Mahler equations for Zeckendorf numeration

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Outline

Prologue: numeration systems and automaticity

Act 1: the classical base-q numeration

Act 2: the Zeckendorf numeration

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Prologue: numeration systems and automaticity

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Act 2: the Zeckendorf numeration

Numeration systems

A numeration system (U, B) consists of

- a sequence of natural numbers $U = (u_n)_{n \geqslant 0}$ with $u_0 = 1$,
- a finite ordered digit set B such that for each $n \in \mathbb{N}$ there are $b_k, \ldots, b_0 \in B$ with $n = \sum_{i=0}^{k} b_i u_i$.

We say that $b_k \cdots b_0$ is the canonical representation of n if $b_k \cdots b_0$ is the greatest representation of n for the lexicographic order.
We write

$$(n)_U := b_k \cdots b_0 \quad \text{and} \quad [b_k \cdots b_0]_U := n$$

Example (The base-q numeration system for $q \geqslant 2$)
$U = (q^n)_{n \geqslant 0}$, $B = \{0, 1, \ldots, q-1\}$. If $q = 3$, then $(26)_3 = 222$.

Zeckendorf numeration

Recall the Fibonacci numbers $(F_n)_{n\geqslant 0}$, defined by

$$\begin{aligned} F_{-2} &= 0, \\ F_{-1} &= 1, \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geqslant 0. \end{aligned}$$

The Zeckendorf numeration system is $Z = ((F_n)_n, B = \{0,1\})$.

The canonical expansion $(n)_Z = b_n \cdots b_0$ satisfies $b_i b_{i+1} = 0$ for each i.

Example

n	55	34	21	13	8	5	3	2	1
61	0	1	0	1	1	1	0	0	1
61	0	1	1	0	0	1	0	0	1
61	1	0	0	0	0	1	0	0	1

$$(61)_Z = 100001001 \text{ and } [10111001]_Z = 61$$

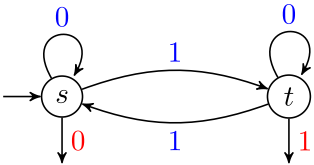
Automaticity in U

A sequence $(a_n)_{n\geqslant 0}$ taking values in a finite set A is U-automatic if there is a deterministic finite automaton whose output is a_n when fed with $(n)_U$.
If U is the base-q numeration, we will say that $(a_n)_{n\geqslant 0}$ is q-automatic.

Example (Thue-Morse $a_n = |(n)_2|_1 \bmod 2$)

$B = \{0, 1\}$

$A = \{0, 1\}$



$$(17)_2 = 10001 \quad \text{and} \quad s \xrightarrow{1} t \xrightarrow{0} t \xrightarrow{0} t \xrightarrow{0} t \xrightarrow{1} s \quad \text{and} \quad s \xrightarrow{0}$$

Thus $a_{17} = 0$. $a_n = 0$ precisely when $(n)_2$ contains an even number of the digit 1.

Outline

Characterisation of q-automaticity for $q = p^n$, p prime

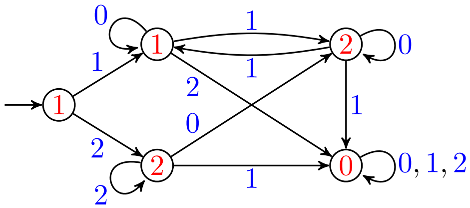
Theorem (Christol 1980)

Let $(a_n)_{n \geqslant 0}$ be a sequence in $\mathbb{F}_q$, with $q = p^n$ for some $n \geqslant 1$. Then $(a_n)_{n \geqslant 0}$ is q-automatic if and only if $f(x) = \sum_{n \geqslant 0} a_n x^n$ is algebraic over $\mathbb{F}_q(x)$.

Example (Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$)

- $y = \sum_{n \geqslant 0} C_n x^n$ satisfies $xy^2 - y + 1 = 0$ over $\mathbb{Q}$,
- $y = \sum_{n \geqslant 0} (C_n \bmod 3) x^n$ satisfies $xy^2 + 2y + 1 = 0$ over $\mathbb{F}_3$,

  and hence $(C_n \bmod 3)_{n \geqslant 0}$ is automatic.



Limitations of Christol's theorem

Christol's theorem does not provide a characterisation of q-automatic sequences if q not a power of a prime.

Question Is there a generalisation of Christol's theorem to all q-automatic sequences?

Answer Yes.

Question Is there a version of Christol's theorem for the Zeckendorf numeration system?

Answer Yes!

