Partial Sums of Binary Subword-Counting Sequences

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A bit about me

I am a graduate student from India. I just completed a BS-MS Dual Degree from the Indian Institute of Science Education and Research (IISER), Pune and will be joining Alejandro Adem at the University of British Columbia (UBC), Vancouver for my PhD this fall.

My primary research interest lies in algebraic topology. I was interested in combinatorics as an undergrad (and still am), which is when I did the work that we will discuss today.

Acknowledgements

This presentation is based on joint work with Shuo Li which was carried out at the Université du Québec à Montréal (UQAM) in Summer 2023. I would like to thank the LACIM group at UQAM for their hospitality during this period.

I would also like to thank Mitacs, whose generous funding and match-making (as part of the Globalink Research Internship Program) made this work possible.

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Example

Consider $w = 10, n = 26$. The binary expansion of 26 is 11010. Hence, $s_{10}(26) = 5$ and $e_{10}(26) = 2$.

We call $\left((-1)^{s_w(n)}\right)_{n\geq 0}$ a subword-counting sequence and $\left((-1)^{e_w(n)}\right)_{n\geq 0}$ a factor-counting sequence.

The goal

We wish to explore when w satisfies the following.

Property P

There exist $\epsilon > 0$ such that

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In the literature, similar questions have been explored for factor-counting sequences since the 1980s. However, subword-counting sequences have been relatively unexplored in this context. Work on subword counting includes that of Narad Rampersad and Jean-Paul Allouche, among others.

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A recurrence relation

For $n \geq 1$, it is easy to see that

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Caution: Some of the above recurrences fail for $n = 0$.

An alternate formulation

For $a \in \{0, 1\}$ and $n \geq 2$, we have

$$s_{wa}(n) = \begin{cases} s_{wa}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + s_w\left(\left\lfloor \frac{n}{2} \right\rfloor\right) & n \equiv a \pmod 2; \\ s_{wa}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) & n \not\equiv a \pmod 2 \end{cases}.$$

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For subword-counting sequences, this yields

$$(-1)^{s_{wa}(n)} = \begin{cases} (-1)^{s_{wa}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + s_w\left(\left\lfloor \frac{n}{2} \right\rfloor\right)} & n \equiv a \pmod 2; \\ (-1)^{s_{wa}\left(\left\lfloor \frac{n}{2} \right\rfloor\right)} & n \not\equiv a \pmod 2 \end{cases}.$$

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Remark: The analogous recurrence for factor-counting sequences does not involve a product of multiple factor-counting sequences (unlike the above).

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$$\begin{aligned} S_{2N+1} &= \sum_{n=0}^{N} (-1)^{s_{01}(2n)} + \sum_{n=0}^{N} (-1)^{s_{01}(2n+1)} \\ &= 2 + \sum_{n=1}^{N} (-1)^{s_{01}(n)} + \sum_{n=1}^{N} (-1)^{s_{01}(n)+s_0(n)} \end{aligned}$$

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Call this new sum T_N, so that $S_{2N+1} = 2 + S_N + T_N$.

$w = 01$ (contd.)

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One can do a similar calculation for T_{2N+1} using

$$s_{01}(2n) + s_0(2n) \equiv s_{01}(n) + s_0(n) + 1 \pmod 2 \text{ and}$$
$$s_{01}(2n+1) + s_0(2n+1) \equiv s_{01}(n) \pmod 2$$

to obtain

$$T_{2N+1} = -2 + S_N - T_N. \tag{2}$$

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From here, it is not hard to see that $S_N = O(\sqrt{N})$ (roughly, quadrupling N leads to doubling S_N). In particular, 01 has *Property P*.

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By considering (1) − (2), one similarly obtains that $T_N = O(\sqrt{N})$, i.e.,

$$\sum_{n=0}^{N} (-1)^{s_{01}(n)+s_0(n)} = O\left(\sqrt{N}\right).$$

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The application on T_N did not produce any new summations, so the resulting system of the two equations

$$S_{2N+1} = 2 + S_N + T_N \text{ and}$$
$$T_{2N+1} = -2 + S_N - T_N$$

in the two unknowns S_N and T_N was sufficient for our purposes.

$w = 011$

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Applying the recurrence relation on $\sum_{n=0}^{N}(-1)^{s_{011}(n)}$ produces four equations

$$S_{2N+1} = S_N + T_N, \quad T_{2N+1} = 2 + T_N + U_N,$$
$$U_{2N+1} = -U_N + V_N, \text{ and } \quad V_{2N+1} = -2 + V_N - S_N$$

in four unknowns

$$S_N = \sum_{n=0}^{N}(-1)^{s_{011}(n)}, \qquad T_N = \sum_{n=0}^{N}(-1)^{s_{011}(n)+s_{01}(n)},$$
$$U_N = \sum_{n=0}^{N}(-1)^{s_{011}(n)+s_0(n)}, \text{ and } \quad V_N = \sum_{n=0}^{N}(-1)^{s_{011}(n)+s_{01}(n)+s_0(n)}.$$

$w = 011$ (contd.)

