Hats, CAPs, and Spectres

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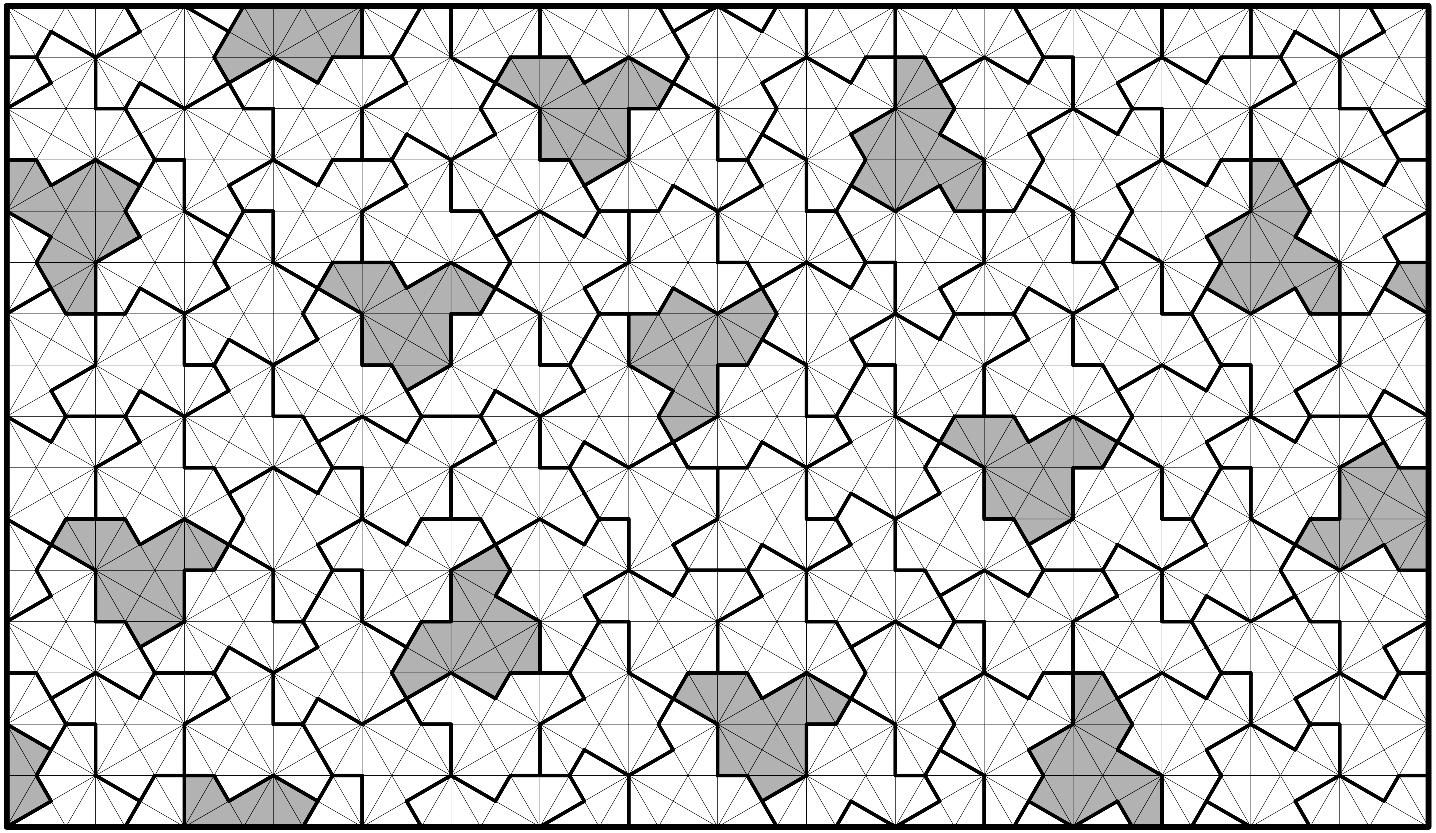
(joint work with Franz Gähler and Lorenzo Sadun)

Menu

- An aperiodic Hat
- Why care ?
- Shape changes
- The CAP tiling
- Embedding & window
- Back to the Hat
- Spectres & shadows
- Some stories repeat ...
- Outlook

Hats and anti-Hats

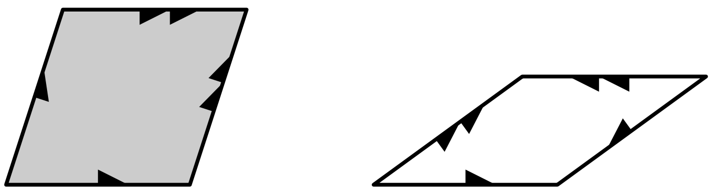


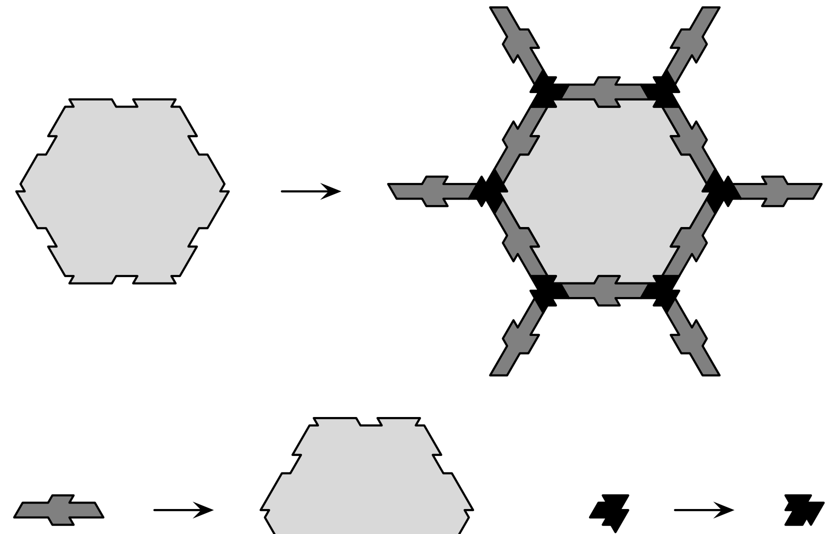
Smith, Myers, Kaplan and Goodman-Strauss, `arXiv:2303.10798`

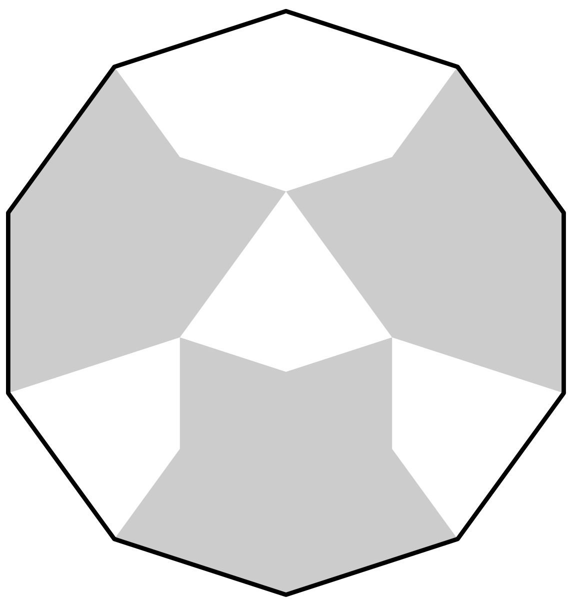
Hats and anti-Hats together can tile the plane, but only aperiodically

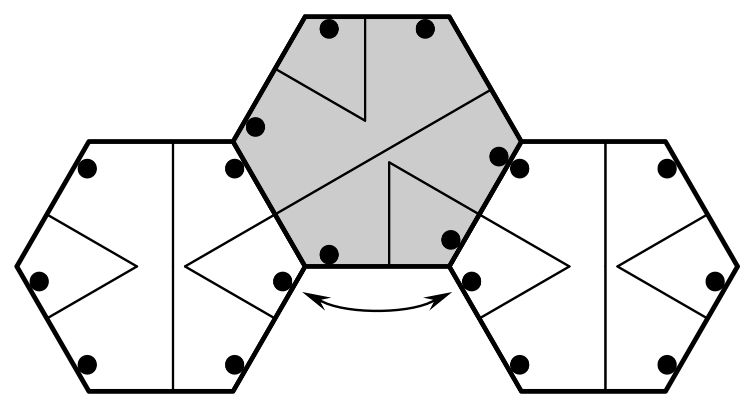
Ratio of Hats to anti-Hats is τ^4, with $\tau = \frac{1}{2}(1 + \sqrt{5}\,)$, so two classes

Predecessors









Why care ?

