Analogs of Markoff and Lagrange spectra on one-sided shift spaces

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1. Motivation: Diophantine approximation problems (simple case)

2. Markoff-Lagrange spectrum of shift spaces with cylinder order

3. Symmetric alphabets

Diophantine approximation related to geometric progressions

Fact

(1) $\alpha \in \mathbb{R}$, $\alpha > 1$: fixed.
$\Rightarrow$ for almost every $\xi \in \mathbb{R}$, $(\xi \alpha^n)_{n \geq 0}$: uniformly distributed modulo one (Weyl, 1916).
(2) $b \in \mathbb{Z}$, $b \geq 2$: fixed. $\xi \in \mathbb{R}, \xi \geq 0$.
Then $(\xi b^n)_{n \geq 0}$: uniformly distributed modulo one if and only if ξ is normal in base b.
However, for each individual ξ, it is generally difficult to investigate the distribution ξb^n modulo one.

Target $(b \in \mathbb{Z}, b \geq 2)$

$\limsup_{n \to \infty} \|\xi b^n\|$, where $\|x\| = \min\{|x - m| \mid m \in \mathbb{Z}\} \in [0, 1/2]$.
Example: $\|5.9\| = |5.9 - 6| = |-0.1| = 0.1$.
$\mathcal{L}(b) := \{\limsup_{n \to \infty} \|\xi b^n\| \mid \xi \in \mathbb{R}\} \subset [0, 1/2]$.

Multiplicative Lagrange spectrum

Target ($b \in \mathbb{Z}, b \geq 2$)

$\limsup_{n\to\infty} \|\xi b^n\|$, where $\|x\| = \min\{|x-m| \mid m \in \mathbb{Z}\} \in [0, 1/2]$.
$\mathcal{L}(b) = \{\limsup_{n\to\infty} \|\xi b^n\| \mid \xi \in \mathbb{R}\} \subset [0, 1/2]$.

Theorem (Main idea was obtained by Dubickas 2006)

$(\exists C(b) \in \mathcal{L}(b))$ satisfying the following:

1. $(0, C(b)) \cap \mathcal{L}(b)$ is an infinite discrete set denoted as $\{d^{(b)}(0) < d^{(b)}(1) < d^{(b)}(2) < \cdots\}$, where $\lim_{m\to\infty} d^{(b)}(m) = C(b)$. In particular, $C(b)$ is the minimal limit point of $\mathcal{L}(b)$.
2. $(\forall m \geq 0) d^{(b)}(m) \in \mathbb{Q}$, $C(b)$ is a transcendental number.

We call $[0, C(b)] \cap \mathcal{L}(b)$ the discrete part of $\mathcal{L}(b)$.

Discrete part of $\mathcal{L}(b)$ and substitution $(b \in \mathbb{Z}, b \geq 2)$

$\mathcal{L}(b) = \{\limsup_{n\to\infty} \|\xi b^n\| \mid \xi \in \mathbb{R}\} \subset [0, 1/2]$. $C(b)$: min. lim. p.t. of $\mathcal{L}(b)$.
$(0, C(b)) \cap \mathcal{L}(b) = \{d^{(b)}(0) < d^{(b)}(1) < d^{(b)}(2) < \cdots\}$

Substitution and additional rule

(1) $\tau : \{0,1\}^* \to \{0,1\}^*$. $\tau(0) = 1, \tau(1) = 100$.
$A_0 := 0$, $A_{m+1} := \tau(A_m)$. $A_1 = 1, A_2 = 100, A_3 = 10011, \ldots$.
$A_\infty := \lim_{m\to\infty} A_m = 10011100100\ldots$: fixed point of τ.
(2) For $\mathbf{x} = x_0 x_1 \ldots \in \{0,1\}^\infty$, put $\Phi(\mathbf{x}) := 10^{x_0}\overline{1}0^{x_1}10^{x_2}\overline{1}0^{x_3}\ldots$, where $\overline{1} = -1$.
If $x_n = 1$, then insert 0. $v_m := \Phi(A_m^\infty) = \Phi(A_m A_m A_m \ldots), v_\infty := \Phi(A_\infty)$.
Example: $A_2^\infty = 100100100\ldots$ $v_2 = 10\overline{1}1\overline{1}01\overline{1}10\overline{1}1\overline{1}\ldots$.

