What I know about Parikh-collinear morphisms

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Based on joint work M. Rigo and M. Stipulanti (ULiège)

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@One World Combinatorics on Words Seminar

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Main characters

- $|w|_a$: Number of occurrences of $a \in A$ in $w \in A^*$
- $\Psi(w) := (|w|_a)_{a \in A}$: Parikh vector of word $w \in A^*$

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Definition (Parikh-collinear morphism)

A morphism $f : A^* \to B^*$ is Parikh-collinear if, $\forall a, b \in A$, $\Psi(f(b)) = r_{a,b}\Psi(f(a))$ for some $r_{a,b} \in \mathbb{Q}$. (i.e., the Parikh-vectors span a 1D subspace.)

Example: $f : 0 \mapsto 000111, 1 \mapsto 0110$ is Parikh-collinear;
$\Psi(f(0)) = (3, 3)$, $\Psi(f(1)) = (2, 2)$.

further examples: Thue–Morse; $0 \mapsto 012$, $1 \mapsto 102$, $2 \mapsto \varepsilon$.

In the literature

Appear in the works of Dekking, Allouche et al., Cassaigne et al.

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- Factor complexity function: $\mathsf{p}_{\mathbf{x}} \colon \mathbb{N} \to \mathbb{N},\ n \mapsto \#\mathcal{L}_n(\mathbf{x})$.
- Abelian equivalence relation: $u \sim_{ab} v$ if $\Psi(u) = \Psi(v)$.
- Abelian complexity function $\mathsf{a}_{\mathbf{x}} \colon \mathbb{N} \to \mathbb{N}, n \mapsto \#\mathcal{L}_n(\mathbf{x})/\sim_{\mathrm{ab}}$.

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Theorem (Cassaigne–Richomme–Saari–Zamboni 2011)

A morphism is Parikh-collinear iff it maps all infinite words to words with bounded a.

Example: C the binary Champernowne word, μ Thue–Morse morphism; $\mathsf{a}_{\mu(C)}(n) = 2$ if n is odd, otherwise $\mathsf{a}_{\mu(C)}(n) = 3$.

Aim/Outline of the talk

- Characterizations of Parikh-collinear morphisms
  - Via binomial complexities
- Discuss automatic properties of fixed points
  - In the sense of Allouche and Shallit
- Binomial complexities under the Thue–Morse morphism

Based on Rigo, Stipulanti, W:

- *Characterizations of families of morphisms and words via binomial complexities*, Eur. J. Comb. (2024).
- *Automaticity and Parikh-collinear Morphisms*, WORDS'23.
- *Automatic Abelian Complexities of Parikh-Collinear Fixed Points*, submitted.

Definitions and notation

- In a word,
  A factor is a contiguous subsequence;
  A (scattered) subword is a subsequence.

Example: $|\texttt{reappear}| = 8$,

$|\texttt{reappear}|_{\texttt{a}} = 2 = |\texttt{reappear}|_{\texttt{e}}$

factor	subword
reappear	reappear
reappear	reappear
reappear	reappear

Binomial equivalence

Definition

Binomial coefficient $\binom{u}{v}$ of $u, v \in A^*$: $\#$ occurrences of v as a subword of u.

Example: $\binom{101001}{101} = 6$

101001 101001 101001

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Definition

Let $k \geq 1$ be an integer.

Words $u, v \in A^*$ are k-binomially equivalent ($u \sim_k v$) if $\binom{u}{x} = \binom{v}{x}$ for all $x \in A^{\leq k}$.

Example: $0110 \sim_2 1001$ but $0110 \not\sim_3 1001$

x	0	1	00	01	10	11	000	001
$\binom{0110}{x}$	2	2	1	2	2	1	0	0
$\binom{1001}{x}$	2	2	1	2	2	1	0	1

Binomial complexity functions

Definition (Rigo–Salimov (2015) (also WORDS 2013))

k-binomial complexity function of $\mathbf{x} \in A^{\mathbb{N}}$: $\mathrm{b}_{\mathbf{x}}^{(k)} : \mathbb{N} \to \mathbb{N},\ n \mapsto \#(\mathcal{L}_n(\mathbf{x})/{\sim_k})$.

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Observation

- $u \sim_{k+1} v \implies u \sim_k v$ for all $k \geq 1$.

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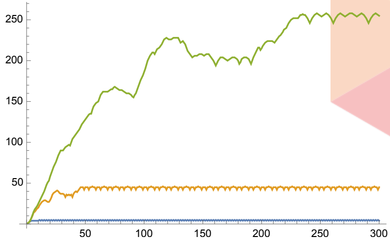
Observation

- $u \sim_{k+1} v \implies u \sim_k v$ for all $k \geq 1$.
- $\mathrm{a}(n) = \mathrm{b}^{(1)}(n) \leq \mathrm{b}^{(2)}(n) \leq \ldots \leq \mathrm{b}^{(k)}(n) \leq \mathrm{b}^{(k+1)}(n) \leq \ldots \leq \mathrm{p}(n)$.



