Factor complexity of the most significant digits and unipotent dynamics on a torus

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One World Combinatorics on Words Seminar

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State of Art

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- F.C. is stand for factor complexity.

1. F.C. of sequences of the most significant digits of the decimal expansion of 2^n.
2. F.C. of sequences of the most significant digits of the decimal expansion of a^n.
3. F.C. of sequences of the most significant digits of the b-expansion of a^n.
4. F.C. of sequences of the most significant digits of the decimal expansion of 3^{n^2}.
5. F.C. of sequences of the most significant digits of the b-expansion of a^{n^d}.

Factor Complexity

Factor Complexity

Morse and Hedlund (1938)

- under the name block growth
- subword complexity (1975)

Definition

The *factor complexity* of $\mathbf{w}$ is the map

$$\begin{aligned} p_{\mathbf{w}} : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto \#(Fac(\mathbf{w}) \cap \mathcal{A}^n). \end{aligned}$$

Let's cite a few of them

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Periodic words	$p(\mathbf{n}) = \mathcal{O}(\mathbf{1})$
Sturmian words	$p(\mathbf{n}) = \mathbf{n} + \mathbf{1}$
Minimal complexity words	$p(\mathbf{n}) = \mathbf{n} + \mathbf{k} - \mathbf{1}$ **over** ***k*** **letters**
Interval exchange transformation	$p(\mathbf{n}) = (\mathbf{k} - \mathbf{1})\mathbf{n} + \mathbf{1}$ **over** ***k*** **intervals**
Cube billiards	$p(\mathbf{n}) = \mathcal{O}(\mathbf{n}^{\mathbf{d}-\mathbf{1}})$ **in** ***d*** **dimension**

And so on and so on ....

The Most Significant Digits

The Most Significant Digits

Definition

For every non-zero real number x, *the most significant (decimal) digit* of x, denoted by $D_b(x)$, is the unique integer $j \in [\![1, b-1]\!]$ satisfying

$$b^k j \leqslant |x| < b^k (j+1)$$

for some (necessarily unique) $k \in \mathbb{Z}$.

Notation

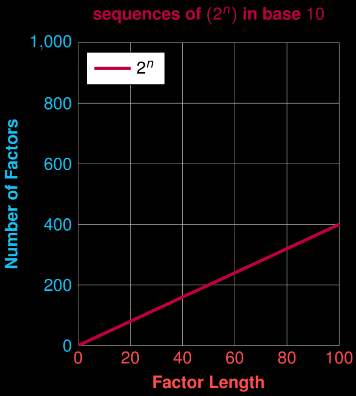
$\mathrm{p}_{a,b}(n)$: factor complexity of the most significant digits of the sequence (a^n) in base b.

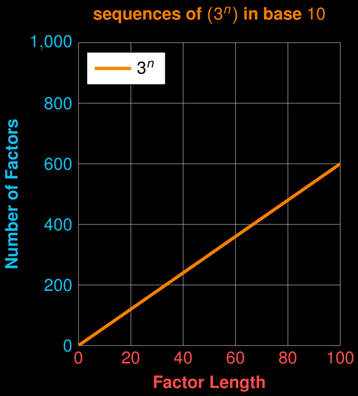
The Most Significant Digits of First 50 Terms (Concatenated)

Sequence	The most significant digits of first 50 terms (concatenated)
(2^n)	2481361251 2481361251 2481361251 2481361251 2481371251
(3^n)	3928272615 1514141313 1392827262 6151514141 3139282727
(4^n)	4162141621 4162141621 4172141731 4172141731 4173141731
(5^n)	5216317319 4216317319 4215217319 4215217319 4215217318
(6^n)	6321742116 3217421163 2174211632 1742116321 8421163218
(7^n)	7432118542 1196432117 5321196432 1175321196 4321175321
(8^n)	8654322111 8654322111 9754322111 9765432211 1865432211
(9^n)	9876554433 3222211111 1987765544 3332222111 1119877655

