(-1)^{t_{2n}} = (-1)^{t_n} \qquad (-1)^{t_{2n+1}} = -(-1)^{t_n}$$

The period-doubling sequence

d_n = the highest power of 2 dividing $n+1$, modulo 2.



The fixed point of the morphism on $\{0,1\}^*$ s.t. $0 \mapsto 01$ and $1 \mapsto 00$

$$(d_n : n \in \mathbb{N}_0) = (0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, \dots)$$

A motivating result

Catalan's constant: $G = \frac{1}{1^2} - \frac{1}{3^2} + \cdots$

$$\prod_{n=1}^{\infty} \left(\left(\frac{n+2}{n} \right)^{n+1} \left(\frac{4n+3}{4n+5} \right)^{4n+4} \right)^{d_n} = \frac{e^{\frac{2G}{\pi}}}{\sqrt{2}}$$

Observe that this is not of the form

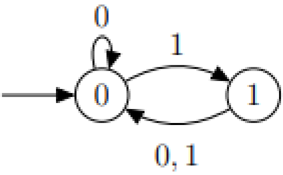
$$\prod_{n} R(n)^{a(n)}$$

for a rational function $R(n)$ and an automatic sequence $a(n)$.

Observe the combination of fundamental constants, which recalls

$$e^{\pi i} + 1 = 0.$$

The recursivity of the period-doubling sequence



$$d_{2n} = 0 \qquad d_{4n+1} = 1 \qquad d_{4n+3} = d_n$$

$$\prod_{n=1}^{\infty} \left(\frac{R(n)}{R(4n+3)} \right)^{d_n} = \prod_{n=0}^{\infty} R(4n+1)$$

A similar approach was applied by Allouche, Riasat, and Shallit to evaluate products of the forms

$$\prod_{n=1}^{\infty} S(n)^{(-1)^{t_n}} \quad \text{and} \quad \prod_{n=1}^{\infty} S(n)^{t_n}.$$

The period-doubling sequence and the Γ-function

$$\prod_{n=1}^{\infty}\left(\frac{R(n)}{R(4n+3)}\right)^{d_n}=\prod_{n=0}^{\infty}R(4n+1)$$

It was known to Euler that convergent infinite products of rational functions admit evaluations in terms of $\Gamma(x)=\int_0^\infty u^{x-1}e^{-u}\,du$
The reflection formula: $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}$

$$\prod_{n=1}^{\infty}\left(\frac{R(n)}{R(4n+3)}\right)^{d_n}=e(n)$$

$$\sum_{n=1}^{\infty}d_n\ln\left(\frac{R(n)}{R(4n+3)}\right)=\ln(e(n))$$

An experimental approach

Set

$$R(n) = 1 - \frac{1}{an^2 + bn + c},$$

where a, b, and c are real parameters such that $a \neq 0$ and $a + b + c \neq 0$. Then $\prod_{n=1}^{\infty} \left(\frac{R(n)}{R(4n+3)} \right)^{d_n} = \prod_{n=0}^{\infty} R(4n+1)$ reduces to

$$\frac{a+b+c-1}{a+b+c} \Gamma \left[\begin{matrix} \frac{-\sqrt{b^2-4ac}+10a+b}{8a}, \frac{\sqrt{b^2-4ac}+10a+b}{8a} \\ \frac{-\sqrt{-4ac+4a+b^2}+10a+b}{8a}, \frac{\sqrt{-4ac+4a+b^2}+10a+b}{8a} \end{matrix} \right].$$

Systematically search for combinations of input parameters that yield a reduction to an elementary function.

A family of products involving d_n

Lemma

Let $a \neq 0$ be a real parameter. The equality

$$\prod_{n=1}^{\infty} \left(\frac{16\left(a + 2an + an^2 - 1\right)}{16a + 32an + 16an^2 - 1} \right)^{d_n} = \cos\left(\frac{\pi}{4\sqrt{a}} \right)$$

then holds.

$b = 2a$ and $c = a$

$$\prod_{n=1}^{\infty} \left(\frac{16(2n+1)(2n+3)}{(8n+7)(8n+9)} \right)^{d_n} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

An integration-based approach

Theorem

The following product evaluation holds:

$$\prod_{n=1}^{\infty}\left(\frac{16(n+2)^{n+2}}{n^n}\frac{(4n+3)^{4n+3}}{(4n+5)^{4n+5}}\right)^{d_n} = \frac{1}{2}e^{\frac{2G}{\pi}}.$$

Set $a = \frac{1}{\alpha^2} > 0$

$$\sum_{n=1}^{\infty} d_n \ln\left(\frac{16(1+n-\alpha)(1+n+\alpha)}{(4+4n-\alpha)(4+4n+\alpha)}\right) = \ln\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)$$