Example: Thue–Morse word

- Thue–Morse morphism: $\mu \colon 0 \mapsto 01,\ 1 \mapsto 10$
- Thue–Morse word $\mathbf{t}$, fixed point of μ: $0110100110010110\cdots$.

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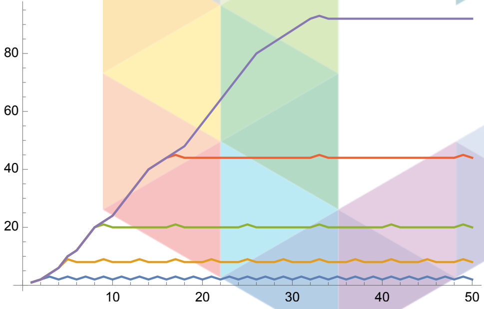
Theorem (Lejeune–Leroy–Rigo 2020)

For all $k \geq 1$,

$$\mathrm{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} \mathrm{p}_{\mathbf{t}}(n), & \text{if } n \leq 2^k - 1; \\ 3 \cdot 2^k - 3, & \text{if } 2^k | n \text{ and } n \geq 2^k; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$$



Example: Sturmian words

Sturmian words:

- Those words $\mathbf{x}$ for which $\mathrm{p}_{\mathbf{x}}(n) = n + 1$ for all n;
- Those aperiodic words $\mathbf{x}$ for which $\mathrm{b}_{\mathbf{s}}^{(1)} = 2$ for all n.
- E.g., Fibonacci word $0100101001001010010100 1\cdots$ (f.p. of $0 \mapsto 01$, $1 \mapsto 0$).

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Theorem (Rigo–Salimov 2015)

For a Sturmian word $\mathbf{s}$ and any $k \geq 2$, $\mathrm{b}_{\mathbf{s}}^{(k)} = \mathrm{p}_{\mathbf{s}}$.

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For a Sturmian word $\mathbf{s}$ and any $k \geq 2$, $\mathrm{b}_{\mathbf{s}}^{(k)} = \mathrm{p}_{\mathbf{s}}$.

Theorem (Rigo–Stipulanti–W. 2024)

An infinite word $\mathbf{s}$ is a Sturmian word if and only if there exists $k \geq 2$ such that $\mathrm{b}_{\mathbf{s}}^{(k)}(n) = n + 1$ for all $n \geq 1$;

Binomial complexities

Tribonacci word $\mathbf{z}$	$\mathrm{b}_{\mathbf{z}}^{(k)} = \mathrm{p}_{\mathbf{z}} \ \forall\, k \geq 2$	Lejeune–Rosenfeld–Rigo 2020
Billiard words $\mathbf{c}$	$\mathrm{b}_{\mathbf{c}}^{(k)} = \mathrm{p}_{\mathbf{c}} \ \forall\, k \geq 2$	Vivion 2024
Parikh-constant morphic words	$\mathrm{b}^{(k)}$ bounded $\forall k \geq 1$	Rigo–Salimov, 2015
Generalized Thue–Morse words	precise values of $\mathrm{b}^{(2)}$	Lü–Chen–Wen–Wu 2024

Work in progress (Golafshan–Rigo–W.): binomial complexities of generalized Thue–Morse words

Characterizing Parikh-collinear morphisms

Theorem (Cassaigne–Richomme–Saari–Zamboni 2011)

A morphism is Parikh-collinear iff it maps all infinite words to words with bounded $\mathsf{b}^{(1)}$.

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Theorem (Rigo–Stipulanti–W. (2024))

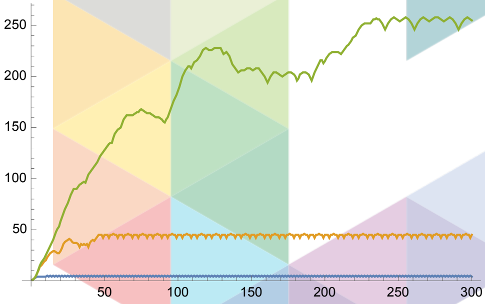
For a morphism $f\colon A^* \to B^*$, *TFAE:*

1. f *is Parikh-collinear;*
2. $\forall k \geq 1,\ u, v \in A^* : u \sim_k v \Rightarrow f(u) \sim_{k+1} f(v)$;
3. $\exists k \geq 1\ \forall u, v \in A^* : u \sim_k v \Rightarrow f(u) \sim_{k+1} f(v)$;
4. f *maps, for all* $k \geq 0$, *all words with bounded* $\mathrm{b}^{(k)}$ *to words with bounded* $\mathrm{b}^{(k+1)}$;
5. f *maps, for some* $k \geq 0$, *all words with bounded* $\mathrm{b}^{(k)}$ *to words with bounded* $\mathrm{b}^{(k+1)}$.