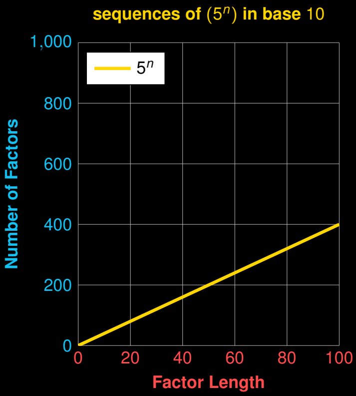
Table: Leading digits (in base 10) of the first 50 terms of the sequences (a^n), $a \in [\![2, 9]\!]$

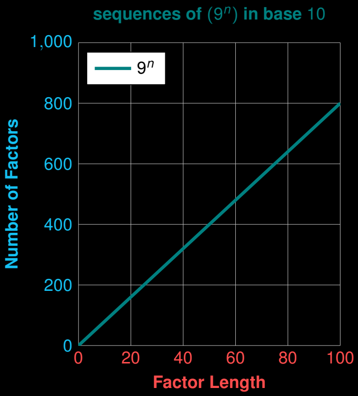
Empirical Factor Complexities Based on the First 100, 000 Terms (I)





Empirical Factor Complexities Based on the First 100, 000 Terms (II)





Admissible Pairs

Definition

A pair (a, b) is called admissible if

i) a is a positive rational number;.

ii) b is a squarefree integer $\geqslant 5$;

iii) a and b are multiplicative independent: $\log_b(a) \notin \mathbb{Q}$.

- A pair (a, b) is called strong admissible if

I) a is a positive real number;.

II) b is a squarefree integer $\geqslant 5$;

III) a and b are multiplicative independent: $\log_b(a) \notin \mathbb{Q}$.

- A pair (a, b) is called weak admissible if

A) a is a positive real number;.

B) $b \geqslant 5$;

C) a and b are multiplicative independent: $\log_b(a) \notin \mathbb{Q}$.

Complexity of Leading Digit Sequences

	(2^n)	(3^n)	(4^n)	(5^n)	(6^n)	(7^n)	(8^n)	(9^n)
b = 5	**2n + 2**	**3n + 1**	**3n + 1**		**4n**	**4n**	**4n**	**3n + 2**
b = 6	**2n + 3**	**2n + 3**	**3n + 2**	**4n + 1**		**5n**	**4n + 1**	**3n + 2**
b = 7	**3n + 3**	**4n + 2**	**5n + 1**	**5n + 1**	**5n + 1**		**6n**	**6n**
b = 10	**4n + 5**	**6n + 3**	**6n + 3**	**4n + 5**	**7n + 2**	**8n + 1**	**7n + 2**	**8n + 1**

Table: Formulas for sequences in different bases

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Theorem 1. (He & Hildebrand & ... 2020)

- Let (a, b) be an admissible pair. Then $\mathrm{p}_{a,b}(n)$ is an affine function for $n \geqslant 1$.

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- Let (a, b) be an admissible pair. Then $\mathtt{p}_{a,b}(n)$ is an affine function for $n \geqslant 1$.

Theorem 2. (P. Alessandri. PhD thesis, 1996)

- A coding of an irrational rotation, the complexity has the form $\mathtt{p}(n) = cn + d$, for n large enough.

Admissibility Requirement

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Multiplicative independent: $\log_b(a) \notin \mathbb{Q}$

$\log_b a \in \mathbb{Q} \implies$ the sequence of the most significant digits: periodic $\implies \mathtt{p}_{a,b}(n)$: *bounded*

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Lower bound of base: $b \geqslant 5$

$b = 3 \implies$ interval $I = [\log_b 1, \log_b 2) \implies \ell(I) > \frac{1}{2}$

• the intersection of the interval $[0, 2/3]$ with its translate by $1/2$ consists of the two disjoint intervals $[0, 1/6]$ and $[1/2, 2/3]$.