An antiderivative with respect to α of the right-hand side is

$$\frac{1}{8}i\pi\alpha^2 - \alpha\ln\left(1+e^{\frac{i\pi\alpha}{2}}\right) + \alpha\ln\left(\cos\left(\frac{\pi\alpha}{4}\right)\right) + \frac{2i\mathrm{Li}_2\left(-e^{\frac{i\pi\alpha}{2}}\right)}{\pi}.$$

An integration-based approach

Lemma

For a real parameter $b \neq 0$, the equality

$$\prod_{n=1}^{\infty}\left(\frac{(4n+3)(4n+5)\left(bn^2+2bn-2\right)}{n(n+2)\left(16bn^2+32bn+15b-2\right)}\right)^{d_n} = \sqrt{2}\cos\left(\frac{\pi}{4}\sqrt{\frac{b+2}{b}}\right)$$

holds.

This can be used to prove the motivating result given earlier. We set b as $\frac{2}{\beta^2-1}$ for $0 < \beta < 1$.

$\longrightarrow$ a series expansion for $\ln\left(\sqrt{2}\cos\left(\frac{\pi\beta}{4}\right)\right)$

$\longrightarrow$ an antiderivative of the form

$$\frac{1}{8}i\pi\beta^2 - \beta\ln\left(1+e^{\frac{i\pi\beta}{2}}\right) + \beta\ln\left(\sqrt{2}\cos\left(\frac{\pi\beta}{4}\right)\right) + \frac{2i\operatorname{Li}_2\left(-e^{\frac{i\pi\beta}{2}}\right)}{\pi}.$$

Apéry's constant

Apéry's constant: $\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$

Theorem

The evaluation

$$\prod_{n=1}^{\infty} \left(\frac{(4n+3)^{16}(4n+5)^{16}}{2^{64} n(n+1)^{30}(n+2)} \right)^{\frac{d_n (n+1)^2}{2}} = \frac{e^{\frac{2G}{\pi} - \frac{21\zeta(3)}{8\pi^2}}}{\sqrt[4]{2}}$$

holds.

We follow a similar approach as before, but with the use of $\beta \int \cdot \, d\beta$ instead of $\int \cdot \, d\beta$.

Further results

Theorem

The evaluation

$$\prod_{n=1}^{\infty}\left(\frac{\left(\frac{(4n+3)(4n+5)}{16}\right)^{(4n+3)(4n+5)}}{(n+1)^{30(n+1)^2}(n(n+2))^{n(n+2)}}\right)^{d_n} = \frac{1}{2}e^{\frac{4G}{\pi}-\frac{21\zeta(3)}{4\pi^2}}$$

holds.

There is a close connection to *log-sine integrals* and the *Clausen function*, and this can be used to obtain the following.

$$\prod_{n=1}^{\infty}\left(\frac{16(n+3)^4(n+2)^{4n+8}}{(n+1)^{3n+3}(n+5)^{n+5}(2n+3)^4}\right)^{d_n} = \frac{\pi^2}{e^2}$$

Future/ongoing research

- Applications via Abel-type summation lemmas

- Mimic or build upon the given techniques concerning t_n and d_n using automatic sequences much more generally

- Develop techniques for evaluating

$$\prod_n (f(n))^{a(n)}$$

for non-rational functions $f(n)$

Future/ongoing research

Recall:

$$\prod_{n=1}^{\infty} \left(\frac{R(n)}{R(4n+3)} \right)^{d_n} = \prod_{n=0}^{\infty} R(4n+1).$$

Instead of restricting R to a rational function in the hope of the right-hand reducing to an *elementary* function, how could we instead make use of known results on integrals as in $\int \ln s(z)\, dz$ for a *special* function $s(z)$?

Gosper's integral:

$$\int_0^{\frac{1}{2}} \ln \Gamma(z+1)\, dz = \frac{\gamma + 3 \ln \pi}{8} - \frac{\ln 2}{6} - \frac{3\zeta'(2)}{4\pi^2} - \frac{1}{2}$$

$$\prod_{n=1}^{\infty} \left(\frac{1}{2n+4} \frac{(2n+2)^{14n+16}(2n+6)^{2n+5}}{(2n+1)^{4n+3}(2n+3)^{12n+17}} \right)^{t(n)} = \frac{A^{12}}{2^{\frac{13}{3}}}$$

A: The **Glaisher–Kinkelin constant**

Future/ongoing research

Recall the above antiderivatives involving

$$\mathrm{Li}_2\left(-e^{\frac{i\pi\alpha}{2}}\right).$$

This motivates an exploration of the relationship between automatic sequences and closed forms for:

- The Legendre χ_2-function,
- The inverse tangent integral Ti_2,
- The Clausen function Cl_2.