- Lots of fun ... for everyone !
- First purely geometric monotile
- Dense sphere packings and beyond
- The ‘crystallisation problem’
- Unique ground states ?
- Dynamics and topology
- Links to harmonic analysis
- Do we understand Euclidean space ?
- ...

The Hat family of tilings

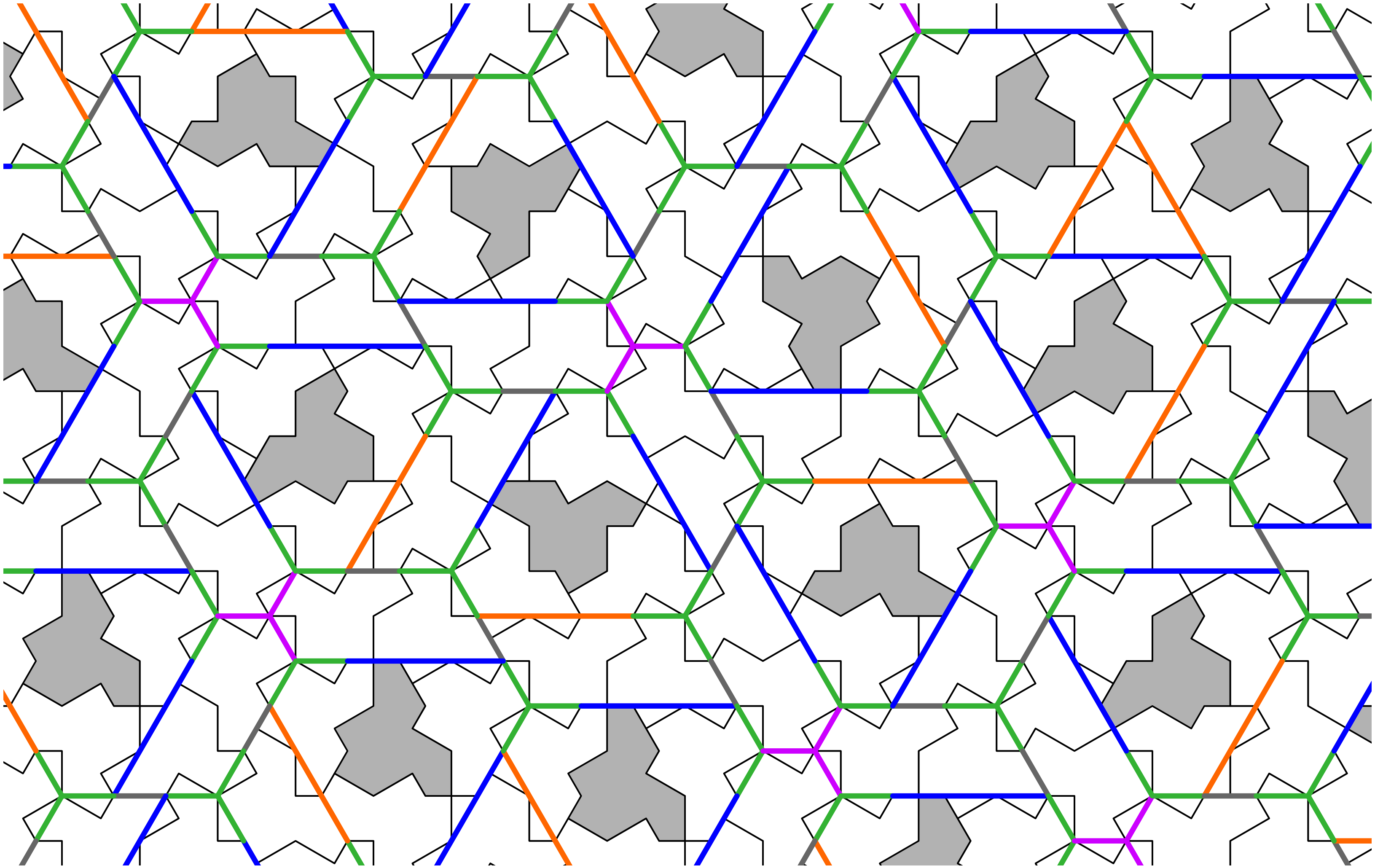
There are many hat-like tilings:

- Hats inevitably combine into clusters, forming meta-tiles
- A combinatorial inflation induces a hierarchical structure
- Many deformations are possible without changing the (local) combinatorial structure: Hats, Turtles, ... and the like

Questions:

- What is the overall structure of such tilings ?
- Is there Bragg diffraction ? Only Bragg diffraction ?
- Are they projection tilings ? Are they quasiperiodic ?

Hats and meta-tiles



Shape changes in $\mathbb{R}^2$

The Hat deformations can be understood as shape changes (SC):

- A shape change is a map f from (classes) of edges to $\mathbb{R}^2$
- SC functions must be closed: $\sum_{e \in \partial T} f(e) = 0$ for any tile T
- Some SC functions induce MLD transformations
- We care about SC modulo MLD transformations

Clark & Sadun: SC / MLD $\simeq \check{H}^1(\mathbb{X}, \mathbb{R}^2)$

$\mathbb{X} = \overline{\{t + \mathcal{T} : t \in \mathbb{R}^2\}}$ is the tiling space of a tiling $\mathcal{T}$

$\check{H}^1(\mathbb{X}, \mathbb{R}^2)$ is a (computable !) cohomology group of $\mathbb{X}$

More shape changes

Types of SCs / cohomology classes:

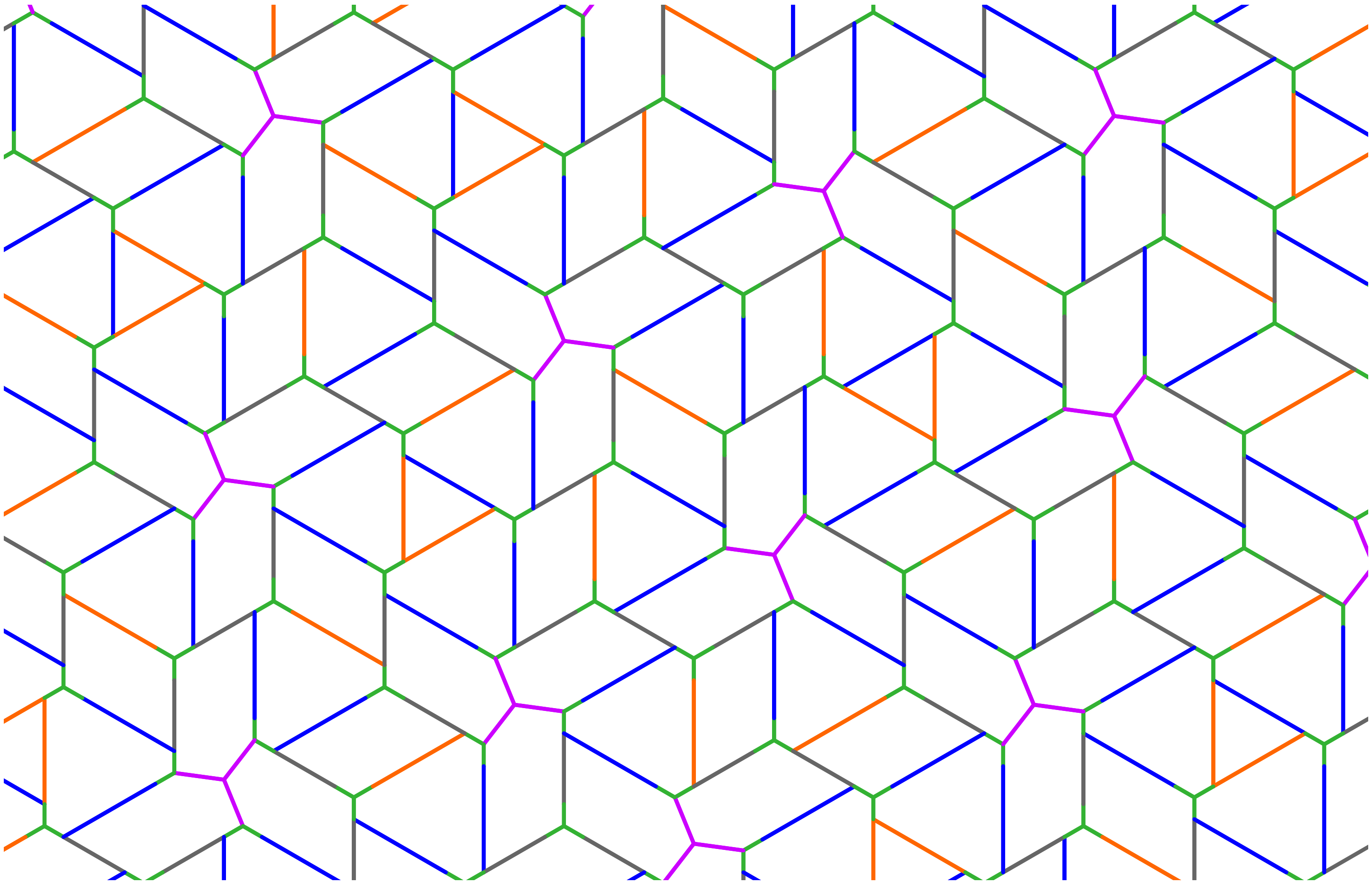
- Linear maps — essential change, but well understood
- Asymptotically negligible SCs — no effect on long-range order
- Other SCs — essential, but absent for the Hat family of tilings !