Naturally, the best way to capture this information is using vectors and matrices. Let

$$v_N = \begin{bmatrix} S_N & T_N & U_N & V_N \end{bmatrix}^T$$

and

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix},$$

so that

$$v_{2N+1} = Mv_N + c \tag{3}$$

(for some constant $c \in \mathbb{R}^4$).

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$$\left\lfloor \frac{N}{2^k} \right\rfloor = 2 \left\lfloor \frac{N}{2^{k+1}} \right\rfloor + a_k.$$

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Let $x_k = v_{\left\lfloor \frac{N}{2^k} \right\rfloor}$ and $\Delta_k = x_k - (Mx_{k+1} + c)$, so that $\|\Delta_k\| \leq 2$ (by (3)).

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Hence,

$$\begin{aligned} v_N = x_0 &= Mx_1 + c + \Delta_0 \\ &= M^2x_2 + (M + I)c + (M\Delta_1 + \Delta_0) \\ &= M^3x_3 + (M^2 + M + I)c + (M^2\Delta_2 + M\Delta_1 + \Delta_0) \\ &\dots \end{aligned}$$

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If all eigenvalues of M have absolute value < 2, then $\|v_N\| = O(N^{1-\epsilon})$ for some $\epsilon > 0$. In particular, 011 would have *Property P* in this case.

$w = 011$ (eigenvalues of M)

It turns out that we do not need to explicitly find the eigenvalues of M for showing this. M can be written as a sum of two "signed" permutation matrices:

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{B}.$$

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$w = 011$ (conclusion)

To recap:

$$v_N = \begin{bmatrix} S_N & T_N & U_N & V_N \end{bmatrix}^T,$$
$$v_{2N+1} = Mv_N + c, \text{ and}$$

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Remark: If λ is the eigenvalue of M with largest absolute value, then $\epsilon < 1 - \log_2 |\lambda|$ works. The eigenvalues of M are $0, 0, \pm\sqrt{2}$, so $\epsilon = \frac{1}{2}$ works in the present case.

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Some definitions

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- For $u \in \{0,1\}^\ell$ and $n \geq 0$, define

$$\begin{bmatrix} w \\ u \end{bmatrix} : \mathbb{N} \to \{0,1\}; n \mapsto \sum_{\substack{i=1 \\ u_i=1}}^{\ell} s_{w_1 \dots w_i}(n) \pmod 2$$

$$\equiv \sum_{i=1}^{\ell} u_i \cdot s_{w_1 \dots w_i}(n) \pmod 2.$$

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- For example,

$$\begin{bmatrix} abcd \\ 0101 \end{bmatrix} (n) \equiv s_{ab}(n) + s_{abcd}(n) \pmod 2.$$

Capturing the recurrence

Let $a \in \{0,1\}$. Applying the recurrence for $s_w(n)$ to subword counters, we obtain a word $S_a(w)(u) \in \{0,1\}^\ell$ and a bit $T_a(w)(u) \in \{0,1\}$ such that

$$\begin{bmatrix} w \\ u \end{bmatrix}(2n+a) \equiv \begin{bmatrix} w \\ S_a(w)(u) \end{bmatrix}(n) + T_a(w)(u) \pmod 2.$$

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Example

$$\begin{bmatrix} 01 \\ 01 \end{bmatrix}(2n+1) \equiv \begin{bmatrix} 01 \\ 11 \end{bmatrix}(n) + 0 \pmod 2 \text{ and}$$

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We will call $S_a(w) : \{0,1\}^\ell \to \{0,1\}^\ell$ an 'a-split function' (for reasons that will be clear soon) and $T_a(w) : \{0,1\}^\ell \to \{0,1\}$ an 'a-carry function'.

Capturing the recurrence (contd.)

$$\begin{bmatrix} w \\ u \end{bmatrix} (2n + a) \equiv \begin{bmatrix} w \\ S_a(w)(u) \end{bmatrix} (n) + T_a(w)(u) \pmod 2.$$

Capturing the recurrence (contd.)

