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Palindromic length of infinite aperiodic words

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Finite and Infinite Words

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Let $u = u_1 u_2 \cdots u_n$ be a nonempty word of length n, where u_i are letters and $i \in \{1, 2, \ldots, n\}$. We say that u is a *palindrome* if $u_1 u_2 \cdots u_n = u_n u_{n-1} \cdots u_1$.

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For example, the words "noon" and "level" are palindromes.

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If x is an infinite word and k is an integer such that $\mathrm{PL}(u) \leq k$ for every factor u of x then x is ultimately periodic.

[Frid, Puzynina, Zamboni: *On palindromic factorization of words*, Adv. Appl. Math., 2013.]

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Given a word $t \in \Sigma^+$, let $\mathrm{NPP}(t)$ denote the set of all nonempty non-periodic palindromic prefixes of t.

We say that a word $t \in \Sigma^+$ is *ordinary* if for every factor u of t we have that $|\,\mathrm{NPP}(t)| \geq |\,\mathrm{NPP}(u)|$.

Let z be an (ordinary) factor with $\mathrm{NPP}(z) = h$. Let $\{z_1, z_2, \ldots, z_h\} = \mathrm{NPP}(z)$, $z = z_h$, and $|z_1| < |z_2| < \cdots < |z_h|$. Let $\widehat{z}_1, \widehat{z}_2, \ldots, \widehat{z}_{h-1} \in \mathrm{Pal}^+$ be such that $z_{i+1} = z_i \widehat{z}_i z_i$, where $i \in \{1, 2, \ldots, h-1\}$.

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Definition

Let $B_1 = \{(1,1)\}$. Given $j \in \mathbb{N}(2,h)$, let

$$\widehat{B}_j = \{(g, \overline{e}) \mid (g,e) \in B_{j-1} \text{ and } \overline{e} = e + |z_{j-1}\widehat{z}_{j-1}|\} \text{ and}$$
$$B_j = \widehat{B}_j \cup B_{j-1} \cup \{(j,1)\}.$$

Let $B = B_h$ and let $\widetilde{B} = \{e \mid (g,e) \in B\}$.
If $(g,e) \in B$ then we say that e is a *base position* of z_g.

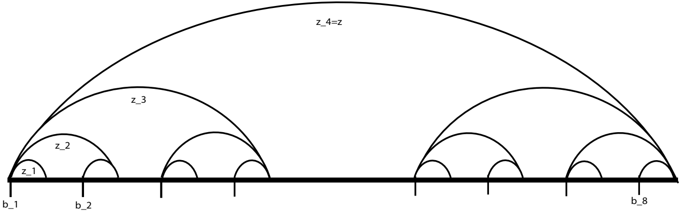

Figure: Base positions

Let $\mathbb{N}(n_1, n_2) = \{n \in \mathbb{N}_0 \mid n_1 \leq n \leq n_2\}$.

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$$\begin{aligned} \mathrm{UC} = \{S \subseteq \mathbb{N}(1, |z|) \mid \text{ if } \mu_1 \leq \mu_2 \in \mathbb{N}(1, |z|) \text{ and} \\ \xi \in \mathrm{Period}(z[\mu_1, \mu_2]) \\ \text{then there is } E \subseteq \mathbb{N}(\mu_1, \mu_2) \text{ such that } |E| \leq c_2 \\ \text{and } S \cap \bigcup_{\delta \in E} \mathbb{N}(\delta, \delta + \xi - 1) = S \cap \mathbb{N}(\mu_1, \mu_2)\}. \end{aligned}$$

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Proposition

We have that $\widetilde{B} \in \mathrm{UC}$.

Suppose $n_1 < n_2 \in \mathbb{N}(1, |z|)$. Let

- $B(n_1, n_2) = \{(g, e) \in B \mid n_1 \leq e \leq e + |z_g| - 1 \leq n_2\}$,
- If $B(n_1, n_2) \neq \emptyset$ then let $\text{height}(n_1, n_2) = \max\{g \mid (g, e) \in B(n_1, n_2)\}$ and $\text{width}(n_1, n_2) = \max\{e_1 + |z_{g_1}| - e_2 \mid (g_1, e_1), (g_2, e_2) \in B(n_1, n_2)\}$.
- If $B(n_1, n_2) = \emptyset$ then let $\text{width}(n_1, n_2) = \text{height}(n_1, n_2) = 0$.

We call $\text{height}(n_1, n_2)$ and $\text{width}(n_1, n_2)$ the *height* and the *width* of (n_1, n_2), respectively.

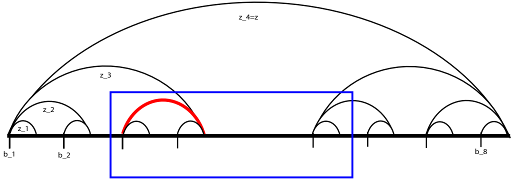

Figure: Height of (n_1, n_2)

We consider the special letter b that we use for "padding" as follows: Let Σ_0 and Σ be two finite alphabets such that $\Sigma_0 \cup \{b\} = \Sigma$ and $b \notin \Sigma_0$.

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Given an infinite word $x_0 \in \Sigma_0^\infty$, let $\mathrm{pad}(x_0) = x \in \Sigma^\infty$ be such that $x[2i-1] = b$ and $x[2i] = x_0[i]$, where $i \in \mathbb{N}_1$.

