Restivo Salemi property for α-power free languages with $\alpha \geq 5$ and $k \geq 3$ letters

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Finite and Infinite Words, Powers

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Finite and Infinite Words, Powers

An α-*power* of a nonempty word r is the word $r^\alpha = rr \cdots rt$ such that $\frac{|r^\alpha|}{|r|} = \alpha$ and t is a prefix of r, where $\alpha > 0$ is a rational number.

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Example

$(1234)^3 = 123412341234$ and $(1234)^{\frac{7}{4}} = 1234123$.

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We say that w is α-*power free* if

$$\{(r, \beta) \in \Theta(w) \mid \beta \geq \alpha\} = \emptyset$$

and we say that w is α^+-power free if

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It is a used convention to define that if the word *w* is "α-*power-free*" then α denotes a number or a "number with $+$".

Powers

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It is also easy to see that the reverse function preserves the power-freeness.

Square free infinite word

[A. Thue,*Über unendliche Zeichenreihen*, Skrifter udgivne af Videnskabsselskabet i Christiania: Mathematisk-naturvidenskabelig Klasse, (1906).]

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Some other generalizations of power-free words are being studied; for example abelian powers, pseudo squares, and reverse powers.

Dejean's Conjecture

We define the *repetition threshold* $RT(k)$ to be the infimum of all rational numbers α such that there exists an infinite α-power-free word over an alphabet with k letters.

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Dejean's conjecture states that $RT(2) = 2$, $RT(3) = \frac{7}{4}$, $RT(4) = \frac{7}{5}$, and $RT(k) = \frac{k}{k-1}$ for each $k > 4$. [Dejean, *Sur un théorème de Thue*, Journal of Combinatorial Theory, 1972.]

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Dejean's conjecture has been proved by the work of several authors. The final step in 2011 by [Currie and Rampersad, *A proof of dejean's conjecture*, Math. Comp., 2011.]

Dejean's Conjecture

Then Dejean's conjecture implies that there are infinitely many finite α-power free words over Σ_k, where $\alpha > RT(k)$.

Restivo and Salemi Problems

In 1985, Restivo and Salemi presented a list of five problems concerning the extendability of power free words. [Restivo and Salemi, *Some decision results on nonrepetitive words*, Combinatorial Algorithms on Words, 1985]

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Problem 1: Given an α-power-free word *u*, decide whether for every positive integer *n* there are words *w*, *v* such that $|w| = |v| = n$ and such that:

- *uv* is α-power-free,
- *wu* is α-power-free, and
- *wuv* is α-power-free.

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Problem 2: Given an α-power-free word u, construct, if it exists, an infinite α-power-free word having u as a prefix.

Restivo and Salemi Problems

Problem 3: Given an arbitrary positive integer k, does there exists an α-power-free word u such that:

- there exists a word v of length k such that uv is α-power-free and
- for every word $\bar{v}$ with $|\bar{v}| > |v|$ we have that $u\bar{v}$ is not α-power-free.

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Problem 4: Given finite α-power-free words *u* and *v*, decide whether there is a transition word *w*, such that *uwu* is α-power free.

Problem 5: Given finite α-power-free words *u* and *v*, find a transition word *w*, if it exists.

Restivo and Salemi Problems

[Petrova and Shur, *Transition property for cube-free words*, Computer Science – Theory and Applications, 2019.]

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For every pair (u, v) of cube free words over an alphabet with k letters, if u can be infinitely extended to the right and v can be infinitely extended to the left respecting the cube-freeness property, then there exists a “transition” word w over the same alphabet such that uwv is cube free.

Restivo and Salemi Property

[Shur,*Two-Sided Bounds for the Growth Rates of Power-Free Languages*, Developments in Language Theory, 2009]

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Conjecture (Conjecture 1)

Let L be a power-free language and let $e(L) \subseteq L$ be the set of words of L that can be extended to a bi-infinite word respecting the given power-freeness. If $u, v \in e(L)$ then $uwv \in e(L)$ for some word w.

Restivo and Salemi Property

[Shallit and Shur, *Subword complexity and power avoidance*, Theoretical Computer Science, 2019.]

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In 2018, the Conjecture 1 appeared again using a “Restivo Salemi property”; it was defined that a language L has the *Restivo Salemi property* if Conjecture 1 holds for the language L.

Restivo and Salemi Problems

[Rukavicka, *Transition property for α-power free languages with $\alpha \geq 2$ and k ≥ 3 letters*, Developments in Language Theory, 2020]

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$$\Upsilon = \{(k,\alpha) \mid k \in \mathbb{N} \text{ and } \alpha \in \mathbb{Q} \text{ and } k = 3 \text{ and } \alpha > 2\}$$
$$\cup\{(k,\alpha) \mid k \in \mathbb{N} \text{ and } \alpha \in \mathbb{Q} \text{ and } k > 3 \text{ and } \alpha \geq 2\}$$
$$\cup\{(k,\alpha^+) \mid k \in \mathbb{N} \text{ and } \alpha \in \mathbb{Q} \text{ and } k \geq 3 \text{ and } \alpha \geq 2\}.$$

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The definition of Υ says that: If $(k,\alpha) \in \Upsilon$ and α is a "number with $+$" then $k \geq 3$ and $\alpha \geq 2$. If $(k,\alpha) \in \Upsilon$ and α is "just" a number then $k = 3$ and $\alpha > 2$ or $k > 3$ and $\alpha \geq 2$.

Restivo and Salemi Problems

Suppose $(\alpha, k) \in \Upsilon$. For every pair (u, v) of α-power free words over an alphabet with k letters, if u can be infinitely extended to the right and v can be infinitely extended to the left respecting the α-freeness property, then there exists a “transition” word w over the same alphabet such that uwv is α-power free. Also it was shown how to construct the word w.