Generalization of Rigo and Salimov:

Corollary

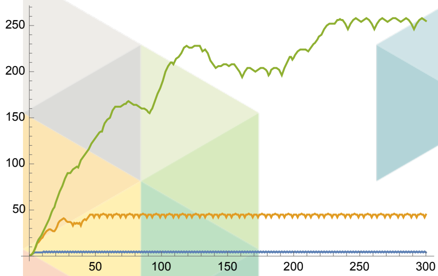
A fixed point of a Parikh-collinear morphism has bounded $\mathrm{b}^{(k)}$ for all $k \geq 1$.



$f : 0 \mapsto 000111, 1 \mapsto 0110$

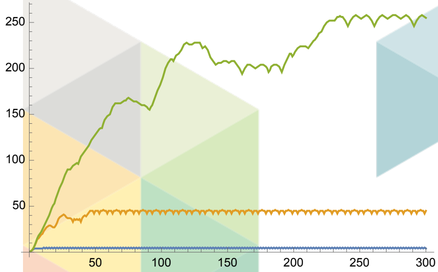
Question by Sportiello and Salo

Is a_x ultimately periodic for f.p. of Parikh-collinears?

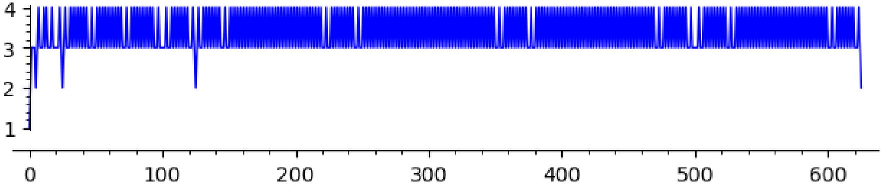


Question by Sportiello and Salo

Is $\mathsf{a_x}$ ultimately periodic for f.p. of Parikh-collinears?



Consider $f^\omega(0)$ with $0 \mapsto 010011$, $1 \mapsto 1001$.



Automatic sequences

A sequence $\mathbf{x} \in A^{\mathbb{N}}$ is k-automatic if and only if...

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- for each letter $a \in A$, the characteristic sequence $\{n \colon \mathbf{x}_n = a\}$ is definable in $Th(\mathbb{N}, +, V_k)$, where $V_k \colon \mathbb{N} \to \mathbb{N}$, $V_k(n) =$ the largest power of k dividing n.

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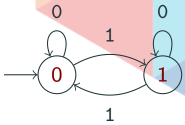
- There is a DFAO A such that $A([n]_k) = x_n$ for all $n \in \mathbb{N}$.

E.g., Thue–Morse is defined with the automaton



Automatic Parikh-collinear fixed points

Theorem (Dekking (1978), Allouche et al. (2020), Rigo–Stipulanti–W. (2023))

Let $\mathbf{x} = f^{\omega}(a)$, *where* f *is Parikh-collinear. Then* $\mathbf{x}$ *is* k*-automatic for* $k = \sum |f(a)|_a$.

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Example

Consider $\mathbf{w} = f^\omega(0)$ with $f(0) = 010011$, $f(1) = 1001$.

The construction gives $\mathbf{w} = \tau(g^\omega(0))$, where

$$g\colon 0 \mapsto 01023,\ 1 \mapsto 14501,\ 2 \mapsto 10102,\ 3 \mapsto 31450,\ 4 \mapsto 45010,\ 5 \mapsto 10231$$

$$\tau\colon 0,2,5 \mapsto 0;\ 1,3,4 \mapsto 1.$$

Automatic abelian complexity functions

Theorem (Rigo–Stipulanti–W. (2023+))

Let $\mathbf{x} = f^{\omega}(a)$, where f is Parikh-collinear. Then $\mathsf{a}_{\mathbf{x}}$ is k-automatic for $k = \sum |f(a)|_a$. Moreover, given f, the morphism generating $\mathsf{a}_{\mathbf{x}}$ can be effectively computed.