Theorem (BGS ‘23)

The Hat tiling possesses no non-trivial, asymptotically relevant SCs. The asymptotically negligible SCs lead to topologically conjugate dynamical systems.

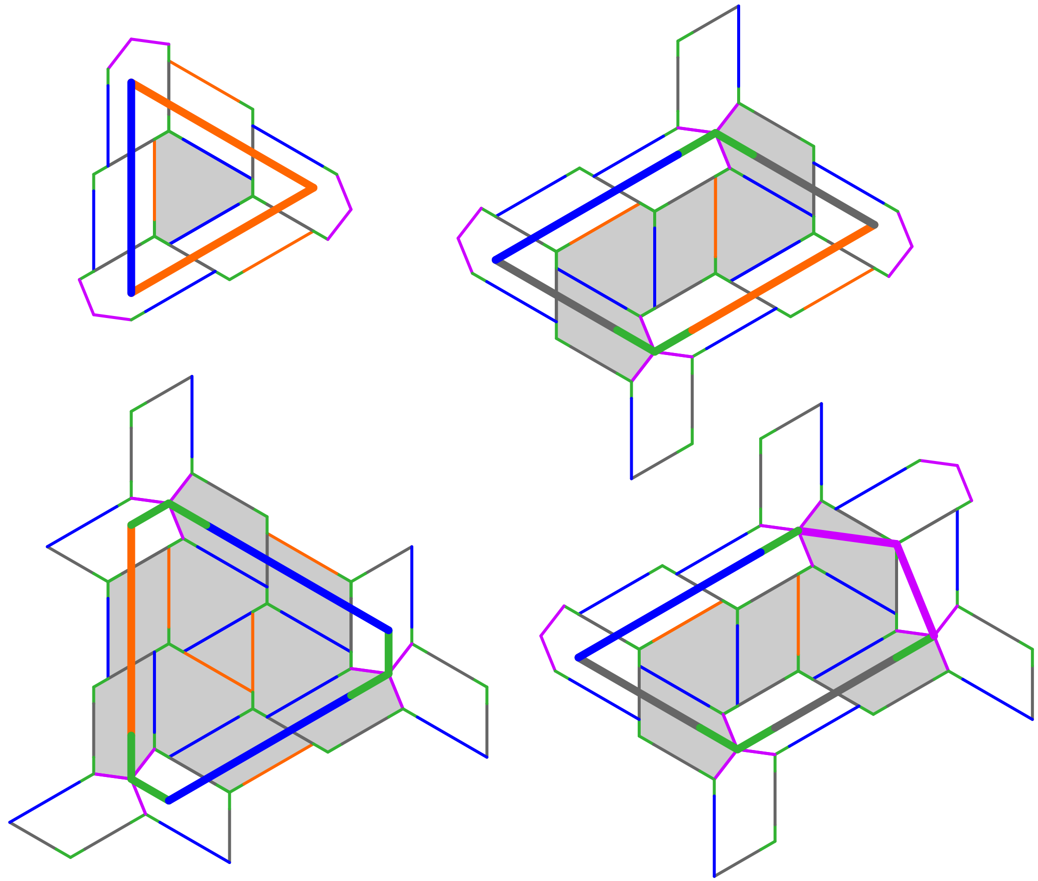
All members of the Hat family of tilings thus share their essential topological and spectral features.

Self-similar meta-tiles: CAP tiling



One particular choice of (deformed) meta-tiles leads to a self-similar tiling, thus with a geometric inflation, not only a combinatorial one

Inflation for CAP meta-tiles



Not a stone inflation, but each tile belongs to a unique super-tile

Spectrum of CAP

Theorem (BGS '23)

The dynamical system of the CAP tiling, under the translation action of $\mathbb{R}^2$, is strictly ergodic, and the spectrum is pure point, with continuous eigenfunctions (topological pure point spectrum).

The same applies to all tilings from the topological conjugacy class.

Proof

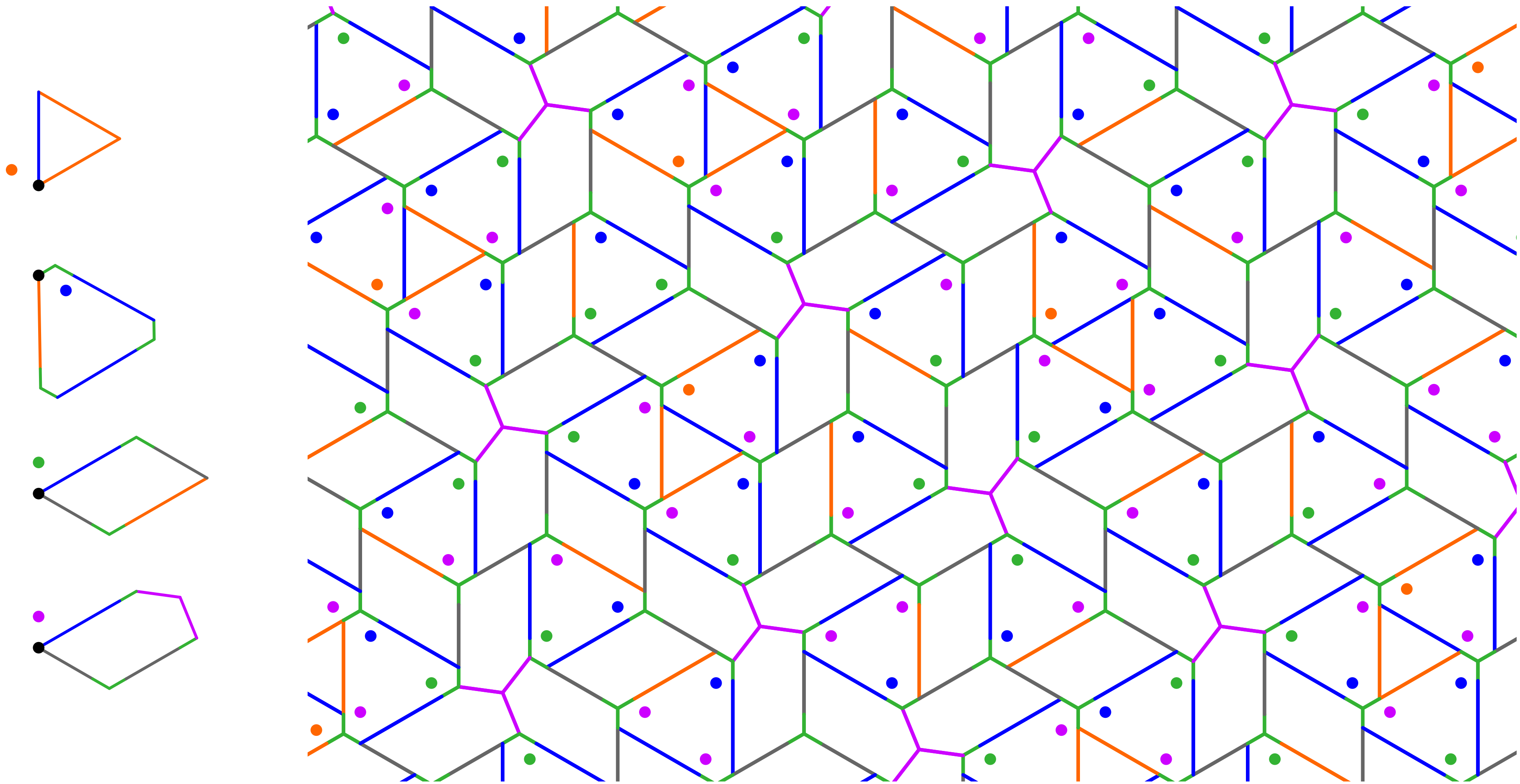
Solomyak's overlap algorithm can be applied, and gives: YES !