"base-b" expansion of $d^{(b)}(m)$ and $C(b)$

$d^{(b)}(m) = (.v_m)_b, C(b) = (.v_\infty)_b$. For instance,
$d^{(b)}(2) = (.\nu_2)_b = (.10\overline{1}1\overline{1}01\overline{1}10\overline{1}1\overline{1}\ldots)_b = b^{-1} - b^{-3} + b^{-4} - b^{-5} + b^{-7} - b^{-8} + \cdots$.

Examples of $\limsup_{n\to\infty} \|\xi b^n\|$ $(b \in \mathbb{Z}, b \geq 2)$

For simplicity, we assume that $b \geq 3$

$\xi = \sum_{m=1}^{\infty} b^{-m^2} = (.100100001000\ldots)_b = \|\xi\|$.
$\xi b = (1.00100001000\ldots)_b$, $\|\xi b\| = (.00100001000\ldots)_b$,
$\xi b^2 = (10.0100001000\ldots)_b$, $\|\xi b^2\| = (.0100001000\ldots)_b \ldots$.

Key obs.: Distribution of $\|\xi b^n\|$ $(n = 0, 1, \ldots) \Rightarrow$ shifts of the word $100100001000\ldots$.
$\limsup_{n\geq 0} \|\xi b^n\| = (.10000\ldots)_b = 1/b$ because $\lim_{m\to\infty}((m+1)^2 - m^2) = \infty$.
Interpretation in terms of seq.: $w := 100100001000\ldots$ (seq. cor. to base-b exp. of ξ).
Then limit sup word (limsup word) of w w.r.t. lexicographical order
is $1000\ldots = 10^\infty$. (cf. $\sup_{n\geq 0} \|\xi b^n\| = \|\xi\| = (.100100001000\ldots)_b$.)

Our goal

Limsup words w.r.t. one-sided spaces with general order (for further application).
In particular, research the minimal limit point in terms of substitutions.

Cylinder order on $\{0,1\}^\infty$

$\mathcal{A} := \{0,1\}$, $\mathcal{A}^\infty := \{\boldsymbol{a} = a_1a_2\cdots \mid a_1, a_2, \ldots \in \mathcal{A}\}$.
$\boldsymbol{a} = a_1a_2\cdots, \boldsymbol{b} = b_1b_2\cdots \in \mathcal{A}^\infty \Rightarrow d(\boldsymbol{a},\boldsymbol{b}) := 2^{-k}$, where $k = \min\{n \geq 1 \mid a_n \neq b_n\}$.
$a_1, \ldots, a_m \in \mathcal{A} \Rightarrow [a_1\cdots a_m] := \{a'_1a'_2\cdots \in \mathcal{A}^\infty \mid a'_1\cdots a'_k = a_1\ldots a_k\}$: cylinder set.

Cylinder order on $\mathcal{A}^\infty$ (or $\{\overline{1},0,1\}^\infty$ in the next section)

$\leq$: total order on $\mathcal{A}^\infty$.
$\leq$ is cylinder order if the following holds:
For any $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathcal{A}^\infty$, $\boldsymbol{a} \leq \boldsymbol{b} \leq \boldsymbol{c}$ implies $d(\boldsymbol{a},\boldsymbol{b}) \leq d(\boldsymbol{a},\boldsymbol{c})$.

Equivalent definition of cylinder order: $k \geq 1$, $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathcal{A}$.
We define $[a_1\cdots a_k] < [b_1\cdots b_k]$ by $\boldsymbol{a} < \boldsymbol{b}$ for any $\boldsymbol{a} \in [a_1\cdots a_k]$ and $\boldsymbol{b} \in [b_1\cdots b_k]$.
Total order on $\mathcal{A}^\infty$ is cylinder order if and only if, for any $k \geq 1$,
$<$ is a total order on $\{[a_1\cdots a_k] \mid a_1, \ldots, a_k \in \mathcal{A}\}$ (the set of cyl. set of length k).
Example: lexicographic order (next page).