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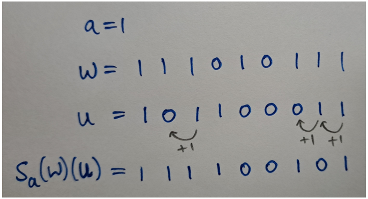
Here's how to calculate $S_a(w)(u)$:



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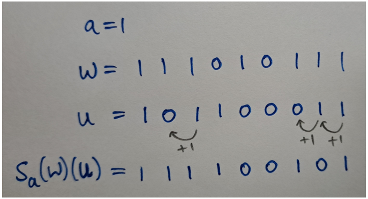

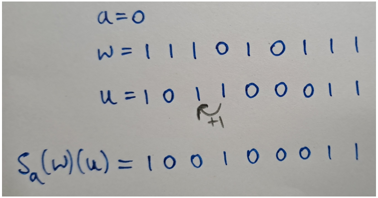

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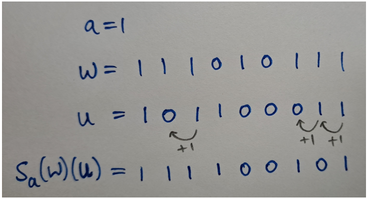

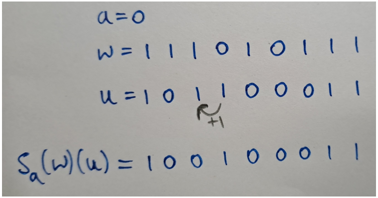

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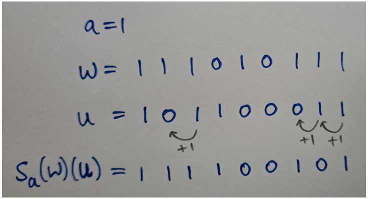

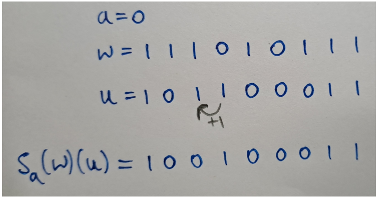

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$T_a(w)(u)$ captures the split which 'spills over' from the leftmost digit of u. In the above example, $T_0(w)(u) = 0$ and $T_1(w)(u) = 1$. An explicit formula:

$$T_a(w)(u)\,S_a(w)(u) = \left[\left(\overline{a}^{\ell} \oplus w\right)0 \wedge u0\right] \oplus 0u.$$

Basic properties of splits and carries

- $S_a(w)(u \oplus u') = S_a(w)(u) \oplus S_a(w)(u')$ and likewise for $T_a(w)$ — split and carry functions are group homomorphisms.

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is a group. Let $\mathcal{O}_w(u)$ be the orbit of u under this group. Special emphasis is given to the case of $u = 0^{\ell-1}1$ by defining $\mathcal{O}_w := \mathcal{O}_w(0^{\ell-1})$.

The 01 and 011 cases

$$\mathcal{O}_{01} = \{01, 11\}$$

$$S_N = \sum_{n=0}^{N} (-1)^{\begin{bmatrix} 01 \\ 01 \end{bmatrix}(n)}$$

$$T_N = \sum_{n=0}^{N} (-1)^{\begin{bmatrix} 01 \\ 11 \end{bmatrix}(n)}$$

$$\mathcal{O}_{011} = \{001, 011, 101, 111\}$$

$$S_N = \sum_{n=0}^{N} (-1)^{\begin{bmatrix} 011 \\ 001 \end{bmatrix}(n)}$$

$$T_N = \sum_{n=0}^{N} (-1)^{\begin{bmatrix} 011 \\ 011 \end{bmatrix}(n)}$$

$$U_N = \sum_{n=0}^{N} (-1)^{\begin{bmatrix} 011 \\ 101 \end{bmatrix}(n)}$$

$$V_N = \sum_{n=0}^{N} (-1)^{\begin{bmatrix} 011 \\ 111 \end{bmatrix}(n)}$$

Matrices and vectors

Let $v(w,u)(n)$ be the $|\mathcal{O}_w(u)|$-dimensional column vector, with entries indexed by $\mathcal{O}_w(u)$, whose u'-th entry is

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For $a \in \{0,1\}$, let $M_a(w,u)$ be the matrix of size $|\mathcal{O}_w(u)| \times |\mathcal{O}_w(u)|$, with entries indexed by $\mathcal{O}_w(u) \times \mathcal{O}_w(u)$, whose (u', u'')-th entry is

$$M_a(w,u)[u',u''] := \begin{cases} (-1)^{T_a(w)(u')} & \text{if } S_a(w)(u') = u''; \\ 0 & \text{otherwise} \end{cases}.$$

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Let $M(w,u) = M_0(w,u) + M_1(w,u)$. It should be no surprise that we obtain the following recurrence for $v(w,u)$:

$$v(w,u)(2n+a) = M_a(w,u)\, v(w,u)(n)$$

for $n \geq 1$.