Strict ergodicity of the system and continuity of eigenfunctions follows from the inflation process. $\square$

Done ... but no real insight ...

Model set approach

Alternative: Construct MLD Delone set of control points, lift to $\mathbb{R}^4$, and prove that they form a one-component regular model set ...



Pure point spectra

CPS

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathbb{R}^m & \xrightarrow{\;\pi_{\text{int}}\;} & \mathbb{R}^m \\
\cup & & \cup & & \cup \;\text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\quad\star\qquad\quad} & & L^{\star}
\end{array}$$

(Meyer 1972)

(Moody 1997)

Model set

$$\boxed{\varLambda = \{ x \in L : x^{\star} \in W \}}$$

(assumed regular)

with $W \subset \mathbb{R}^m$ compact, $\lambda_{\mathrm{L}}(\partial W) = 0$

Diffraction

$$\boxed{\widehat{\gamma} = \sum_{k \in L^{\circledast}} |A(k)|^2 \, \delta_k}$$

pure point !!

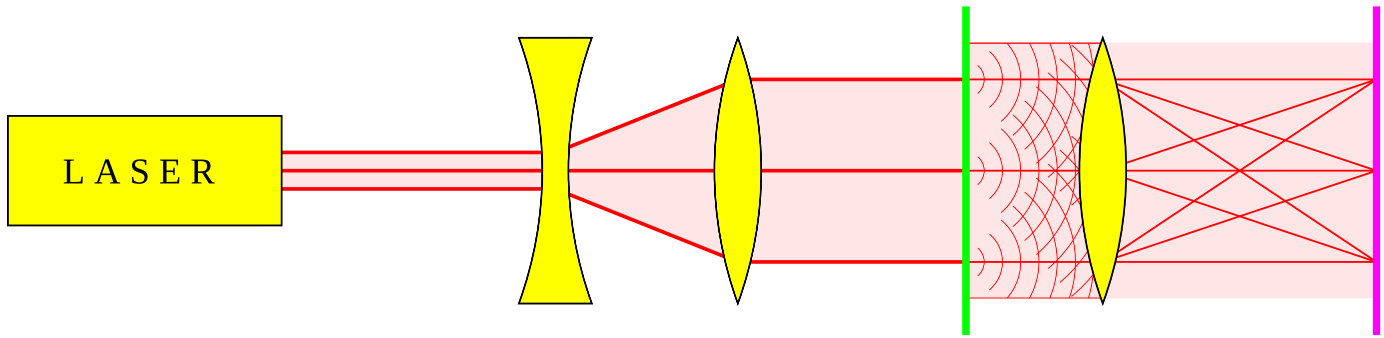
$(\omega = \delta_{\varLambda})$

with $L^{\circledast} = \pi(\mathcal{L}^{*})$

(Fourier module of $\varLambda$)

and amplitude $A(k) = \frac{\mathrm{dens}(\varLambda)}{\mathrm{vol}(W)} \, \widehat{1_W}(-k^{\star})$

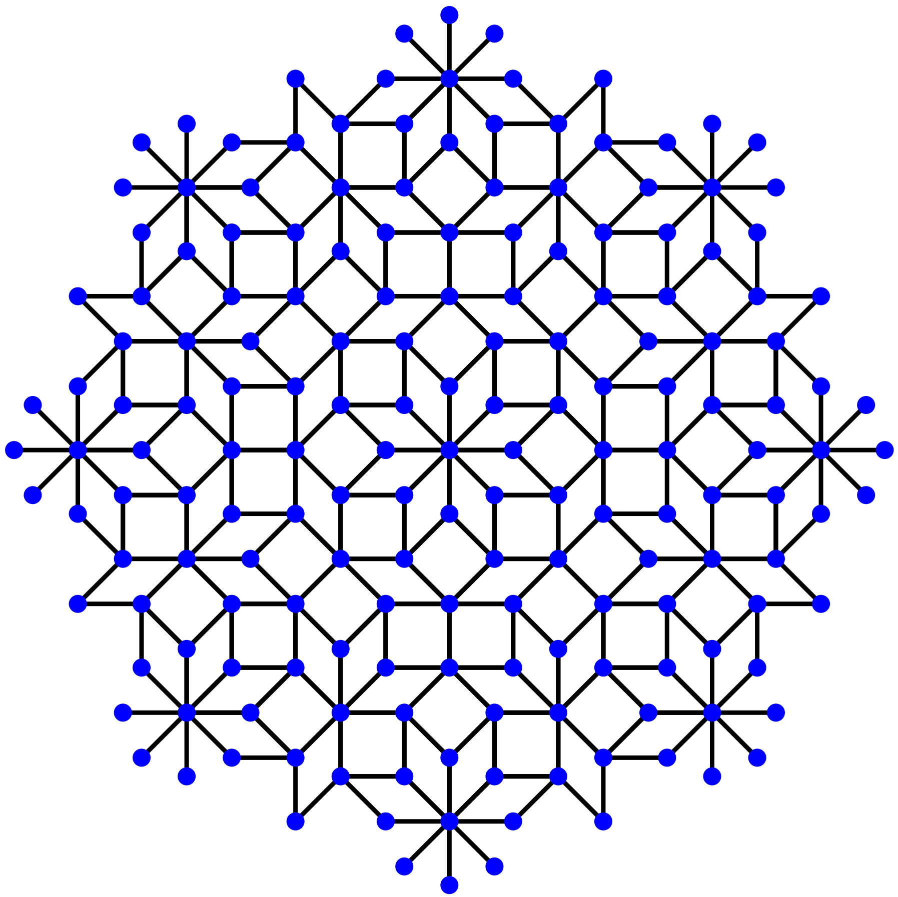
Diffraction and spectra



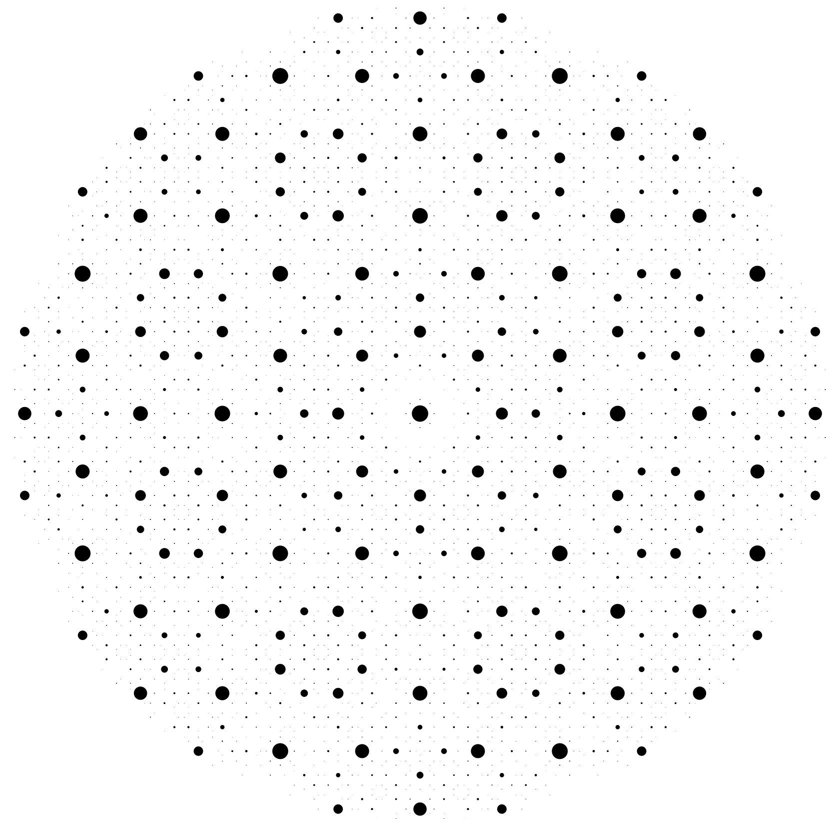
Wiener's diagram obstacle $f(x)$, with $\widetilde{f}(x) := \overline{f(-x)}$