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Squarefree bases

$b \geqslant 5$: non-squarefree integer, q: prime, $q^2 \mid b$, $a = q$

Case I: $b \neq q^n \implies \mathtt{p}_{a,b}(n)$: *NOT affine*

Case II: $b = q^n \implies$ *periodic* $\implies \mathtt{p}_{a,b}(n)$: *bounded*

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Question: The following number made by the most significant digits of 2^n should be transcendental?

$$A := 0.124813612512\cdots$$

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Theorem 3. (Adamczewski & Bugeaud. 2004)

- Let $b \geqslant 2$ be an integer. The factor complexity of the b-ary expansion of every irrational algebraic number satisfies

$$\liminf_{n\to\infty} \frac{\mathrm{p}(n)}{n} = \infty$$

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Example

Let $a = 2$ and $b = 10$. Then $\mathrm{p}_{2,10}(n) = 4n + 5$. Hence, A is a transcendental number.

Main Result (arXiv:2402.16210v2)

Theorem 4. (M. G., Kanel-Belov., Kondakov., & Mitrofanov. 2021)
Let $P(n)$ be a polynomial with an irrational leading coefficient. Let $\mathbf{w}$ be an infinite word where $\mathbf{w}_n = [2\{P(n)\}]$. Then there is a polynomial $Q(k)$ that depends only on $\deg(P)$, such that $Q(k) = p_{\mathbf{w}}(k)$ for all sufficiently large k.

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Theorem 5. (M. G. & Mitrofanov. 2024)
Let $d \in \mathbb{Z}_{>0}$, let $b \geqslant 5$ be an integer, and let $a > 0$ be a real number such that a and b are multiplicatively independent. Consider the sequence w, where w_n is the most significant digit of a^{n^d} when expressed in base b.

Then, there exists a polynomial $\mathrm{P}(k)$ of degree $\frac{d(d+1)}{2}$ such that:

$$\mathrm{P}(k) = p_w(k) \quad \text{for large enough} k.$$

Equidistributed Sequences (1)

Definition on Real Intervals

- A sequence of real numbers $(s_i)_{i \in \mathbb{N}_{>0}}$ is *equidistributed* on a non-degenerate interval $[a, b]$ if, for any sub-interval $[c, d] \subset [a, b]$:

$$\lim_{n \to \infty} \frac{\#\left(\{s_1, s_2, \ldots, s_n\} \cap [c, d]\right)}{n} = \frac{d - c}{b - a}.$$

- **Intuition:** The terms of the sequence spread out uniformly over $[a, b]$ as n approaches infinity.

Equidistributed Sequences (2)

Equidistribution Modulo 1

- A sequence $(s_i)_{i \in \mathbb{N}_{>0}}$ is *equidistributed modulo* 1 if the sequence of fractional parts $\{s_i\}$ is equidistributed in $[0, 1]$.
  - *Fractional Part:* $\{s_i\} = s_i - \lfloor s_i \rfloor$.
- **Visual Interpretation:** When plotted on the unit interval, the fractional parts fill the interval uniformly.

Equidistributed Sequences (3)

- Let X be a topological space with a measure μ.

Generalization to Topological Spaces

- A sequence $(a_n)_{n\in\mathbb{N}_{>0}} \subset X$ is *equidistributed* if, for any open set $U \subset X$:

$$\lim_{n\to\infty} \frac{\#\left(\{a_1, a_2, \ldots, a_n\} \cap U\right)}{n} = \frac{\mu(U)}{\mu(X)}.$$

- **Implication:** The sequence distributes across X proportionally to the measure μ.

Weyl's Theorem

Weyl's Theorem

- Let $\mathbb{T}^d$ be the torus $\mathbb{R}^d/\mathbb{Z}^d$.

Theorem 6. (Weyl. 1916)
Let $\mathrm{P}(t)$ be a be a polynomial with at least one irrational coefficient. Then the sequence of fractional parts $(\{\mathrm{P}(i)\})_{i\in\mathbb{N}_{>0}}$ is equidistributed (and in particular dense) in $\mathbb{T}^1$.

Theorem 7. (Weyl. 1916)
Let $\mathrm{P}(t) = a_0 t^d + \cdots + a_d$ be a polynomial with real coefficients, where a_0 is irrational. Then the sequence of d-tuples

$$(\{\mathrm{P}(n)\}, \{\mathrm{P}(n+1)\}, \cdots, \{\mathrm{P}(n+d-1)\})$$

is equidistributed in $\mathbb{T}^d$.