limsup words

$\mathcal{A} = \{0, 1\}$, $\mathcal{A}^\infty = \{\boldsymbol{a} = a_1a_2\cdots \mid a_1, a_2, \ldots \in \mathcal{A}\}$. $[a_1\cdots a_k]$: cylinder set.
$\leq$: cylinder order $\Leftrightarrow$ $(\forall k \geq 1)$ $<$ is a total order on $\{[a_1\cdots a_k] \mid a_1, \ldots, a_k \in \mathcal{A}\}$.
Examples: $\boldsymbol{a} = a_1a_2\cdots, \boldsymbol{b} = b_1b_2\cdots \in \mathcal{A}^\infty$ with $\boldsymbol{a} \neq \boldsymbol{b}$. $m := \min\{n \geq 1 \mid a_n \neq b_n\}$.
(1) Lexicographic order $\leq_{lex}$: $\boldsymbol{a} <_{lex} \boldsymbol{b} \Leftrightarrow a_m < b_m$.
$\Rightarrow [0] <_{lex} [1]$, $[00] <_{lex} [01] <_{lex} [10] <_{lex} [11]$.
(2) Alternating lexicographic order $\leq_{alt}$: $\boldsymbol{a} <_{alt} \boldsymbol{b} \Leftrightarrow \begin{cases} a_m < b_m & m \text{ is odd,} \\ a_m > b_m & m \text{ is even.} \end{cases}$
$\Rightarrow [0] <_{alt} [1]$, $[01] <_{alt} [00] <_{alt} [11] <_{alt} [10]$, $[100] <_{alt} [101]$.

limsup words, sup words

$\boldsymbol{a} = a_1a_2\cdots \in \mathcal{A}^\infty$.
$\ell_{\leq}(\boldsymbol{a}) := \limsup_{n\to\infty} a_na_{n+1}a_{n+2}\cdots$, $s_{\leq}(\boldsymbol{a}) := \sup_{n\geq 1} a_na_{n+1}a_{n+2}\cdots$.
Example: $\boldsymbol{a} = 1001000010000010\cdots$ ($a_n = 1$ iff. n is a square number c^2.)
$\ell_{\leq_{lex}}(\boldsymbol{a}) = 1000000\cdots = 10^\infty$, $s_{\leq_{lex}}(\boldsymbol{a}) = \boldsymbol{a}$.

Analogs of Markoff-Lagrange spectrum

$\mathcal{A} = \{0,1\}$, $\mathcal{A}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{A}\}$. $[a_1 \cdots a_k]$: cylinder set.
$\leq$: cylinder order $\Leftrightarrow$ $(\forall k \geq 1)$ $<$ is a total order on $\{[a_1 \cdots a_k] \mid a_1, \ldots, a_k \in \mathcal{A}\}$.
$\boldsymbol{a} = a_1 a_2 \cdots \in \mathcal{A}^\infty$.
$\ell_\leq(\boldsymbol{a}) = \limsup_{n\to\infty} a_n a_{n+1} a_{n+2} \cdots$, $s_\leq(\boldsymbol{a}) = \sup_{n\geq 1} a_n a_{n+1} a_{n+2} \cdots$.

Definition

$\mathcal{L}_\leq := \{\ell_\leq(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathcal{A}^\infty\}$: Lagrange spectrum w.r.t. $\leq$.
$\mathcal{M}_\leq := \{s_\leq(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathcal{A}^\infty\}$: Markoff spectrum w.r.t. $\leq$.

Theorem($\leq$: cylinder order)

$\mathcal{L}_\leq = \mathcal{M}_\leq =$ closure$\{s_\leq(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathcal{A}^\infty$ is purely periodic.$\}$.
Recall that $\mathcal{A}^\infty$ is a metric space with metric $d(\boldsymbol{a}, \boldsymbol{b}) = 2^{\min\{n\geq 1 \mid a_n \neq b_n\}}$.

Next target: minimal limit point $\boldsymbol{m}_\leq$ of $\mathcal{L}_\leq = \mathcal{M}_\leq$.