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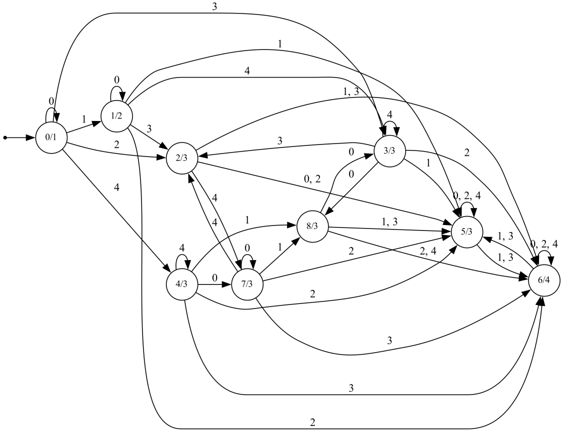
- Rigo's conjecture: $\mathsf{a}_\mathbf{x}$ is k-**regular** whenever $\mathbf{x}$ is k-automatic.
- Guo, Lü, Wen[1]: Thm holds for Parikh-**constant** f.
- Allows to query properties about $\mathsf{a}_\mathbf{x}$ using software like `Walnut`.

[1] On the boundary sequence of an automatic sequence. Discrete Math. 345(1) (2022)

Proof of aperiodicity with Walnut

Abelian complexity $a_{\mathbf{w}}$ of $\mathbf{w} = f^{\omega}(0)$, $0 \mapsto 010011$, $1 \mapsto 1001$:



Walnut evaluates the following query to True:

```
eval isAper "?msd_5 ~(Ei,p (p>0)
  & Aj ((j>=i) =>
  (abCompW[j] = abCompW[j+p])))";
```

Proof sketch

Theorem (Rigo–Stipulanti–W. (2023+))

Let $\mathbf{x} = f^{\omega}(a)$, where f is Parikh-collinear. Then $\mathsf{a}_{\mathbf{x}}$ is k-automatic for $k = \sum |f(a)|_a$. Moreover, given f, the morphism generating $\mathsf{a}_{\mathbf{x}}$ can be effectively computed.

Consider two natural factorizations of $\mathbf{w} = f^{\omega}(0)$; $f(0) = 010011$, $f(1) = 1001$.

1. $\mathbf{w} = 010011|1001|010011|010011|1001|1001|1001|010011|010011\cdots$
2. $\mathbf{w} = 01001|11001|01001|10100|11100|11001|10010|10011|01001|1\cdots$

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2. $\mathbf{w} = 01001|11001|01001|10100|11100|11001|10010|10011|01001|1\cdots$

Definition (Cutting set w.r.t. f)

$\mathrm{CS}_{f,a} := \{|f(\mathrm{pref}_n(\mathbf{x}))| : \ n \geq 0\}$

$010011|1001|010011|010011|1001|1001|1001|010011|010011\cdots$
$\mathrm{CS}_{f,0} = \{0, 6, 10, 16, 22, 26, 30, 34, \ldots\}$

Proof sketch continued

Proposition

The cutting set of a Parikh-collinear f.p. $f^{\omega}(a)$ is effectively definable in $Th(\mathbb{N}, +, V_k)$ for $k = \sum_{a \in A} |f(a)|_a$. In other words, it is k-automatic.

Use effective version of Mossé's recognisability result á la Béal–Durand–Perrin (personal communication)

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Corollary

For each $a \in A$, there exists a 2-tape DFA accepting the pairs $([n]_2, [|\operatorname{pref}_n(\mathbf{x})|_a]_2)$. (I.e., the sequence $(|\operatorname{pref}_n(\mathbf{x})|_a)_{n\geq 0}$ is k-synchronised.)

Proof sketch continued

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The cutting set of a Parikh-collinear f.p. $f^\omega(a)$ *is effectively definable in* $Th(\mathbb{N}, +, V_k)$ *for* $k = \sum_{a\in A} |f(a)|_a$. *In other words, it is* k*-automatic.*

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Conclude by invoking result of J. Shallit (2021).

Words sharing binomial complexities with t

Thue–Morse word: $\mathbf{t} = 01101001100101101001011001101001\cdots$
f.p. of the morphism $\mu : 0 \mapsto 01,\ 1 \mapsto 10$.

Theorem (Thue–Morse complexities (Lejeune–Leroy–Rigo 2020))

For all $k \geq 1$, $\mathrm{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} \mathrm{p}_{\mathbf{t}}(n), & \text{if } n \leq 2^k - 1; \\ 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$

Definition

Let $k \geq 1$. A word $\mathbf{x}$ has Property $\mathcal{P}_k$ if $\mathrm{b}_{\mathbf{x}}^{(j)} = \mathrm{b}_{\mathbf{t}}^{(j)}$ for all $1 \leq j \leq k$.