Unipotent Dynamics on a Torus

Remark: Here we mention the case for 3^{n^2} in decimal expansion. The main result concerns the general case a^{n^d} in base integer $b \geqslant 5$.

- The most significant digit $\mathbf{w}_n$ of the number 3^{n^2} in base 10: t.
- This condition can be expressed as:

$$\log_{10}(t) \leqslant \{\zeta n^2\} < \log_{10}(t+1),$$

where $\zeta = \log_{10}(3)$ and $\{\cdot\}$ denotes the fractional part.

- Map each natural number n to a point $v_n \in \mathbb{T}^2$ with coordinates:

$$v_n = \left(\{\zeta n\}, \{\zeta n^2\}\right).$$

Dynamics under the Map f

- Define the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ by:

$$v_{n+1}^{(1)} = \{v_n^{(1)} + \zeta\},$$
$$v_{n+1}^{(2)} = \{v_n^{(2)} + 2v_n^{(1)} + \zeta\}.$$

Note 1: f is a self-homeomorphism of $\mathbb{T}^2$.

Note 2: f captures the unipotent dynamics of the system.

Note 3:$v_n = f^n(0, 0)$.

Critical Sets and Regions

- Define the critical subsets S_t for $t \in [\![1, 9]\!]$ as:

$$S_t = \left\{ x \in \mathbb{T}^2 : x^{(2)} = \log_{10}(t) \right\}.$$

- These critical sets divide $\mathbb{T}^2$ into regions:

$$U_t = \left\{ x \in \mathbb{T}^2 : \log_{10}(t) \leqslant x^{(2)} < \log_{10}(t+1) \right\}.$$

- The digit w_k is determined by the index t such that $v_k \in U_t$.

Partitions of $\mathbb{T}^2$

- Define the preimages of the critical sets:

$$S_{t,k} = f^{-k}(S_t).$$

- The sets $S_{t,k}$ for all t and $k = 1, \ldots, n$ divide $\mathbb{T}^2$ into connected regions:
  - forming the partition $\mathcal{M}_n$.

- For a digit sequence $\mathbf{u} = u_1 u_2 \ldots u_n$, define:

$$U_{\mathbf{u}} = U_{u_1} \cap f^{-1}(U_{u_2}) \cap \cdots \cap f^{-(n-1)}(U_{u_n}).$$

- The collection $\mathcal{U}_n = \{U_{\mathbf{u}} \mid |\mathbf{u}| = n\}$ forms a partition of $\mathbb{T}^2$, with $\mathcal{M}_n$ being a sub-partition of $\mathcal{U}_n$.

Density and Factorization in $\mathbb{T}^2$

Lemma I.
The sequence $(v_i)_{i\in\mathbb{N}_{>0}}$ is dense in $\mathbb{T}^2$.

Lemma II.
A finite sequence of digits $\mathbf{u}$ is a factor of $\mathbf{w}$ if and only if the set $U_{\mathbf{u}} \subset \mathbb{T}^2$ has non-empty interior.

Stable Vectors

- Let $x, y \in \mathbb{R}^2$. It is easy to see that $A(y) - A(x) = M(y - x)$, where

$$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

- We call a vector $r \in \mathbb{R}^2$ stable when for any $k \in \mathbb{N}_{>0}$ the last coordinate of $M^k(r)$ is an integer.

General case

- Let $x, y \in \mathbb{R}^d$. It is easy to see that $A(y) - A(x) = M(y - x)$, where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 \\ 3 & 3 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ d & \binom{d}{2} & \cdots & \binom{d}{2} & d & 1 \end{bmatrix}$$

Stable Vectors and Their Group Structure

Notation

- $\mathfrak{R}$: the set of all stable vectors in torus $\mathbb{T}^2$

Lemma III.

The set of all stable vectors forms a discrete subgroup in $\mathbb{R}^2$.

