Palindromic Periodicities

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This talk is based on the paper Palindromic Periodicities which is in arxiv and which has been submitted to Theoretical Computer Science.

We use the usual notation for the combinatorics of words, in particular $R(w)$ is the reverse of the word w, and w is a palindrome if $R(w) = w$. Thus *aba* and *cc* are palindromes. The empty word is ϵ. It is a palindrome.

If p and s are palindromes and w a word of length at least $|ps|$ which is a factor of $(ps)^\omega$ then w is a *palindromic periodicity*.

EXAMPLE If $p = aba$ and $s = cc$ then $(ps)^\omega = abaccabaccaba\ldots$ so that, for example, *cabacca* is a palindromic periodicity.

A palindrome is *odd* or *even* depending on its length. Let $w[i..j]$ be a palindrome. Its *centre* is $c = (i + j)/2$ which is an integer for an odd palindrome and an integer plus a half for an even palindrome. If $i \le k \le j$ then

$$w[2c - k] = w[k].$$

If $0 < k \le (j - i)/$ then $w[i + k..j - k]$ is a palindrome *nested* in $w[i..j]$. For example $cdedc$ is nested in $abcdedcba$.

We saw that *cabacca* is a palindromic periodicity as it's a factor of $(ps)^\omega$ with if $p = aba$ and $s = cc$. It's also a factor then $(ps)^\omega$ with $p = b$ and $s = acca$.

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So knowing a palindromic periodicity doesn't determine p and s. It does, however, determine the centres of the palindromes, their parity, and the distance between them which is $|ps|/2$. We call this the *half period* of the periodicity.

We call centres of p and s the *essential centres* of the palindromic periodicity. Each essential centre is the centre of a palindromic prefix or a palindromic suffix.

EXAMPLE dabbcbbadabbcbbadabbc

Here the each c and each d is an essential centre. There are also palindromic centres betweeen each pair of b's. These are not essential and are not the centres of palindromic suffixes or prefixes.

Where do palindromic periodicities come from?

Lemma

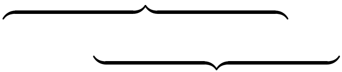
If a palindrome is periodic it is a palindromic periodicity.

For example,
$abcbaabcbaabcba$ is a length 15 palindrome with period 5,
$bcbadabcbadabcb$ is a length 15 palindrome with period 6.

Overlapping palindrome theorem

If two palindromes contain each other's centers and neither is a proper factor of the other then their union is a palindromic periodicity with half period equal to the difference between their centres.

Idea of the proof:



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$abcd$

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Idea of the proof:

$$\color{red}{abcd}\color{blue}{dcba}\color{red}{abcd}\color{blue}{dcba}\color{red}{abc}$$

Last theorem said two palindromes can make a periodicity. We can also have two periodicities making a palindrome.

Theorem

If a word w has periods p and q, where $\gcd(p, q) = 1$, and length $p + q - 2$ then it is a palindrome.

Note that the length here is one letter too short for Fine and Wilf's Periodicity Lemma to apply.

EXAMPLE With $p = 5$ and $q = 9$ we get the following length 12 word.

$$aaabaaaabaaa$$

Towards a Palindomic Periodicity Periodicity Lemma

We can describe the structure of a palindromic periodicity by giving the positions of its essential centres. These are h apart where h is the half period. Thus they are part of an arithmetic progress $\{k + ih : i \in \mathbb{Z}\}$. We call k the *offset* of the palindromic periodicity.

Note that k and h are in $\mathbb{Z}/2$.

A *double periodic periodicity* is a word which is a periodic periodicity in two ways. That is, it has essential centres at positions in the sets $\{k_1 + ih_1 : i \in \mathbb{Z}\}$ and $\{k_2 + ih_2 : i \in \mathbb{Z}\}$ for some parameters $\{k_1, h_1, k_2, h_2\}$.

Recall Fine and Wilf's Periodicity Lemma.

Lemma If a word w has periods p and q and length at least $p+q-\gcd(p,q)$ then it has period $\gcd(p,q)$.