$$\begin{array}{ccc} f & \xrightarrow{\;*\;} & f * \widetilde{f} \\ \mathcal{F}\big\downarrow & & \big\downarrow\mathcal{F} \\ \widehat{f} & \xrightarrow{\;|.|^2\;} & |\widehat{f}\,|^2 \end{array}$$

Example: Ammann–Beenker tiling

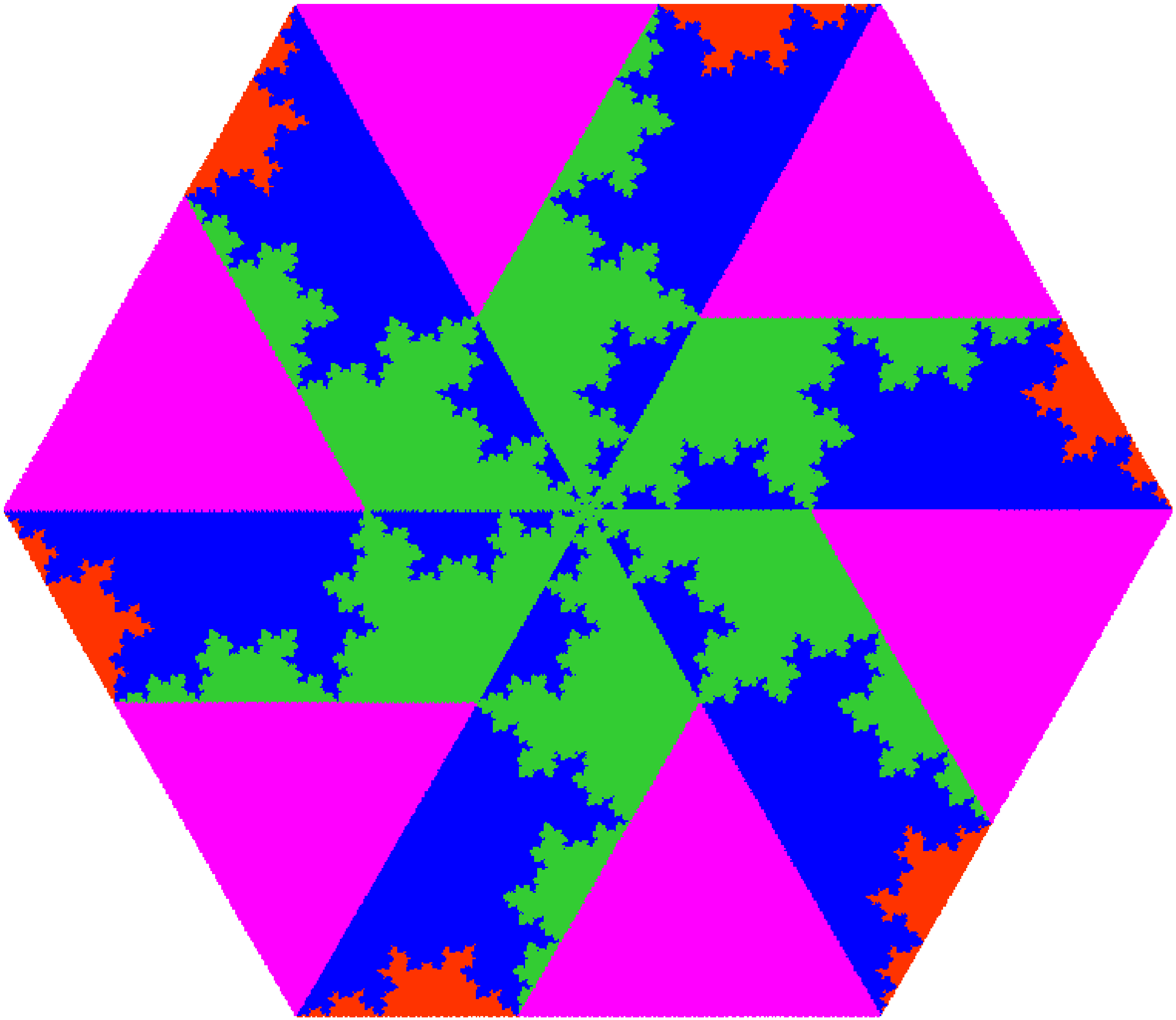


Tiling and point set



Diffraction

The CAP window



The model set structure

Inflation Delone set $\varLambda$ is a subset of a regular model set $\varLambda'$, where the latter defines a system with topological pure point spectrum.

Inflation determines $\rho = \mathrm{dens}(\varLambda)$ via relative tile frequencies (from PF theory) and tile areas (from geometry), giving $\rho = \frac{2}{3}\tau^2\sqrt{3}$.

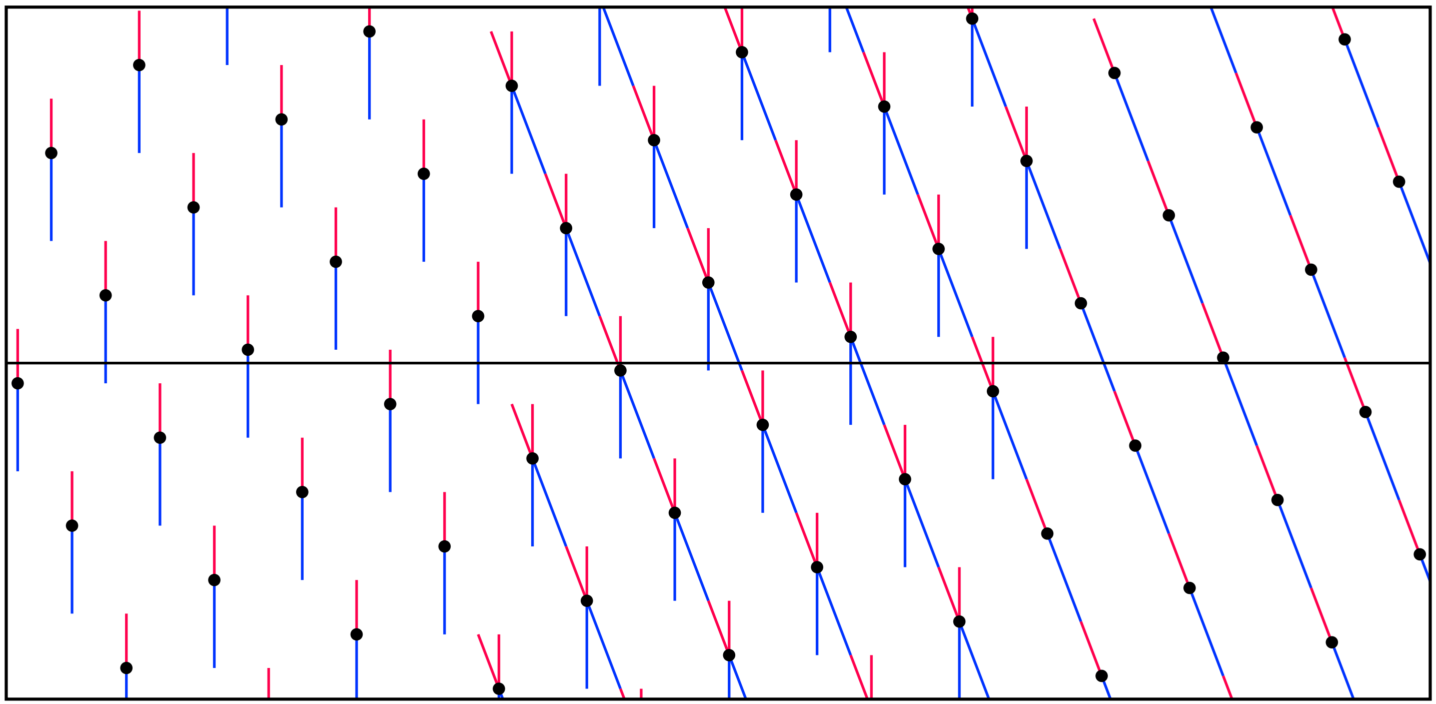
Projection gives $\rho' = \mathrm{dens}(\varLambda')$ as window area times density of embedding lattice, $\rho' = \frac{2}{3}\tau^2\sqrt{3}$, hence $\rho' = \rho$.