Partitioning the Torus into Convex Polygones

Lemma IV.
There exists N such that for all $n > N$, $\mathcal{M}_n$ is a partition of $\mathbb{T}^2$ into convex polygones and every U from $\mathcal{U}_n$ consists of exactly $\#\mathfrak{R}$ parts from $\mathcal{M}_n$.

$$\boxed{\mathtt{pw}(k) = \frac{\#\mathcal{M}_k}{\#\mathfrak{R}} \quad \text{for} \quad k > N}$$

Counting the Number of Regions in a $\boxed{\textit{Line}}$ Torus Partition

Counting the Number of Regions in a $\boxed{\textit{Line}}$ Torus Partition

First:

- Assume that each point belong to at most 2 lines
- Assuming that all the parts in the partition are convex, the number of regions in the partitions is equal to number of points, that belong to exactly 2 lines
- This number is equal to the determinant of some matrix 2×2
- This determinant is a polynomial $P(n)$.

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Second:

- Some points belong to more than 2 lines. So our real number of parts differs from $P(n)$ by some error.
- All such points can be classified, the number of such points is affine in n.
- This error is an affine function in n.

Mapping on the Torus

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Lemma V.

- Consider the torus $\mathbb{T}^2 = [0,1)^2$ and the mapping $f : \mathbb{T}^2 \to \mathbb{T}^2$, as in the previous lemma. Let k be a natural number.
- The union of the pre-images of all critical sets under the maps

$$f^0, f^{-1}, \dots, f^{-(k-1)}$$

  divides $\mathbb{T}^2$ into $N(k)$ connected regions.
- Then $N(k)$ starting from some k_0 is a polynomial in k and the degree of this polynomial is 3.

The factor complexity of the most significant digits of the decimal expansion of the sequence 3^{n^2} for large enough length is a polynomial with degree 3.

Let's recaps for general case

Lemma 1. The sequence $(v_i)_{i \in \mathbb{N}_{>0}}$ is dense in $\mathbb{T}^d$.

Lemma 2. A finite sequence of digits $\mathbf{u}$ is a factor of $\mathbf{w}$ if and only if the set $U_{\mathbf{u}} \subset \mathbb{T}^d$ has non-empty interior.

Lemma 3. The set of all stable vectors forms a discrete subgroup in $\mathbb{R}^d$.

Lemma 4. There exists N such that for all $n > N$, $\mathcal{M}_n$ is a partition of $\mathbb{T}^d$ into convex polyhedrons and every U from $\mathcal{U}_n$ consists of exactly $\#\mathfrak{R}$ parts from $\mathcal{M}_n$.

Counting the Number of Regions in a *Hyperplane* Torus Partition

Counting the Number of Regions in a $\boxed{\textit{Hyperplane}}$ Torus Partition

First:

- Assume that each point belong to at most d hyperplanes
- Assuming that all the parts in the partition are convex, the number of parts in the partitions is equal to number of points, that belong to exactly d hyperplanes
- This number is equal to the determinant of some matrix $n \times n$
- This determinant is a polynomial $P(n)$.

Counting the Number of Regions in a $\boxed{\textit{Hyperplane}}$ Torus Partition

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- Assuming that all the parts in the partition are convex, the number of parts in the partitions is equal to number of points, that belong to exactly d hyperplanes
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Lemma 5. Consider the torus $\mathbb{T}^d = [0,1)^d$ and the mapping $f : \mathbb{T}^d \to \mathbb{T}^d$, as in the previous lemma. Let k be a natural number. The union of the pre-images of all critical sets under the maps

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divides $\mathbb{T}^d$ into $N(k)$ connected regions. Then $N(k)$ starting from some k_0 is a polynomial in k and the degree of this polynomial is $\boxed{d(d+1)/2}$

Conclusions (I)

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- $(2, 10)$: admissible pair $\Longrightarrow \mathrm{p}_{2,10}(k) = 4k + 5, \quad$ for all k

2. F.C. of sequences of the most significant digits of the decimal expansion of a^n.

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3 F.C. of sequences of the most significant digits of the b-expansion of a^n.

- (a, b): admissible pair $\implies \mathrm{p}_{a,b}(k) = ck + d$, for all k
- (a, b): strong admissible pair $\implies \mathrm{p}_{a,b}(k) = ck + d$, for large enough k

Conclusions (II)

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- $(3, 10)$: admissible pair $\Longrightarrow$ $\mathtt{p}(k) = P(k)$,

$$\deg(P) = 3, \quad \text{for large enough} \quad k$$

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5 F.C. of sequences of the most significant digits of the *b*-expansion of a^{n^d}.