Christol's theorem, example

Let us find the annihilating polynomial for the Thue-Morse sequence, defined as

$a_n = 0$ precisely when $(n)_2$ contains an even number 1s.

$$\begin{aligned} f(x) &= \sum_n a_{2n} x^{2n} + \sum_n a_{2n+1} x^{2n+1} \\ &= \sum_n a_n x^{2n} + x \sum_n (a_n + 1) x^{2n} \end{aligned}$$

so, with $s(x) = \frac{1}{1+x}$,

$$f(x) = (1+x) f(x^2) + x s(x^2) \ \ \& \ \ s(x) = (1+x) s(x^2) \tag{1}$$

and

$$f(x^2) = (1+x^2) f(x^4) + x^2 s(x^4), \ \ \& \ \ s(x^2) = (1+x^2) s(x^4) \tag{2}$$

Substituting (2) in (1) we get that $f(x)$ is a root of the Ore polynomial

$$x f(x) = (1+x) f(x^2) + (1+x)^4 f(x^4).$$

From automatic to regular sequences

Definition

A sequence $(a_n)_{n\geqslant 0}$ taking values in a finite set A is U-automatic if there is a deterministic finite automaton whose output is a_n when fed with $(n)_U$.

Definition (Allouche-Shallit)

A sequence $(a_n)_{n\geqslant 0}$ taking values in a ~~finite set A~~ commutative ring R is ~~U-automatic~~ U-regular if there is a ~~deterministic finite automaton~~ weighted automaton whose output is a_n when fed with $(n)_U$.

Theorem (Allouche-Shallit 1992)

A sequence is q-regular and takes on finitely many values if and only if it is q-automatic.

Examples

All from Allouche-Shallit's article, 1992:

- $a_n = \#$ 1's in $(n)_2$ defines a 2-regular sequence
- the sequence
$$0, 2, 6, 8, 20, 24, \dots,$$
which lists the numerators of the left endpoints of the Cantor set, is 2-regular.
- $a_n = (n^j)_{n \geqslant 0}$ is 2-regular,
- $a_n = \sum_{i=1}^{n} \lfloor \log_a i \rfloor$ is q-regular.
- The number of comparisons required to mergesort n items,
- For $a \in \mathbb{R}$, $(a^n)_{n \geqslant 0}$ is q-regular if and only if $a = 0$ or a is a root of unity.

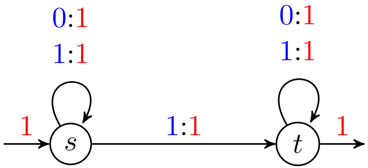
Weighted automata

A weighted automaton $\mathcal{A}$ with weights in a ring R consists of

- a finite state set S,
- an alphabet B
- a transition weight function $\Delta : S \times B \times S \to R$ which assigns a weight to each labelled edge, denoted $s \xrightarrow{b:r} s'$, and
- initial and final weight functions $I : S \to R$ and $F : S \to R$.

Example

Let $B = \{0, 1\}$ and $R = \mathbb{F}_2$.



Generating sequences using weighted automata

In a weighted automaton, there may be many paths that a given word labels. We are interested in the sum of the weights of all paths that this word labels.

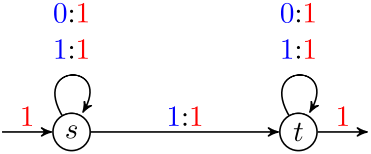
The word 10110 labels three different paths, each of weight 1:

$$\xrightarrow{1} s \xrightarrow{1:1} t \xrightarrow{0:1} t \xrightarrow{1:1} t \xrightarrow{1:1} t \xrightarrow{0:1} t \xrightarrow{1}$$

$$\xrightarrow{1} s \xrightarrow{1:1} s \xrightarrow{0:1} s \xrightarrow{1:1} t \xrightarrow{1:1} t \xrightarrow{0:1} t \xrightarrow{1}$$

$$\xrightarrow{1} s \xrightarrow{1:1} s \xrightarrow{0:1} s \xrightarrow{1:1} s \xrightarrow{1:1} t \xrightarrow{0:1} t \xrightarrow{1}$$

and since $(22)_2 = 10110$ and $R = \mathbb{F}_2$, then $u_{22} = 3 \bmod 2 = 1$.



q-Mahler equations

Let R be any commutative ring and let $q \geqslant 2$. Define the linear operator $\Phi : R[\![x]\!] \to R[\![x]\!]$ as

$$\Phi(f(x)) = f(x^q).$$

Let $A_i(x) \in R[x]$ be polynomials. The equation

$$P(x, y) = \sum_{i=0}^{d} A_i(x)\Phi^i(y) = 0$$

is called a q-Mahler equation.
If $f \in R[\![x]\!]$ satisfies $P(x, f(x)) = 0$, then it is called q-Mahler.