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Less formally said, these results solve Problem 4 and Problem 5 for a wide variety of power free languages.

Restivo and Salemi Problems

The very basic idea of our proof is that if *u*, *v* are α-power free words and *x* is a letter such that *x* is not a factor of both *u* and *v*, then clearly *uxv* is α-power free on condition that $\alpha \geq 2$.

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Generalization: Consider a finite number of occurrences of the letter x instead of a single occurrence.

Restivo and Salemi Problems

An essential observation from 2020: If v is a right (left) infinite α-power word with a factor w and x is a letter, then there is a right (left) infinite α-power free word $\widetilde{v}$ such that $\widetilde{v}$ contains v as a factor and x is not recurrent in $\widetilde{v}$.

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An essential observation from 2020: If v is a right (left) infinite α-power word with a factor w and x is a letter, then there is a right (left) infinite α-power free word $\widetilde{v}$ such that $\widetilde{v}$ contains v as a factor and x is not recurrent in $\widetilde{v}$.

The infinite α-power free words with the non-recurrent letter x have then been used to construct then transition words. The results were shown for α-power free words over an alphabet with k letters, where $(k, \alpha) \in \Upsilon$.

Bi-infinite power free words

[Rukavicka, *Construction of a Bi-infinite Power Free Word with a Given Factor and a Non-recurrent Letter*, DCFS 2023.]

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Let

$$\widetilde{\Upsilon} = \{(k, \alpha) \mid k \in \mathbb{N} \text{ and } \alpha \in \mathbb{Q} \text{ and } k \geq 3 \text{ and } \alpha \geq 5\}.$$

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$$\widetilde{\Upsilon} = \{(k, \alpha) \mid k \in \mathbb{N} \text{ and } \alpha \in \mathbb{Q} \text{ and } k \geq 3 \text{ and } \alpha \geq 5\}.$$

Note that $\widetilde{\Upsilon} \subset \Upsilon$.

Bi-infinite power free words

Let $L^{\mathbb{Z}}_{k,\alpha}$ ($L^{\mathbb{N},L}_{k,\alpha}$, $L^{\mathbb{N},R}_{k,\alpha}$) denote the set of all bi-infinite (left infinite, right infinite) α-power free words over an alphabet with *k* letters, where α is a positive rational number and *k* is positive integer.

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It was proved that if $(k, \alpha) \in \widetilde{\Upsilon}$, $v \in L^{\mathbb{Z}}_{k,\alpha}$, and w is a finite factor of v, then there are $\widetilde{v} \in L^{\mathbb{Z}}_{k,\alpha}$ and a letter x such that w is a factor of $\widetilde{v}$ and x has only a finitely many occurrences in $\widetilde{v}$.

Bi-infinite power free words

Lemma

If $k \geq 3$ and $\alpha > 2$ then $L_{k-1,\alpha}^{\mathbb{N},R} \neq \emptyset$.

Bi-infinite power free words

Lemma

If $k \geq 3$ *and* $\alpha > 2$ *then* $L^{\mathbb{N},R}_{k-1,\alpha} \neq \emptyset$.

Theorem

(reformulation of a result from DLT 2020) If $(k, \alpha) \in \Upsilon$, $v \in L^{\mathbb{N},L}_{k,\alpha}$, $z \in Suf(v)$, $x \in F_{rec}(v) \cap \Sigma_k$, $s \in L^{\mathbb{N},L}_{k,\alpha}$, *and* $x \notin F(s)$, *then there is a finite word* $u \in \Sigma_k^*$ *such that* $z \in Suf(su)$ *and* $su \in L^{\mathbb{N},L}_{k,\alpha}$.

Bi-infinite power free words

We define two technical sets Γ and Δ.

Definition

Let Γ be a set of triples defined as follows. We have that $(w, \eta, u) \in \Gamma$ if and only if

- $w \in \Sigma_k^+$, $\eta, u \in \Sigma_k^*$, and
- if $|u| \leq |w|$ then $|\eta| \geq (\alpha + 1)\alpha^{|w|-|u|}|w|$.

Bi-infinite power free words

Definition

Let Δ be a set of 6-tuples defined as follows. We have that $(s, \sigma, w, \eta, x, u) \in \Delta$ if and only if

1. $s \in \Sigma_k^{\mathbb{N},L}$, $\sigma, \eta, u \in \Sigma_k^*$, $w \in \Sigma_k^+$, $x \in \Sigma_k$,
2. $s\sigma w\eta xu \in L_{k,\alpha}^{\mathbb{N},L}$,
3. $(w, \eta, u) \in \Gamma$,
4. $occur(s\sigma w, w) = 1$, and
5. $x \notin F(s) \cup F(u)$.

Bi-infinite power free words

Theorem

If $(s, \sigma, w, \eta, x, \epsilon) \in \Delta$*,* $t \in L_{k,\alpha}^{\mathbb{N},R}$*, and* $x \notin F(t)$ *then there is* $\widehat{\eta} \in Prf(\eta)$ *such that* $s\sigma w\widehat{\eta}xt \in L_{k,\alpha}^{\mathbb{Z}}$*.*

Bi-infinite power free words

Main theorem:

Theorem

If $v \in L^{\mathbb{Z}}_{k,\alpha}$, $w \in F(v) \setminus \{\epsilon\}$, then there there are $\overline{v} \in L^{\mathbb{Z}}_{k,\alpha}$ and $x \in \Sigma_k$ such that $w \in F(\overline{v})$ and $x \notin F_r(\overline{v})$.

Thank you