Notation for substitution

$\mathcal{A} = \{0, 1\}$, $\mathcal{A}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{A}\}$.
$\leq$: cylinder order $\Leftrightarrow$ $(\forall k \geq 1)$ $<$ is a total order on $\{[a_1 \cdots a_k] \mid a_1, \ldots, a_k \in \mathcal{A}\}$.
$\mathcal{L}_\leq = \mathcal{M}_\leq = \{\sup_{n\geq 1} a_n a_{n+1} \cdots \mid a_1 a_2 \cdots \in \mathcal{A}^\infty\}$.

Definition

(1) $\boldsymbol{m}_\leq$: minimal limit point of $\mathcal{L}_\leq = \mathcal{M}_\leq$.
(2) $0 \leq j < k \Rightarrow \tau_{j,k} : \{0, 1\}^* \to \{0, 1\}^*$: substitution with $\tau_{j,k}(0) = 10^j, \tau_{j,k}(1) = 10^k$.
Note that τ $(\tau(0) = 1, \tau(1) = 100)$ in the result of Dubickas 2006 is $\tau_{0,2}$.
(3) $\mathcal{S} = \{\tau_{j,k} \mid 0 \leq j < k\}$.
(4) $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1} \in \mathcal{S}^\infty$. $\sigma_{[1,n]} := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$, $\boldsymbol{\sigma} := \lim_{n\to\infty} \sigma_{[1,n]}$.

Next page: We describe $\{\boldsymbol{a} \in \mathcal{M}_\leq \mid \boldsymbol{a} \leq \boldsymbol{m}_\leq\}$: Analog of discrete part.
We may assume that $[0] < [1]$.

Discrete part

$\mathcal{A} = \{0, 1\}$, $\mathcal{A}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{A}\}$. $\leq$: cylinder order with $[0] < [1]$.
$\boldsymbol{m}_\leq$: min. lim. p.t. of $\mathcal{L}_\leq = \mathcal{M}_\leq = \{\sup_{n\geq 1} a_n a_{n+1} \cdots \mid a_1 a_2 \cdots \in \mathcal{A}^\infty\}$.
$\mathcal{S} = \{\tau_{j,k} \mid 0 \leq j < k\}$, $\tau_{j,k}(0) = 10^j$, $\tau_{j,k}(1) = 10^k$.
$\sigma = (\sigma_n)_{n\geq 1} \in \mathcal{S}^\infty$. $\sigma_{[1,n]} := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$, $\sigma := \lim_{n\to\infty} \sigma_{[1,n]}$.

Theorem (there is an algorithm to determine $\sigma_1, \sigma_2, \ldots$)

Exactly one of the following holds:
(1) ($\exists \boldsymbol{\sigma} = (\sigma_n)_{n\geq 1} \in \mathcal{S}^\infty$) s.t. $\boldsymbol{m}_\leq = \boldsymbol{\sigma}(1^\infty)$. Moreover, the following holds:
$\{\boldsymbol{a} \in \mathcal{M}_\leq \mid \boldsymbol{a} < \boldsymbol{m}_\leq\} = \{(\sigma_{[1,n]}(0))^\infty \mid n \geq 0\}$, where if $n = 0$, $(\sigma_{[1,n]}(0))^\infty = 0^\infty$.
(2) There exists $h \geq 0$ and $\sigma_1, \ldots, \sigma_h \in \mathcal{S}$ such that $\boldsymbol{m}_\leq = \sigma_{[1,h]}(10^\infty)$, where $\sigma_{[1,h]}(10^\infty) = 10^\infty$ when $h = 0$. Moreover, $\{\boldsymbol{a} \in \mathcal{M}_\leq \mid \boldsymbol{a} < \boldsymbol{m}_\leq\}$ is as follows:

$$\{(\sigma_{[1,n]}(0))^\infty \mid 0 \leq n \leq h\} \cup \{\sigma_{[1,h]}((10^j)^\infty) \mid j \geq 0, \sigma_{[1,h]}((10^j)^\infty) < \boldsymbol{m}_\leq\}.$$

Discrete part (Example)

$\mathcal{A} = \{0, 1\}$, $\mathcal{A}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{A}\}$. $\leq$: cylinder order with $[0] < [1]$.
$\boldsymbol{m}_\leq$: min. lim. p.t. of $\mathcal{L}_\leq = \mathcal{M}_\leq$. $\mathcal{S} = \{\tau_{j,k} \mid 0 \leq j < k\}$, $\tau_{j,k}(0) = 10^j, \tau_{j,k}(1) = 10^k$.
$\boldsymbol{\sigma} = (\sigma_n)_{n \geq 1} \in \mathcal{S}^\infty$. $\sigma_{[1,n]} = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$, $\boldsymbol{\sigma} = \lim_{n\to\infty} \sigma_{[1,n]}$.