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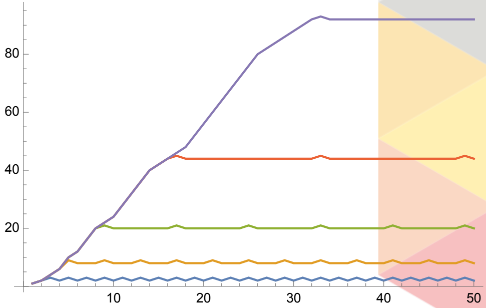


Theorem (Richomme–Saari–Zamboni 2011)

An aperiodic word $\mathbf{x}$ *has Property* $\mathcal{P}_1$ *iff* $\mathbf{x}$ *is a suffix of* $\mu(\mathbf{y})$ *for some binary word* $\mathbf{y}$.

Thue–Morse morphism: $\mu : 0 \mapsto 01,\ 1 \mapsto 10$.

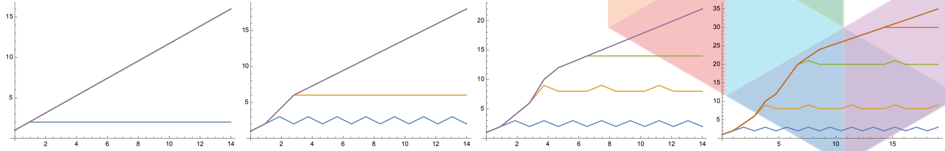
Proposition

Let $\mathbf{y}$ *be an aperiodic binary word. Then* $\mu^k(\mathbf{y})$ *and any of its suffixes has Property* $\mathcal{P}_k$.

Proof is by extending results from Lejeune–Leroy–Rigo 2020.

Example: Let $\mathbf{s} = 010010100100\cdots$ be the Fibonacci word; f.p. of $0 \mapsto 01, 1 \mapsto 0$.

$\mathbf{s} = \mu^0(\mathbf{s})$ $\quad$ $\mu^1(\mathbf{s})$ $\quad$ $\mu^2(\mathbf{s})$ $\quad$ $\mu^3(\mathbf{s})$



Words sharing binomial complexities with t

Definition: An infinite word $\mathbf{x}$ is recurrent, if each factor appears infinitely often.

A partial converse:

Proposition

Let $\mathbf{x}$ be a recurrent and assume it has Property $\mathcal{P}_k$. Then $\mathbf{x}$ is a suffix of the word $\mu^k(\mathbf{y})$ for some aperiodic binary word $\mathbf{y}$.

Words sharing binomial complexities with t

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Let **x** *be a recurrent and assume it has Property* $\mathcal{P}_k$. *Then* **x** *is a suffix of the word* $\mu^k(\mathbf{y})$ *for some aperiodic binary word* **y**.

Work in progress

Replacing "recurrent" with "aperiodic", thus obtaining a generalization of Richomme–Saari–Zamboni.

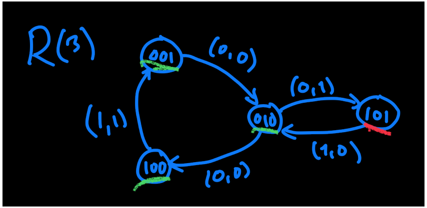
Tools for the proof: abelian Rauzy graphs

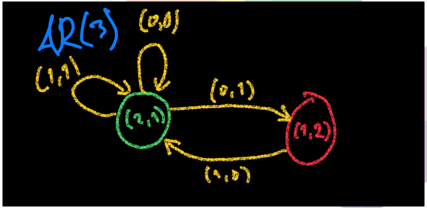
Definition ((Abelian) Rauzy graphs (Richomme–Saari–Zamboni 2010))

For $\mathbf{x}$, the Rauzy graph $R_{\mathbf{x}}(n)$ of order n is the labelled digraph (V, E): $V = \mathcal{L}_n(\mathbf{x})$.
We have $u \to v \in E$ with label (a, b) iff $\exists w : u = aw \& v = wb$ and $awb \in \mathcal{L}(\mathbf{x})$.

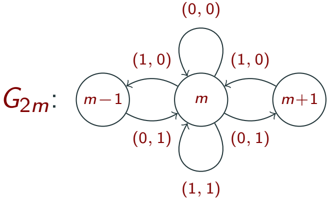
The abelian Rauzy graph $AR_{\mathbf{x}}(n) = R_{\mathbf{x}}(n)/\sim_1$.

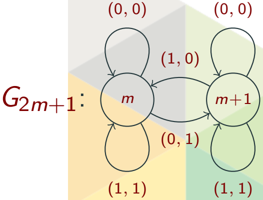
Example: Fibonacci word $\mathbf{f} = 0100101001001010010100\cdots$





Example: Abelian Rauzy graphs of the Thue–Morse word.