$\Longrightarrow$ $\varLambda$ has pure point diffraction

$\Longrightarrow$ $(\mathbb{X}_{\varLambda}, \mathbb{R}^2)$ has pure point dynamical spectrum

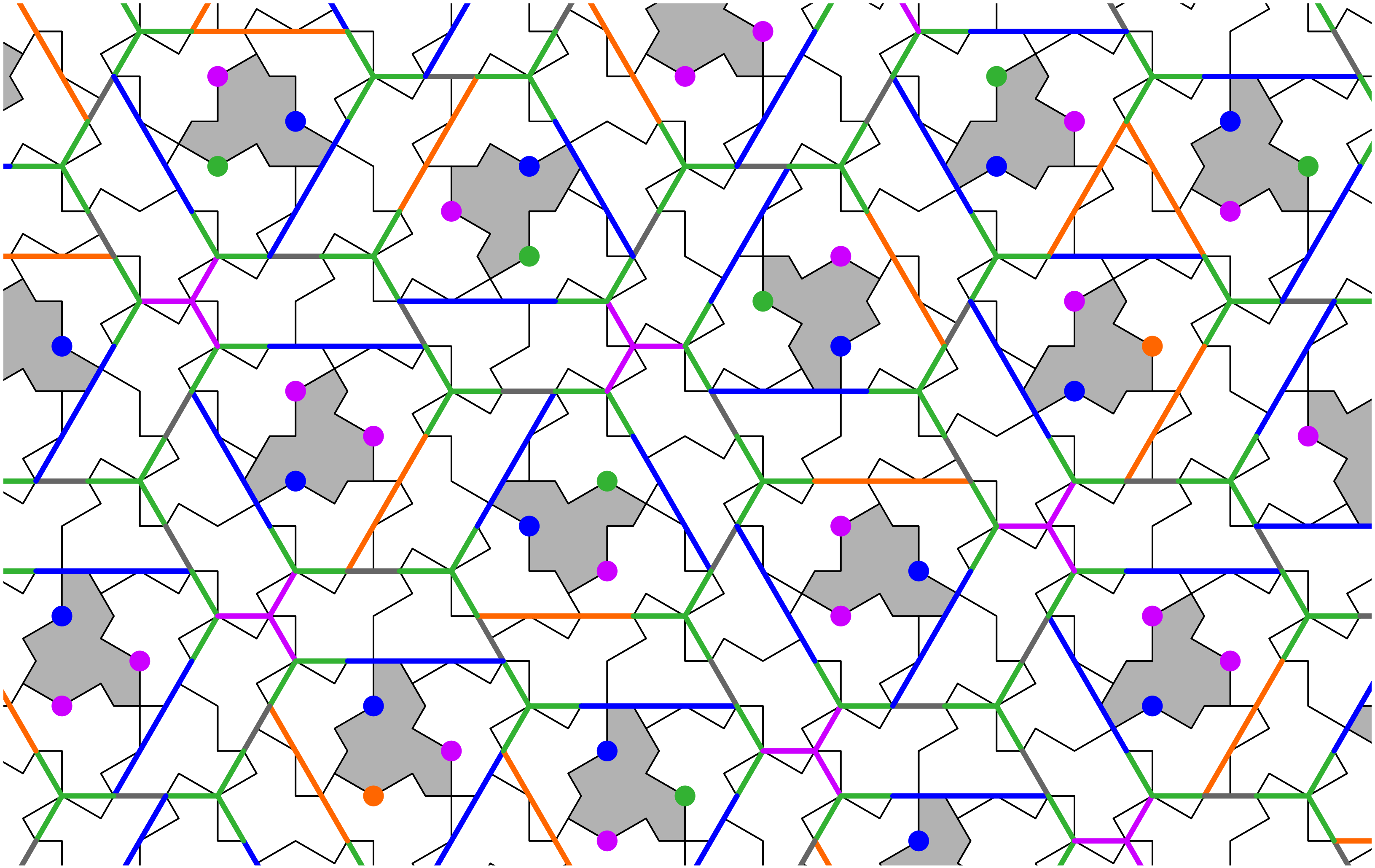
$\Longrightarrow$ Topological pure point spectrum

Example: Fibonacci shape change

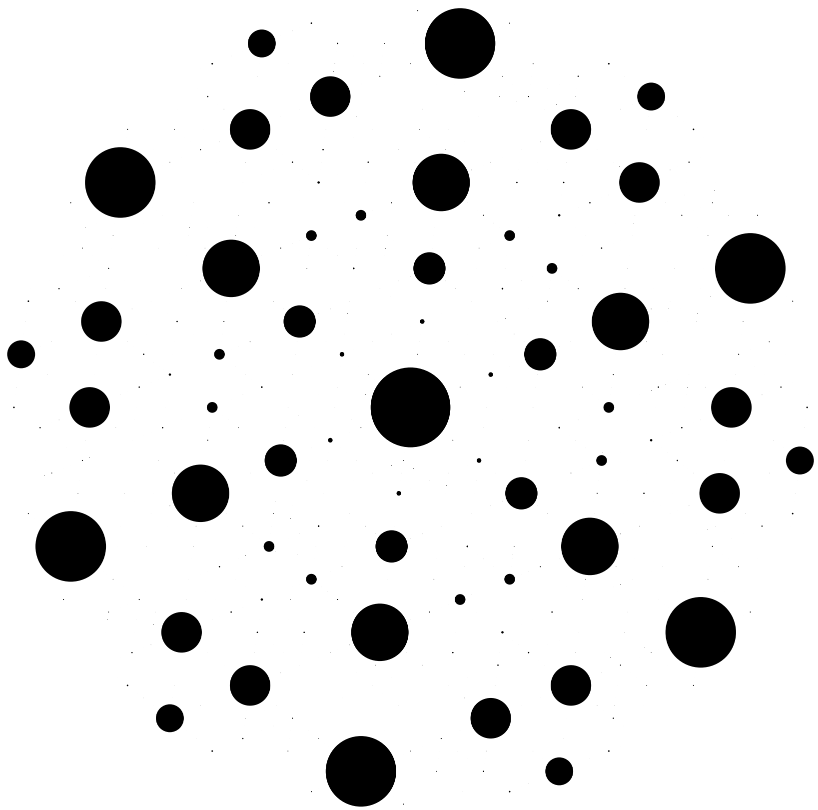


Shape changes	Change of relative interval lengths
Projection	Shear of window gives shape change
Meaning	Shape change is re-projection