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Lemma If a word w has periods p and q and length at least $p + q - \gcd(p, q)$ then it has period $\gcd(p, q)$.

We now present a similar result for a double palindromic periodicity.

Lemma If a word w is a double palindromic periodicity with parameters $\{k_1, h_1, k_2, h_2\}$ and

$$|w| \geq 2h_1 + 2h_2 - \gcd(2(k_2 - k_1), 2h_1, 2h_2)$$

then w is a palindromic periodicity with period

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Note that using the rule that $\gcd(a + kb, b) = \gcd(a, b)$ for any integer k the term $\gcd(2(k_2 - k_1), 2h_1, 2h_2)$ can be replaced with

$$\gcd(2((k_2 + m_2h_2) - (k_1 + m_1h_1), 2h_1, 2h_2)$$

for any integers m_1 and m_2. This means k_1 and k_2 can be replaced by any essential centres of the two palindromic periodicities.

lengths

k_1	k_2	16	15	14	13	12	11	10	9	8	7	6
0	0	4										
0	2	4										
0	4	4										
1	1	4										
1	3	4										
1	5	4										
2	0	4										
2	2	4										
2	4	4										
3	1	4										
3	3	4										
3	5	4										

Figure: Periods of double palindromic periodicities with parameters $h_1 = 4$, $h_2 = 6$ and offsets and lengths as shown. In each case $\gcd(2(k_2 - k_1), 2h_1, 2h_2) = 4$.

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0	4	4	4	4	4	4	8	8	8	8	6	6
1	1	4	4	4	4	4	4	4	4	8	7	6
1	3	4	4	4	4	4	4	4	4	4	4	4
1	5	4	4	4	4	8	8	8	8	8	7	6
2	0	4	4	4	8	8	8	8	8	8	7	6
2	2	4	4	4	4	4	4	4	8	8	7	6
2	4	4	4	4	4	4	4	4	4	4	4	4
3	1	4	4	8	8	8	8	8	8	8	7	6
3	3	4	4	4	4	4	4	8	8	8	7	6
3	5	4	4	4	4	4	4	4	4	4	4	4

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Areas for further investigation

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1. Strengthen the periodicity lemma.

2. Look for palindromic periodicities in Fibonacci words, Thue-Morse word, Sturmian words,...

3. Obtain an efficient algorithm for searching for palindromic periodicities.

X

4. It's well known and easily proved that a word of length n contains at most n distinct non-empty palindromes, and this bound is best possible.

In "Palindromes in circular words", Theoretical Computer Science, 2014, I tried to get an equivalent result for circular words.

I showed that a word of length n contains less than $5n/3$ distinct non-empty palindromes.
If n is a multiple of 3 a word of length n this means it contains at most $5n/3 - 1$ distinct non-empty palindromes.
' By construction we know it can contain $5n/3 - 2$ distinct non-empty palindromes.

The paper used an earlier weaker version of the overlapping palindromes theorem. Maybe with the new improved version the gap can be closed.

5. A *maximal periodicity*, or *run* is a periodic factor, having length at least twice its period, which cannot be extended to the left or right without altering its period. Let $\rho(n)$ be the maximum number of runs can occur in a word of length n.

1999 Kolpakov and Kucherov showed $\rho(n) = O(n)$ and conjectured $\rho(n) < n$.
2006 Rytter showed $\rho(n) < 5n$.
2006 Puglisi, Simpson, Smyth showed $\rho(n) < 3.48n$.
2008 Crochemore and Ilie showed $\rho(n) < 1.6n$.
2008 Giraud showed $\rho(n) < 1.52n$ then 2009 $\rho(n) < 1.29n$.
2012 Crochemore and Ilie showed $\rho(n) < 1.029n$.
2015 Bannai, I, Inenaga, Nakashima, Takeda, Tsuruta $\rho(n) < n$.
2015 Fischer, Holub, I, Levenstein, $\rho(n) < 0.957n$.

What happens with *maximal palindromic periodicities*?

Thank you for attending!