If $q = p^k$, then a q-Mahler equation over $\mathbb{F}_q$ is just a polynomial:

$$\left(\sum_{n \geqslant 0} f_n x^n\right)^q = \sum_{n \geqslant 0} f_n^q x^{qn} = \sum_{n \geqslant 0} f_n x^{qn}$$

Theorem (Christol 1980)

Let q be a power of a prime, and let $(u_n)_{n\geqslant 0}$ be a sequence over $\mathbb{F}_q$. The $(u_n)_{n\geqslant 0}$ is q-automatic if and only if $f(x) = \sum_{n\geqslant 0} a_n x^n$ is algebraic over $\mathbb{F}_q(x)$.

Theorem (Becker 1992, Dumas 1993)

Let $q \geqslant 2$, and let $(u_n)_{n\geqslant 0}$ be a sequence over a commutative ring R.

- *If $(u_n)_{n\geqslant 0}$ is q-regular sequence then $f(x) = \sum_{n\geqslant 0} a_n x^n$ is the solution of a q-Mahler equation,*
- *if $f(x) = \sum_{n\geqslant 0} a_n x^n$ is the solution of an isolating q-Mahler equation, i.e., of the form $y = \sum_{i=1}^{d} A_i(x)\Phi^i(y)$, then $(u_n)_{n\geqslant 0}$ is q-regular.*

From isolating Mahler equations to weighted automata

Theorem (C., Yassawi, 2024)

Let $q \geqslant 2$ be a natural number. There exists a universal q-automaton $\mathcal{A}$, such that any isolating q-Mahler equation $P(x,y)$ over a commutative ring R with initial condition f_0 provides weights for $\mathcal{A}$, so that the corresponding weighted automaton generates the solution $f(x)$ of $P(x,y)$ with $f(0) = f_0$.

- The universal q-automaton $\mathcal{A}$ consists of a countable states set S and transitions in $S \times \{0, 1, \dots, q-1\} \times S$.
- Given an isolating q-Mahler equation $P(x,y) = y - \sum_{i=1}^{d} \left(\sum_{j=0}^{h} \alpha_{i,j} x^j \right) \Phi^i(y)$, we use its coefficients $\alpha_{i,j}$ as weights, setting other edge weights to zero, so reducing $\mathcal{A}$ to a weighted automaton.

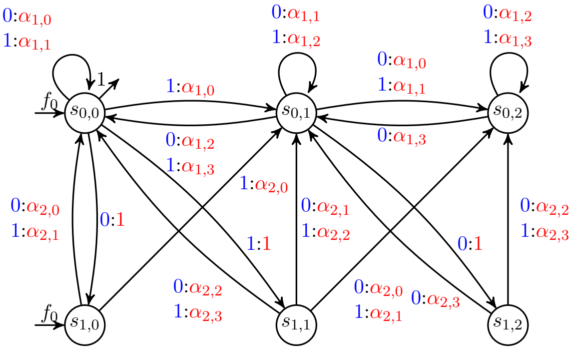
Example

For a 2-Mahler equation with height 3 and exponent 2

$$f(x) = A_1(x) f(x^2) + A_2(x) f(x^4)$$

$$\text{where} \quad A_1(x) = \alpha_{1,0} + \alpha_{1,1} x + \alpha_{1,2} x^2 + \alpha_{1,3} x^3$$

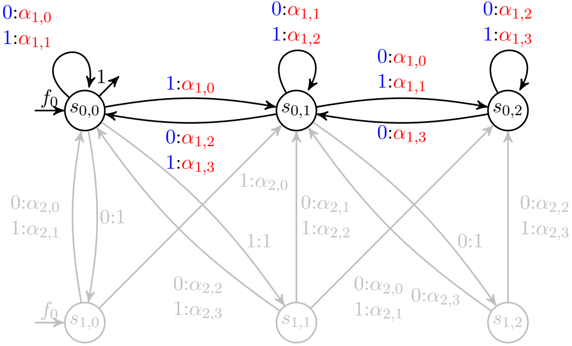
$$\text{and} \quad A_2(x) = \alpha_{2,0} + \alpha_{2,1} x + \alpha_{2,2} x^2 + \alpha_{2,3} x^3$$



Example

whilst for a 2-Mahler equation with height 3 and exponent 1

$$f(x) = (\alpha_{1,0} + \alpha_{1,1}x + \alpha_{1,2}x^2 + \alpha_{1,3}x^3)f(x^2)$$



Outline


ZECKENDORF BLVD
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From q-Mahler equations to Z-Mahler equations

Theorem (Becker 1992, Dumas 1993)

Let $q \geqslant 2$, and let $(u_n)_{n \geqslant 0}$ be a sequence over a commutative ring R.

- *If (u_n) is q-regular, then it is the solution of a q-Mahler equation, and*
- *if (u_n) is a solution of an isolating q-Mahler equation, then it is q-regular.*

Theorem (C., Yassawi 2024)

Let $(u_n)_{n \geqslant 0}$ be a sequence over a commutative ring R.