Theorem (The second statement is omitted.)

$(\exists \boldsymbol{\sigma} = (\sigma_n)_{n\geq 1} \in \mathcal{S}^\infty)$ s.t. $\boldsymbol{m}_\leq = \boldsymbol{\sigma}(1^\infty)$. Moreover, the following holds:
$\{\boldsymbol{a} \in \mathcal{M}_\leq \mid \boldsymbol{a} < \boldsymbol{m}_\leq\} = \{(\sigma_{[1,n]}(0))^\infty \mid n \geq 0\}$, where if $n = 0$, $(\sigma_{[1,n]}(0))^\infty = 0^\infty$.

Alternating lexicographic order $\leq_{alt}$

$\boldsymbol{a} = a_1 a_2 \cdots, \boldsymbol{b} = b_1 b_2 \cdots \in \mathcal{A}^\infty$ with $\boldsymbol{a} \neq \boldsymbol{b}$. $m := \min\{n \geq 1 \mid a_n \neq b_n\}$.
$\boldsymbol{a} <_{alt} \boldsymbol{b} \Leftrightarrow a_m < b_m (m \text{ is odd}), a_m > b_m (m \text{ is even})$.
$(\forall n \geq 1)\ \sigma_n = \tau_{0,2}$. $\sigma_{[1,n]} = \tau_{0,2}^n$, $\boldsymbol{\sigma}(1^\infty) = \lim_{n\to\infty} \tau_{0,2}^n(1^\infty) = \lim_{n\to\infty} \tau_{0,2}^n(1)$.

Unimodal order (If we have no time, then we omit this part)

$\mathcal{A} = \{0, 1\}$, $\mathcal{A}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{A}\}$. $\leq$: cylinder order with $[0] < [1]$.
$\boldsymbol{m}_\leq$: min. lim. p.t. of $\mathcal{L}_\leq = \mathcal{M}_\leq$. $\mathcal{S} = \{\tau_{j,k} \mid 0 \leq j < k\}$, $\tau_{j,k}(0) = 10^j, \tau_{j,k}(1) = 10^k$.
$\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1} \in \mathcal{S}^\infty$. $\sigma_{[1,n]} = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$, $\boldsymbol{\sigma} = \lim_{n\to\infty} \sigma_{[1,n]}$.

Unimodal order $\leq_{uni}$ (related to the dynamics of unimodal maps, e.g. tent map.)

$\boldsymbol{a} = a_1 a_2 \cdots, \boldsymbol{b} = b_1 b_2 \cdots \in \mathcal{A}^\infty$ with $\boldsymbol{a} \neq \boldsymbol{b}$. $m := \min\{n \geq 1 \mid a_n \neq b_n\}$.
$N :=$ Card$\{1 \leq n \leq m - 1 \mid a_n = 1\}$ (if $m = 1$, then $N := 0$).
$\boldsymbol{a} <_{uni} \boldsymbol{b} \Leftrightarrow a_m < b_m (N$ is even$), a_m > b_m (N$ is odd$)$.

Minimal limit point $\boldsymbol{m}_{\leq_{uni}}$

$\sigma_1 := \tau_{0,1}$, $(\forall n \geq 2)$ $\sigma_n := \tau_{0,2}$.
$\sigma_{[1,1]} = \tau_{0,1}$, $(\forall n \geq 2)$ $\sigma_{[1,n]} = \tau_{0,1} \circ \tau_{0,2}^{n-1}$, $\boldsymbol{\sigma}(1^\infty) = \lim_{n\to\infty} \tau_{0,1} \circ \tau_{0,2}^{n-1}(1)$.