Conclusions (II)

4 F.C. of sequences of the most significant digits of the decimal expansion of 3^{n^2}.

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- (a, b): weak admissible pair $\Longrightarrow$ $\mathtt{p}(k) = P(k)$,

$$\deg(P) = d(d+1)/2, \quad \text{for large enough } k$$

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Appendix (I)

Definition

A connected Lie group is a group G equipped with the structure of a smooth manifold such that:

1. **Smooth Group Operations:**
   - The multiplication map $m : G \times G \to G,\ m(g, h) = g \cdot h$, is smooth.
   - The inversion map $\mathrm{inv} : G \to G,\ \mathrm{inv}(g) = g^{-1}$, is smooth.
2. **Connectedness:**
   - The manifold G is connected as a topological space. That is, there do not exist two non-empty, disjoint open subsets U and V of G such that $G = U \cup V$.

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Example

- **General Linear Group** $GL(n, \mathbb{R})$: This group is not connected; it has two connected components corresponding to matrices with positive and negative determinants.
- **Special Orthogonal Group** $SO(n)$: This group is connected for all $n \geqslant 2$.
- **Heisenberg Group**: This is an example of a connected, simply connected Lie group.

Appendix (II)

- Let G be a Lie group.

Definition

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Example

1. **Integer Lattice in $\mathbb{R}^n$:**
   - Consider $G = \mathbb{R}^n$, which is an abelian Lie group under addition.
   - The subgroup $\Gamma = \mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ because the quotient $\mathbb{R}^n/\mathbb{Z}^n$ is compact (specifically, it is the n-dimensional torus).
2. **Special Linear Group $SL(n, \mathbb{Z})$ in $SL(n, \mathbb{R})$:**
   - Let $G = SL(n, \mathbb{R})$, the group of $n \times n$ real matrices with determinant 1.
   - The subgroup $\Gamma = SL(n, \mathbb{Z})$ consists of matrices with integer entries and determinant 1.
   - Γ is a lattice in G because G/Γ has finite volume with respect to the Haar measure.
3. **Heisenberg Group:**
   - The discrete Heisenberg group can serve as a lattice in the continuous Heisenberg group, which is a nilpotent Lie group.

Appendix (III)

Theorem 8. [invariant measures](Ratner. 1990)

Let G be a connected Lie group, H a lattice subgroup of G (i.e., G/H has finite volume), and U a subgroup generated by unipotent elements acting on G/H. Then, any U-invariant and ergodic probability measure on G/H is homogeneous; that is, it is the Haar measure on a closed orbit of a subgroup containing U.

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Let G be a connected Lie group, H a lattice subgroup of G (i.e., G/H has finite volume), and U a subgroup generated by unipotent elements acting on G/H. Then, any U-invariant and ergodic probability measure on G/H is homogeneous; that is, it is the Haar measure on a closed orbit of a subgroup containing U.

- **Invariant Measures:** The theorem classifies all measures on G/H that are invariant and ergodic under the action of unipotent flows.
- **Homogeneity:** Such measures are supported on closed orbits of subgroups, meaning the behavior of unipotent flows is highly regular and structured.

Theorem 8. [orbit closures](Ratner. 1990)

For the action of a unipotent flow U on G/H, the closure of any orbit Ux (for $x \in G/H$) is homogeneous; that is, it is the orbit of a closed subgroup containing U.

Appendix (IV)

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For the action of a unipotent flow U on G/H, the closure of any orbit Ux (for $x \in G/H$) is homogeneous; that is, it is the orbit of a closed subgroup containing U.

Equidistribution: Ratner's theorems imply that sequences generated by unipotent flows become uniformly distributed over certain subsets of G/H.

Number Theory: These results have applications in solving problems related to Diophantine approximations and the distribution of integer points on algebraic varieties.

Dynamics on the Torus: In the context of the original problem, unipotent flows can be used to study the distribution properties of sequences like $\{n^k\alpha\} \mod 1$, where α is irrational, and $k \geq 2$.