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Given a finite word $u \in \Sigma_0^+$, let $\mathrm{pad}(u) = u_1 b u_2 b \ldots b u_n \in \Sigma^+$, where $u = u_1 u_2 \cdots u_n$, $n = |u|$, and $u_1, u_2, \ldots, u_n \in \Sigma_0$.

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We call $\mathrm{pad}(x_0)$ and $\mathrm{pad}(u)$ *padded* words. Note that every second letter in padded words is the letter b.

For $u \in \mathsf{pad}(\Sigma_0^+)$ we define the *padded palindromic length* $\mathsf{PPL}(u)$ to be the minimal number k such that $u = p_1 b p_2 b \cdots b p_k$, where $p_1, p_2, \ldots, p_k \in \Sigma^+$ are palindromes.

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Given $x_0 \in \Sigma_0^\infty$, let

$$\mathrm{maxPL}(x_0) = \max\{\mathrm{PL}(u) \mid u \in \Sigma_0^+ \text{ is a factor of } x_0\} \text{ and}$$
$$\mathrm{maxPPL}(x_0) = \max\{\mathrm{PPL}(\mathrm{pad}(u)) \mid u \in \Sigma_0^+ \text{ is a factor of } x_0\}.$$

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Obviously $u \in \Sigma_0^+$ is a palindrome if and only if $\mathrm{pad}(u) \in \Sigma^+$ is a palindrome.

Lemma

Suppose $x_0 \in \Sigma^\infty$. We have that $\mathrm{maxPL}(x_0) < \infty$ if and only if $\mathrm{maxPPL}(x_0) < \infty$.

Suppose $w_0 \in \Sigma_0^\infty$ to be an infinite aperiodic word such that $\mathrm{maxPL}(w_0) < \infty$ and $\mathrm{maxPPL}(w_0) = k < \infty$. Let $w = \mathrm{pad}(w_0) \in \Sigma^\infty$.

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Let $c_1 = 5$, $c_2 = 8$, and $c_3 = 10$ be constants. Given $h, m \in \mathbb{N}_1$, let $\lambda(h, m) = c_2^m (2c_1 c_3 h)^{m^2} \in \mathbb{N}_1$ be a function.

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Let $h_0 \in \mathbb{N}_1$ be such that if $h \geq h_0$ then

$$2^{h-1} > k(c_3h)^m\lambda(h,m) \text{ for all } m \in \mathbb{N}(1,k). \tag{1}$$

Obviously such h_0 exists, since $\lim_{h\to\infty} \frac{k(c_3h)^k\lambda(h,k)}{2^h} = 0$.

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We show that w contains an ordinary palindromic factor z such that b is a prefix of z and $|\,\mathrm{NPP}(z)| \geq h_0$

Given $m \in \mathbb{N}_1$, let

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Let $D \subseteq \mathbb{N}_1$, $\xi \in \mathbb{N}_1$, and $\overline{D} \subseteq D$. If $\overline{D} = \emptyset$ or $\max(\overline{D}) - \min(\overline{D}) + 1 \leq \xi$ then we call $\overline{D}$ a ξ-*cut* of D.

Let

$$\widetilde{\mathsf{NPS}} = \{(D, \xi) \mid D \subseteq \mathbb{N}(1, |z|) \text{ and } \xi \in \mathbb{N}_1 \text{ and } D \neq \emptyset \text{ and }$$
$$D = \mathsf{Spread}(D, \xi) \cap \mathsf{Close}(D) \text{ and } \xi \in \mathsf{Period}(z[\min(D), \max(D)])\}.$$

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Given $(D, \xi) \in \widetilde{\mathrm{NPS}}$, let $\varphi(D, \xi) = D$. Given $M \subseteq \widetilde{\mathrm{NPS}}$, let $\varphi(M) = \{\emptyset\} \cup \{D \mid (D, \xi) \in M\}$ and let $\widetilde{\varphi}(M) = \bigcup_{D \in \varphi(M)} D$.

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Given $(D, \xi) \in \widetilde{\mathrm{NPS}}$, let $\mathrm{Cut}(D, \xi) = \{\overline{D} \subseteq D \mid \overline{D} \text{ is a } \xi\text{-cut of } D\}$.

We define the *nested periodic structure*:

Definition

Let $\mathrm{NPS}(0) = \{(D, \xi) \in \widetilde{\mathrm{NPS}} \mid |D| = 1\}$. Given $m \in \mathbb{N}_1$, let

$$\mathrm{NPS}(m) = \{(D, \xi) \in \widetilde{\mathrm{NPS}} \mid \text{ if } \overline{D} \in \mathrm{Cut}(D, \xi) \text{ then there is } M \subseteq \mathrm{NPS}(m-1) \text{ such that } |M| \leq \theta(m) \text{ and } \overline{D} \subseteq \widetilde{\varphi}(M) \subseteq \mathrm{Close}(\overline{D})\}.$$

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We call $(D,\xi) \in \mathrm{NPS}(m)$ a *nested periodic structure* (NPS) of *degree* m and we call D an *NPS cluster* of *degree m*.

Given $m \in \mathbb{N}_1$, we show that there is $M \subseteq \mathsf{NPS}(m)$ such that

$$V(m) \subseteq \bigcup_{(D,\xi) \in M} D \text{ and } |M| \leq (c_3 h)^m. \tag{2}$$

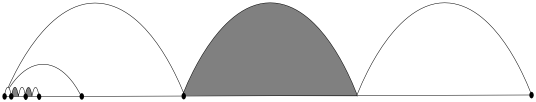

Figure: Palindromic extensions

Thank you