Consistent cylinder order on $\{\bar{1}, 0, 1\}^\infty$

$\mathcal{B} := \{\bar{1}, 0, 1\}$, $\mathcal{B}^\infty := \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{B}\}$.
Model (motivation): $\limsup_{n\to\infty} \left|\sum_{m=1}^\infty a_{n+m}\beta^{-m}\right|$ $(\beta \geq 3)$.
More generally, $(e_m)_{m=1,2,\ldots} \in \{\bar{1}, 1\}^\infty$: fixed sequence of signature.
$\Rightarrow \limsup_{n\to\infty} \left|\sum_{m=1}^\infty e_m \cdot a_{n+m}\beta^{-m}\right|$ $(\beta \geq 3)$.

Absolute values of sequences

$$\boldsymbol{a} \in \mathcal{B}^\infty \;\Rightarrow\; \mathrm{abs}(\boldsymbol{a}) := \begin{cases} \boldsymbol{a} & \text{if } \boldsymbol{a} \geq_{lex} 0^\infty, \\ -\boldsymbol{a} & \text{if } \boldsymbol{a} \leq_{lex} 0^\infty. \end{cases}$$

Definition

$\preceq$: cylinder order on $\mathcal{B}^\infty$. $\preceq$ is consistent if
$(\forall w \in \mathcal{B}^*)$ "$[w\bar{1}] \prec [w0] \prec [w1]$" or "$[w1] \prec [w0] \prec [w\bar{1}]$".

Markoff-Lagrange spectrum on $\{\bar{1}, 0, 1\}^\infty$

$\mathcal{B} = \{\bar{1}, 0, 1\}$, $\mathcal{B}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{B}\}$. $\preceq$: consistent cyl. order on $\mathcal{B}^\infty$:
$(\forall w \in \mathcal{B}^*)$ "$[w\bar{1}] \prec [w0] \prec [w1]$" or "$[w1] \prec [w0] \prec [w\bar{1}]$".
$\boldsymbol{a} \in \mathcal{B}^\infty$: $\mathrm{abs}(\boldsymbol{a}) = \boldsymbol{a}$ if $\boldsymbol{a} \geq_{lex} 0^\infty$, $\mathrm{abs}(\boldsymbol{a}) = -\boldsymbol{a}$ if $\boldsymbol{a} \leq_{lex} 0^\infty$.

Definition ($\preceq$: cylinder order)

(1) $\boldsymbol{a} = a_1 a_2 \cdots \in \mathcal{B}^\infty$.
$\ell_\preceq^{\mathrm{abs}}(\boldsymbol{a}) := \limsup_{n\to\infty} \mathrm{abs}(a_n a_{n+1} a_{n+2} \cdots)$, $s_\preceq^{\mathrm{abs}}(\boldsymbol{a}) := \sup_{n\geq 1} \mathrm{abs}(a_n a_{n+1} a_{n+2} \cdots)$.
(2) $\mathcal{L}_\preceq^{\mathrm{abs}} := \{\ell_\preceq^{\mathrm{abs}}(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathcal{B}^\infty\}$. $\mathcal{M}_\preceq^{\mathrm{abs}} := \{s_\preceq^{\mathrm{abs}}(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathcal{B}^\infty\}$.

Theorem ($\preceq$: cylinder order, not necessary consistent)

$\mathcal{L}_\preceq^{\mathrm{abs}} = \mathcal{M}_\preceq^{\mathrm{abs}} =$ closure$\{s_\preceq^{\mathrm{abs}}(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathcal{B}^\infty$ is purely periodic$\}$,
where $\mathcal{B}^\infty$ is a metric space with $d(\boldsymbol{a}, \boldsymbol{b}) = 2^{\min\{n\geq 1 | a_n \neq b_n\}}$.

Next target: minimal limit point $\mu_\preceq$ of $\mathcal{L}_\preceq^{\mathrm{abs}} = \mathcal{M}_\preceq^{\mathrm{abs}}$.