Definition

Define $\equiv_R$ and $\equiv_L$ on $E_{\mathbf{y}}(n)$ respectively as the equivalence kernels[a] of the functions

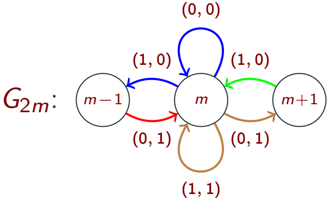
$$\left(\vec{x} \xrightarrow{(a,b)} \vec{y}\right) \mapsto (\vec{x}, b) \qquad \left(\vec{x} \xrightarrow{(a,b)} \vec{y}\right) \mapsto (a, \vec{y}).$$

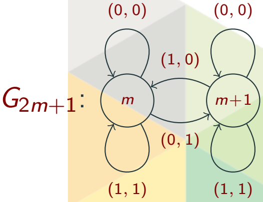
Let $Y = E_{\mathbf{y}}(n)/{\equiv_R} \cup E_{\mathbf{y}}(n)/{\equiv_L}$.

[a] Equivalence kernel $\sim_f$ of a function: $x \sim_f y \Leftrightarrow f(x) = f(y)$

Tools for the proof: abelian Rauzy graphs

Example: Abelian Rauzy graphs of the Thue–Morse word.





Definition

Define $\equiv_R$ and $\equiv_L$ on $E_{\mathbf{y}}(n)$ respectively as the equivalence kernels[a] of the functions

$$\left(\vec{x} \xrightarrow{(a,b)} \vec{y}\right) \mapsto (\vec{x}, b) \qquad \left(\vec{x} \xrightarrow{(a,b)} \vec{y}\right) \mapsto (a, \vec{y}).$$

Let $Y = E_{\mathbf{y}}(n)/{\equiv_R} \cup E_{\mathbf{y}}(n)/{\equiv_L}$.

[a] Equivalence kernel $\sim_f$ of a function: $x \sim_f y \Leftrightarrow f(x) = f(y)$

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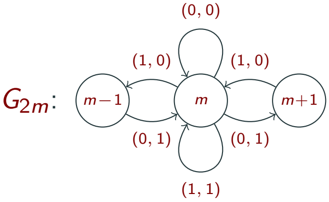

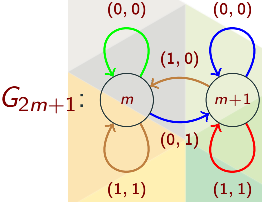

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[a] Equivalence kernel $\sim_f$ of a function: $x \sim_f y \Leftrightarrow f(x) = f(y)$

Tools for the proof: formula for $b^{(k+1)}$

Proposition

Let $\mathbf{x} = \mu^k(\mathbf{y})$*. We have* $b_{\mathbf{x}}^{(k+1)}(r) = p_{\mathbf{t}}(r)$ *for all* $0 \leq r < 2^k$*. For all* $n \geq 1$ *we have*

$$b_{\mathbf{x}}^{(k+1)}(2^k n) = (2^k - 1)\#E_{\mathbf{y}}(n) + b_{\mathbf{y}}^{(1)}(n) - \begin{cases} 2^k, & \text{if } 0^{n+1}, 1^{n+1} \in \mathcal{L}(\mathbf{y}); \\ 1, & \text{if } \substack{0^{n+1}, 1^n \in \mathcal{L}(\mathbf{y}) \\ \text{and } 1^{n+1} \notin \mathcal{L}(\mathbf{y})} \text{ (or symm.)}; \\ 0, & \text{otherwise.} \end{cases}$$

For all $n \geq 1$ *and* $0 < r < 2^k$*, setting*
$Z(n,r) := (r-1)\#E_{\mathbf{y}}(n+1) + (2^k - r - 1)\#E_{\mathbf{y}}(n) + \#Y_{\mathbf{y}}(n)$*, we have*
$b_{\mathbf{x}}^{(k+1)}(2^k n + r) = Z(n,r) - \ldots$

Proof sketch by induction on k.

Proposition

Let $\mathbf{x}$ be a recurrent and assume it has Property $\mathcal{P}_k$. Then $\mathbf{x}$ is a suffix of the word $\mu^k(\mathbf{y})$ for some aperiodic binary word $\mathbf{y}$.

Proof sketch by induction on k.

Proposition

Let $\mathbf{x}$ *be a recurrent and assume it has Property* $\mathcal{P}_k$*. Then* $\mathbf{x}$ *is a suffix of the word* $\mu^k(\mathbf{y})$ *for some aperiodic binary word* $\mathbf{y}$*.*

Base case $k = 1$: Richomme–Saari–Zamboni 2011.