- *If (u_n) is ~~q-regular~~ Z-regular, then it is the solution of a ~~q-Mahler~~ Z-Mahler equation,*
- *if (u_n) is a solution of an isolating ~~q-Mahler~~ Z-Mahler equation, then it is ~~q-regular~~ Z-regular.*

New ingredients

Our proof strategy was to emulate our proof in the case of q-numeration, i.e.,

- to define the linear Z-version of the map $m \mapsto qm$, and
- to define the appropriate concept of a Z-Mahler equation,

in order to construct a weighted Z-automaton directly from an isolating Z-Mahler equation.

The Zeckendorf analogue of $n \mapsto qn$

The map $f(n) = qn$ can be written $f(n) := [w0]_q$ where $w = (n)_q$. So, for $(n)_Z = w$, define $\phi : \mathbb{N} \to \mathbb{N}$ as

$$\phi(n) := [w0]_Z.$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi(n)$	0	2	3	5	7	8	10	11	13	15	16	18	20	21

Issue: ϕ is not linear.

For example,

$$3 = \phi(2) = \phi(1+1) \neq 2\phi(1) = 4.$$

Dealing with the nonlinearity of ϕ

Recall $\phi(n) := [(n)_Z\, 0]_Z$. Define the linearity defect δ by

$$\delta(m, n) := \phi(m + n) - \phi(m) - \phi(n).$$

A simple application of Binet's formula gives

Lemma

For natural numbers m, n, we have $-1 \leqslant \delta(m, n) \leqslant 1$.

In other words, ϕ is almost linear.

We would like to track the linearity defect.

Regularity of Z-expansions

Given a finite set $C \subset \mathbb{Z}$, consider

$$\mathcal{L}_C := \{w \in C^* : [w]_Z = 0\}$$

Example ($C = \{0, 1, -1\}$)

The following words belong to $\mathcal{L}_C$:

55	34	21	13	8	5	3	2	1
0	1	−1	−1	0	−1	1	1	0
1	−1	0	−1	0	−1	−1	0	0

Theorem (Frougny)

For $C \subset \mathbb{Z}$ finite, $\mathcal{L}_C$ is a regular set of words.

Corollary

There exists a deterministic automaton, which on input of $(m)_Z$ and $(n)_Z$, outputs the linearity defect $\delta(m-n, n)$ for $m \geqslant n \geqslant 0$.

Going back to our strategy

We have defined the almost linear map $m \mapsto \phi(m)$.

We now define the Z-version of $\Phi_q(\sum_n f_n x^n) = \sum_n f_n x^{qn}$.

Define the Z-Mahler operator $\Phi : R[\![x]\!] \to R[\![x]\!]$ as

$$\Phi\Big(\sum_{n\geqslant 0} f_n x^n\Big) := \sum_{n\geqslant 0} f_n x^{\phi(n)}.$$

The equation

$$P(x,y) = \sum_{i=0}^{d} A_i(x)\Phi^i(y) = 0$$

with $A_i(x) \in R[x]$, is a Z-Mahler equation.

If $f \in R[\![x]\!]$ satisfies $\sum_{i=0}^{d} A_i(x)\Phi^i(f) = 0$, then it is Z-Mahler.

Results

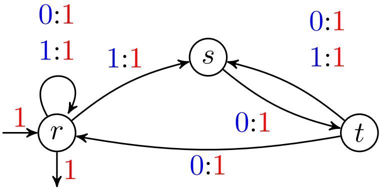
Theorem (C., Yassawi 2024)

Let $(u_n)_{n\geqslant 0}$ be a sequence over a commutative ring R.

- *If (u_n) is Z-regular, then it is the solution of a Z-Mahler equation*
- *if (u_n) is a solution of an isolating Z-Mahler equation, then it is Z-regular.*

Example ($a_n = \#$ of Zeckendorf expansions of n)

$$f(x) = \sum_n a_n x^n = \prod_n (1 + x^{F_n}) \quad \text{and} \quad f(x) = (1+x)\Phi(f(x)).$$



Open questions

- Allouche and Shallit show that for $a \in \mathbb{R}$, $(a^n)_{n \geqslant 0}$ is q-regular if and only if $a = 0$ or a is a root of unity. Is there a similar result for Z-regular sequences?
- For $R = \mathbb{C}$, Bell, Chyzak, Coons, & Dumas characterise q-regular series in terms of the q-Mahler equations they satisfy. Is there a similar characterisation for Zeckendorf numeration?
- Adamczewski-Bell and Shäfke-Singer show that a sequence which is both k- and ℓ-Mahler over a field of characteristic zero, with k and ℓ multiplicatively independent, must be rational. Which series are both k- and Z-Mahler?