Discrete part

$\mathcal{B} = \{\overline{1}, 0, 1\}$, $\mathcal{B}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{B}\}$. $\preceq$: consistent cyl. order on $\mathcal{B}^\infty$:
$(\forall w \in \mathcal{B}^*)$ "$[w\overline{1}] \prec [w0] \prec [w1]$" or "$[w1] \prec [w0] \prec [w\overline{1}]$".
$\boldsymbol{a} \in \mathcal{B}^\infty : \mathrm{abs}(\boldsymbol{a}) = \boldsymbol{a}$ if $\boldsymbol{a} \geq_{lex} 0^\infty$, $\mathrm{abs}(\boldsymbol{a}) = -\boldsymbol{a}$ if $\boldsymbol{a} \leq_{lex} 0^\infty$.
$\mu_\preceq$: min. lim. p.t. of $\mathcal{L}_\preceq^{\mathrm{abs}} = \{\limsup_{n\to\infty} \mathrm{abs}(a_n a_{n+1} a_{n+2} \cdots) \mid \boldsymbol{a} \in \mathcal{B}^\infty\}$.
Recall for $\mathbf{x} = x_0 x_1 \ldots \in \{0,1\}^\infty$ that $\Phi(\mathbf{x}) = 10^{x_0}\overline{1}0^{x_1}10^{x_2}\overline{1}0^{x_3}\ldots$.
For simplicity, we only consider the case where $[1\overline{1}] \prec [10]$ and $[10\overline{1}] \prec [100]$ hold.

Theorem ($\preceq$: with assumption above.)

There exists a cylinder order $\leq$ on $\mathcal{A}^\infty = \{0,1\}^\infty$ such that $\mu_\preceq = \Phi(\boldsymbol{m}_\leq)$,
where $\boldsymbol{m}_\leq$ is the min. lim. p.t. of $\mathcal{L}_\leq$. Moreover, the following holds:
$\{\boldsymbol{a} \in \mathcal{L}_\preceq^{\mathrm{abs}} \mid \boldsymbol{a} \prec \mu_\preceq\} = \{0^\infty\} \cup \{\Phi(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathcal{L}_\leq, \boldsymbol{b} < \boldsymbol{m}_\leq\}$.
Note: there exists an algorithm to determine $\leq$ by using $\preceq$.

Example: $\preceq_{lex}$: lexicographic order on $\mathcal{B}^\infty$: next page.

Example

$\mathcal{B} = \{\bar{1}, 0, 1\}$, $\mathcal{B}^\infty = \{\boldsymbol{a} = a_1 a_2 \cdots \mid a_1, a_2, \ldots \in \mathcal{B}\}$.
$\preceq$: consistent cyl. order on $\mathcal{B}^\infty$. $\mu_\preceq$: min. lim. p.t. of $\mathcal{L}_\preceq^{\mathrm{abs}}$.
$\mathbf{x} = x_0 x_1 \ldots \in \{0,1\}^\infty \Rightarrow \Phi(\mathbf{x}) = 10^{x_0}\bar{1}0^{x_1}10^{x_2}\bar{1}0^{x_3}\ldots$.

Theorem (assumption: $\preceq$ satisfies $[1\bar{1}] \prec [10]$, $[10\bar{1}] \prec [100]$)

There exists a cylinder order $\leq$ on $\mathcal{A}^\infty = \{0,1\}^\infty$ such that $\mu_\preceq = \Phi(\boldsymbol{m}_\leq)$, where $\boldsymbol{m}_\leq$ is the min. lim. p.t. of $\mathcal{L}_\leq$. Moreover, the following holds:
$\{\boldsymbol{a} \in \mathcal{L}_\preceq^{\mathrm{abs}} \mid \boldsymbol{a} \prec \mu_\preceq\} = \{0^\infty\} \cup \{\Phi(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathcal{L}_\leq, \boldsymbol{b} < \boldsymbol{m}_\leq\}$.

Example: $\preceq_{lex}$: lexicographic order on $\mathcal{B}^\infty \Rightarrow \leq_{alt}$: alternative lex. order.
$\mu_{\preceq_{lex}} = \Phi(\boldsymbol{m}_{\leq_{alt}}) = \Phi(\lim_{n\to\infty} \tau_{0,2}^n(1))$ (Result by Dubickas 2006).
We also have other examples (alt. lex. order on $\mathcal{B}^\infty$ and bimodal order on $\mathcal{B}^\infty$).