If $\mathbf{x}$ has Property $\mathcal{P}_{k+1}$, then by induction $\mathbf{x}$ is a suffix of $\mu^k(\mathbf{y})$.

We show that $\mathbf{y}$ must have Property $\mathcal{P}_1$ by analyzing the possible abelian Rauzy graphs and using formula above.

Question A

Binomial complexities are increasing nested:

$$\mathsf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathsf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathsf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathsf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}.$$

Notation: For two functions $\mathsf{f}, \mathsf{g} \colon \mathbb{N} \to \mathbb{N}$, we write $\mathsf{f} \prec \mathsf{g}$ when

- $\mathsf{f}(n) \leq \mathsf{g}(n)$ for all $n \in \mathbb{N}$ and
- $\mathsf{f}(n) < \mathsf{g}(n)$ for infinitely many $n \in \mathbb{N}$.

Question A

Does there exist an infinite word $\mathbf{w}$ such that $\mathsf{b}_{\mathbf{w}}^{(k)} \prec \mathsf{b}_{\mathbf{w}}^{(k+1)}$ for all $k \geq 1$?

This question was inspired by Lejeune's PhD thesis.

"Structured" words answering Question A

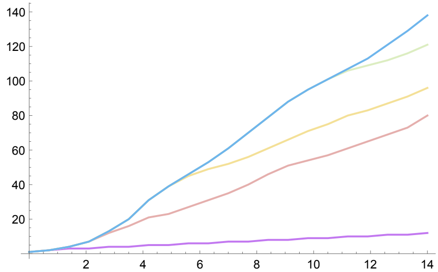
The word $\mathbf{v} = \tau(g^\omega(a))$, where

$$g \colon a \mapsto a0\alpha, 0 \mapsto 01, 1 \mapsto 10, \alpha \mapsto \alpha^2$$

$$g^\omega(a) := \lim_{n\to\infty} g^n(a)$$

$$\tau \colon a \mapsto \varepsilon, 0 \mapsto 0, 1 \mapsto 1, \alpha \mapsto 1,$$

- has $\mathsf{b}_{\mathbf{v}}^{(1)}$ unbounded
- is binary and morphic



Grillenberger's construction gives a word $\mathbf{w} = 0100010101100111\cdots$ which

- has $\mathsf{b}_{\mathbf{w}}^{(1)}$ unbounded
- is binary and uniformly recurrent*

*each factor occurs infinitely many times within bounded gaps.

Question B

Binomial complexities are increasing:

$$\mathrm{b}_{\mathbf{x}}^{(1)}(n) \leq \mathrm{b}_{\mathbf{x}}^{(2)}(n) \leq \cdots \leq \mathrm{b}_{\mathbf{x}}^{(k)}(n) \leq \mathrm{b}_{\mathbf{x}}^{(k+1)}(n) \leq \cdots \leq \mathrm{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}.$$

Question B (Stabilization)

For each $k \geq 1$, does there exist a word $\mathbf{w}_k$ such that

$$\mathrm{b}_{\mathbf{w}_k}^{(1)} \prec \mathrm{b}_{\mathbf{w}_k}^{(2)} \prec \cdots \prec \mathrm{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathrm{b}_{\mathbf{w}_k}^{(k)} = \mathrm{p}_{\mathbf{w}_k}?$$

This question was inspired by Lejeune's PhD thesis.

Towards a second answer

Recall for the Thue–Morse word $\mathbf{t}$, fixed point of $\varphi : 0 \mapsto 01, 1 \mapsto 10$:

Theorem (Lejeune–Leroy–Rigo 2020)

$$\text{For all } k \geq 1, \quad \mathrm{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} \mathrm{p}_{\mathbf{t}}(n), & \text{if } n \leq 2^k - 1; \\ 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$$

Theorem

The word $\mathbf{x} = \varphi^k(\mathbf{y})$ *has Property* $\mathcal{P}_k$.

Corollary

$\mathrm{b}_{\mathbf{x}}^{(1)} \prec \ldots \prec \mathrm{b}_{\mathbf{x}}^{(k)}$. *Can be further shown that* $\mathrm{b}_{\mathbf{x}}^{(k)} \prec \mathrm{b}_{\mathbf{x}}^{(k+1)}$.

Answering Question B

Theorem (Rigo–Stipulanti–W. (2024))

Let k be an integer.

Let $\mathbf{s}$ be a Sturmian word and let $\mathbf{s}_k = \varphi^k(\mathbf{s})$.