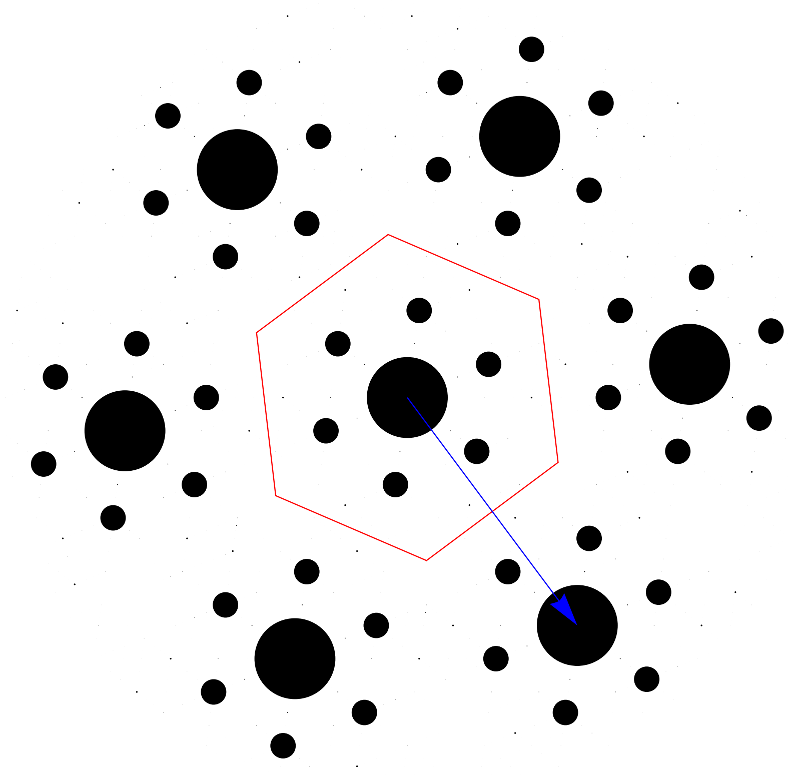
Back to the Hat



Diffraction spectra

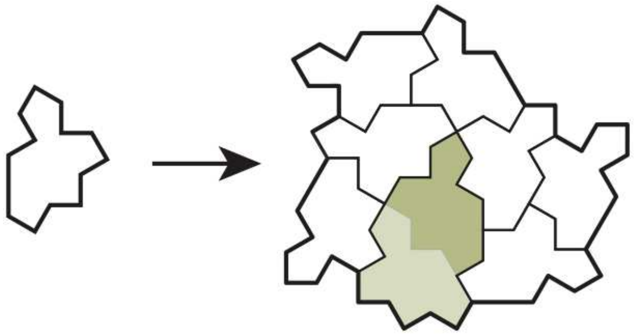


CAP



Hat

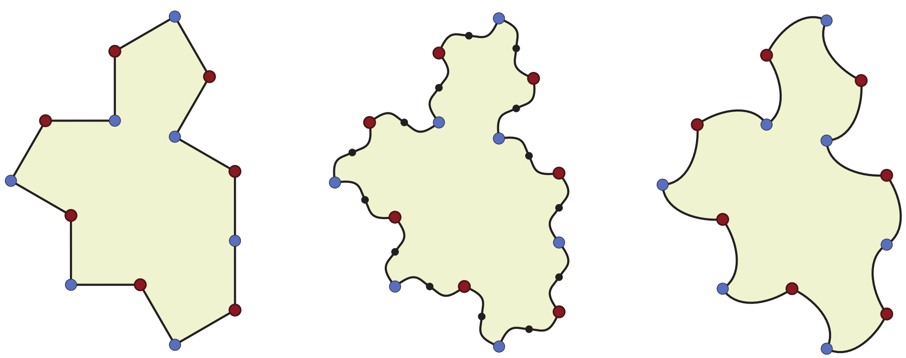
Spectres and shadow Spectres



Smith, Myers, Kaplan and Goodman-Strauss, `arXiv:2303.17743`

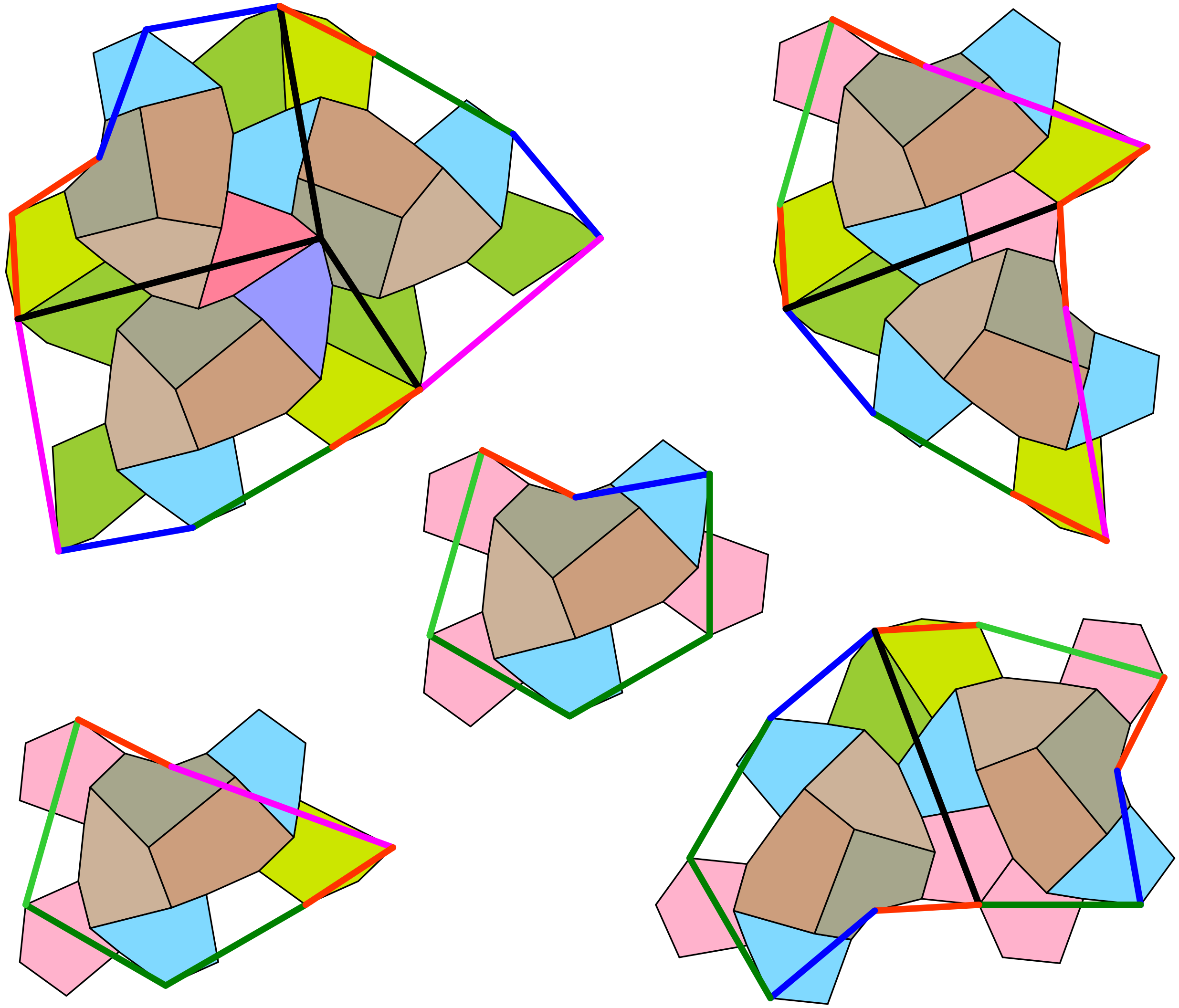
`https://cs.uwaterloo.ca/~csk/spectre/`

Frequency ratio of Spectres to shadow-Spectres is $4 + \sqrt{15}$

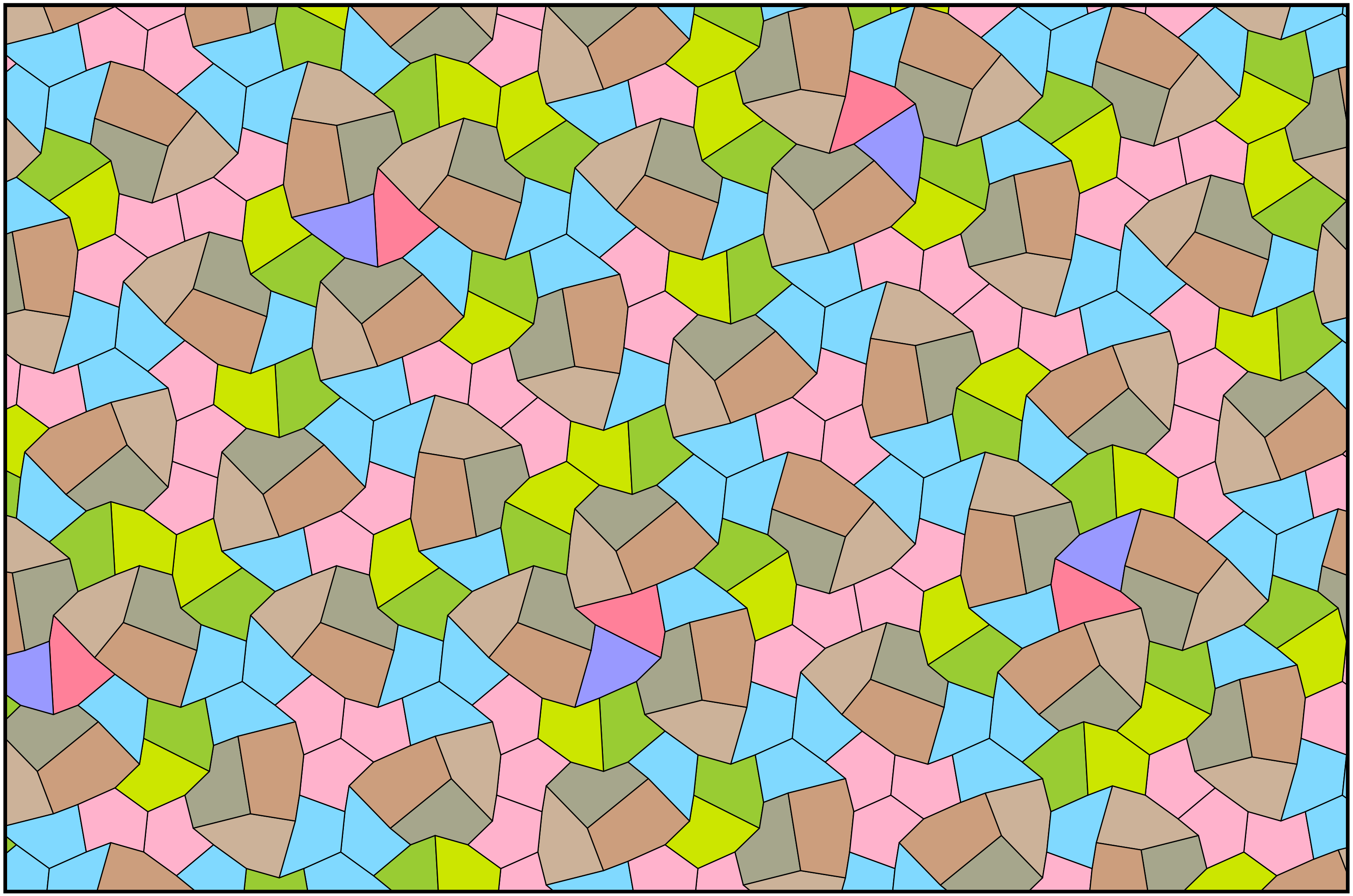


Modification of a lattice polygon to a real monotile