Then $\mathrm{b}_{\mathbf{s}_k}^{(1)} \prec \mathrm{b}_{\mathbf{s}_k}^{(2)} \prec \cdots \prec \mathrm{b}_{\mathbf{s}_k}^{(k+1)} \prec \mathrm{b}_{\mathbf{s}_k}^{(k+2)} = \mathrm{p}_{\mathbf{s}_k}$.

Remark: Such words considered by Frid (1999)

Proof sketch: By the previous corollary, $\mathrm{b}_{\mathbf{s}_k}^{(1)} \prec \mathrm{b}_{\mathbf{s}_k}^{(2)} \prec \cdots \prec \mathrm{b}_{\mathbf{s}_k}^{(k+1)}$.

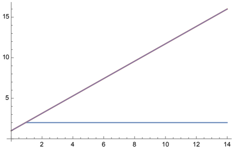
Can be shown that $\mathrm{b}_{\mathbf{s}_k}^{(k+1)} \neq \mathrm{p}_{\mathbf{s}_k}$ by analysing abelian Rauzy graphs.

$\mathrm{b}_{\mathbf{s}_k}^{(k+2)} = \mathrm{p}_{\mathbf{s}_k}$: Use fact that in $\mathbf{s}$: $u \sim_2 v \Rightarrow u = v$ (Rigo–Salimov 2015)

Example: Let $\mathbf{s} = 010010100100\cdots$ be the Fibonacci word

(fixed point of $0 \mapsto 01, 1 \mapsto 0$).

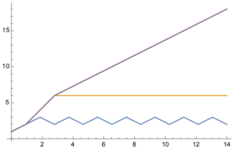
$\mathbf{s} = \varphi^0(\mathbf{s})$



$b^{(2)} = p$

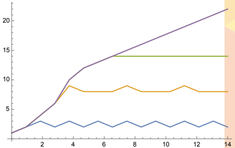
(Rigo–Salimov, 2013)

$\varphi^1(\mathbf{s})$



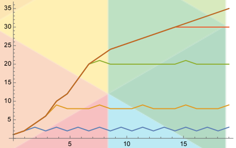
$b^{(3)} = p$

$\varphi^2(\mathbf{s})$



$b^{(4)} = p$

$\varphi^3(\mathbf{s})$



$b^{(5)} = p$

Conclusions

Considered Parikh collinear morphisms w.r.t. binomial complexities.

- Characterisations of Parikh-collinear morphisms using binomial equivalence and binomial complexities.
- Discussed automatic properties of Parikh-collinear fixed points
- Application of Thue–Morse to words.

Conclusions

Considered Parikh collinear morphisms w.r.t. binomial complexities.

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- Discussed automatic properties of Parikh-collinear fixed points
- Application of Thue–Morse to words.

Future prospects:

- Complete full generalization of Richomme–Saari–Zamboni (2011)
- Binomial complexities of other families of words?
- Possible behviours of "consecutive" binomial complexities.
- Automaticity of binomial complexities?
- Automaticity of cutting sets?

Thank you!

Unbounded complexities for Question B?

But... the binomial complexities $\mathsf{b}^{(1)}_{\mathbf{s}_k}, \ldots, \mathsf{b}^{(k+1)}_{\mathbf{s}_k}$ are bounded.

Question C

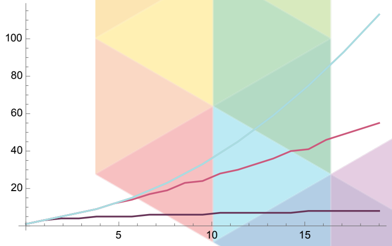
For each $k \geq 1$, does there exist a word $\mathbf{w}_k$ such that $\mathsf{b}^{(1)}_{\mathbf{w}_k}$ is unbounded and $\mathsf{b}^{(1)}_{\mathbf{w}_k} \prec \mathsf{b}^{(2)}_{\mathbf{w}_k} \prec \cdots \prec \mathsf{b}^{(k-1)}_{\mathbf{w}_k} \prec \mathsf{b}^{(k)}_{\mathbf{w}_k} = \mathsf{p}_{\mathbf{w}_k}$?

Answer for $k = 3$:

$\mathbf{h} = 01121221222122221222\cdots$

(fixed point of $0 \mapsto 01, 1 \mapsto 12, 2 \mapsto 2$)

- has $\mathsf{b}^{(1)}_{\mathbf{h}}$ unbounded
- has $\mathsf{b}^{(1)}_{\mathbf{h}} \prec \mathsf{b}^{(2)}_{\mathbf{h}} \prec \mathsf{b}^{(3)}_{\mathbf{h}} = \mathsf{p}_{\mathbf{h}}$.



What about larger values of k? The question remains open...

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