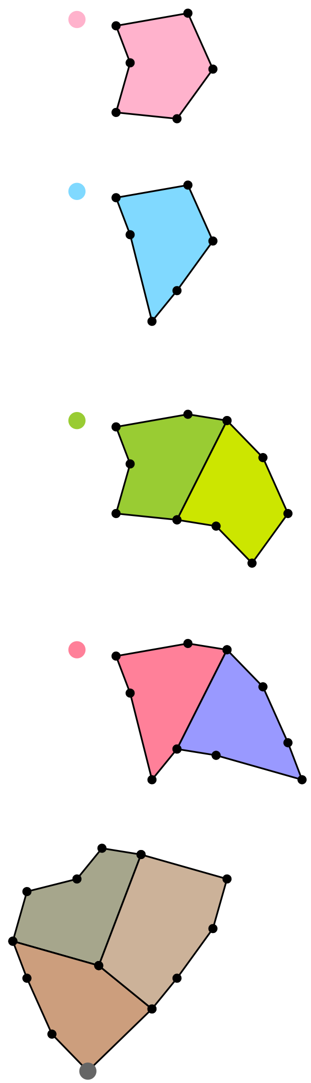
Same old story ...

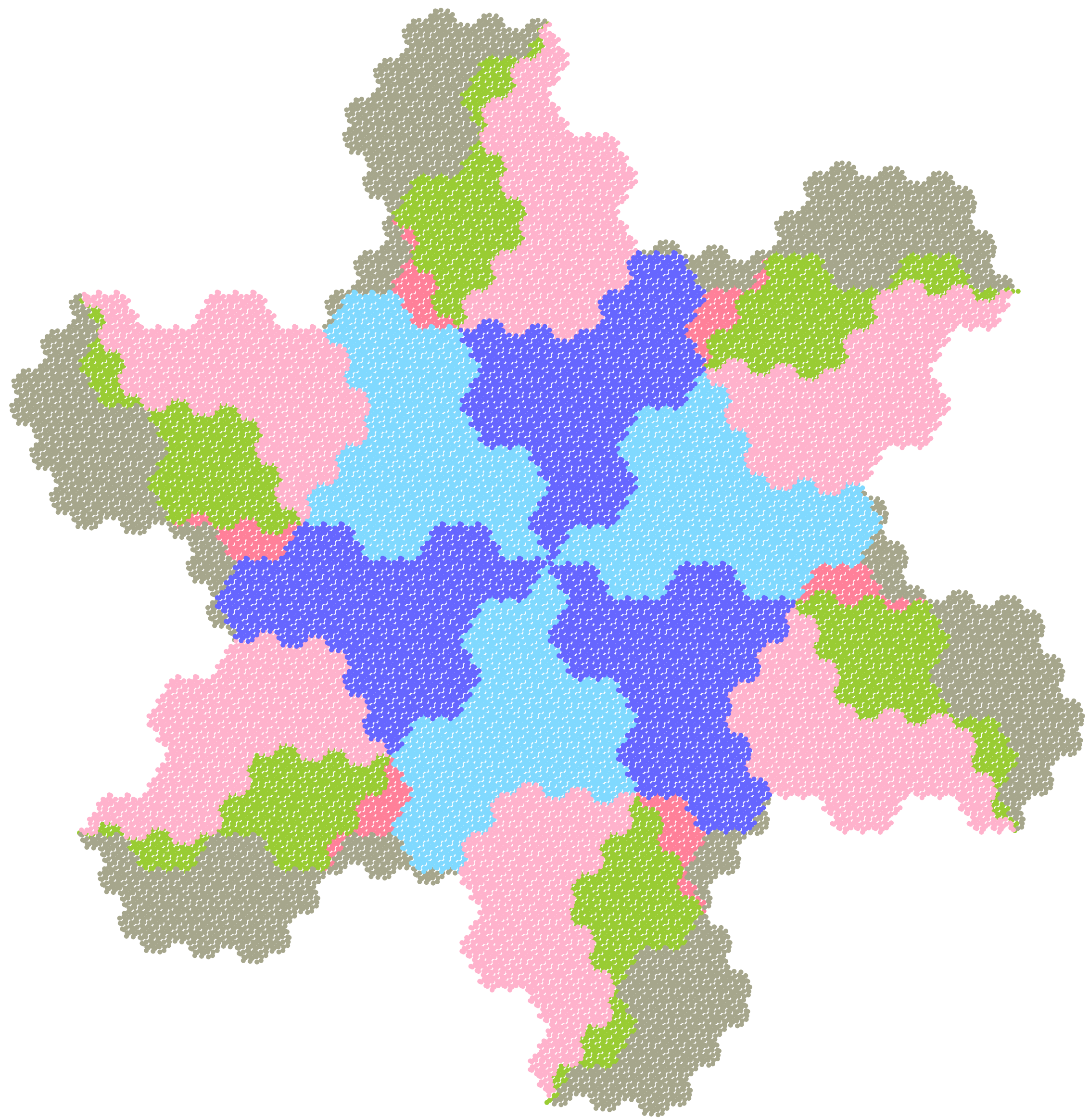


Same old story ... only even more so



Control points and window





Outlook

- Calculate eigenfunctions !
- Harmonic analysis of Rauzy fractals
- Interesting Markov partitions
- FB coefficients via matrix cocycles
- Other monotiles ?
- Monotiles for $d \geqslant 3$?
- Role of almost periodicity ?
- Do we understand Euclidean space ?
- This is only the beginning ...

References

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