Matrices and vectors (contd.)

Now, define

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if all eigenvalues of $M(w,u)$ have absolute value < 2. We say that the pair (w,u) has *Property Q* if (4) holds. If $(w, 0^{\ell-1}1)$ has *Property Q*, then w has *Property P*.

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Previously, we showed that $(01, 01)$ and $(011, 001)$ have *Property Q*, and hence concluded that 01 and 011 have *Property P*.

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$$V(001, 011)(N) = \begin{matrix} 001 \\ 011 \\ 101 \\ 111 \end{matrix} \begin{bmatrix} S_N \\ T_N \\ U_N \\ V_N \end{bmatrix}$$

$$V(01, 01)(N) = \begin{matrix} 01 \\ 11 \end{matrix} \begin{bmatrix} S_N \\ T_N \end{bmatrix}$$

$$M(01, 01) = \begin{matrix} & 01 & 01 \\ 01 & 1 & 1 \\ 11 & 1 & -1 \end{matrix}$$

$$M(011, 001) = \begin{matrix} & 001 & 011 & 101 & 111 \\ 001 & 1 & 1 & 0 & 0 \\ 011 & 0 & 1 & 1 & 0 \\ 101 & 0 & 0 & -1 & 1 \\ 111 & 1 & 0 & 0 & -1 \end{matrix}$$

In the old notation, $v_N = V(011, 001)(N)$, $M = M(011, 001)$, $A = M_0(011, 001)$, and $B = M_1(011, 001)$.

Basic properties of $M(w, u)$

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The overarching strategy in our work is to convert these linear-algebraic statements into combinatorial ones. This leads to a sufficient condition for (w, u) to satisfy *Property Q*, and hence for w to satisfy *Property P*.

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One run

Theorem

Let $a \in \{0, 1\}$. *Property P* is satisfied by a^ℓ if and only if ℓ is a power of 2.

Two runs

Theorem

Let $a \in \{0, 1\}$ and $j, k \geq 1$. *Property P* is satisfied by $a^j \overline{a}^k$. More generally, for $u \in \{0, 1\}^k$ ($u \neq 0^k$), *Property Q* is satisfied by $(a^j \overline{a}^k, 0^j u)$.

This includes the cases of 01 and 011 from before!

A surprising example of *Property Q*

Theorem

Let $a \in \{0,1\}$ and $w \in \{0,1\}^*$ (possibly with $|w| < 2$). Let $k > 1$ be a power of 2 and $1 < j \leq k$. The pair $\left(a^k w \bar{a} a^{j-1}, 0^{k-1} 1 0^{|w|+j-1} 1\right)$ satisfies *Property Q*.

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Example

Set $a = 1$ and $k = j = 2$. Let $v = 11w01$. By the theorem, $\left(v, 010^{|w|+1}1\right)$ satisfies *Property Q*. In particular,

$$\sum_{n=0}^{N} (-1)^{s_v(n)+s_{01}(n)} = O\left(N^{1-\epsilon}\right)$$

for some $\epsilon > 0$.

Long first run

Theorem

Let $a \in \{0,1\}$, $w \in \{0,1\}^*$ (possibly with $|w| < 2$), and k be a power of 2 such that a^k is not a factor of w. *Property P* is satisfied by $a^k\overline{a}w$.

More generally, if $u \in \{0,1\}^{|w|}$, $u \neq 0^{|w|}$, and $b \in \{0,1\}$ such that $S_b(w)(u) = u$ and $T_b(w)(u) = 0$, then $\left(a^k\overline{a}w, 0^{k+1}u\right)$ satisfies *Property Q*.

Can you see why the second statement implies the first?

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Example

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Limitations

- **Necessary and sufficient conditions:** Our main techniques are not applicable for showing that a given word *w does not* satisfy property P. This is a major hindrance when it comes to obtaining a necessary and sufficient condition for *Property P* to be satisfied. Perhaps our proof of the 'one run' theorem could be used as a jumping-off point.

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- **Necessary and sufficient conditions:** Our main techniques are not applicable for showing that a given word w *does not* satisfy property P. This is a major hindrance when it comes to obtaining a necessary and sufficient condition for *Property* P to be satisfied. Perhaps our proof of the 'one run' theorem could be used as a jumping-off point.
- **Property P without Property Q:** In principle, it is possible for there to exist a word w such that w satisfies *Property* P but $(w, 0^{|w|-1}1)$ does not satisfy *Property* Q. Our method cannot detect such a case.
- **Eigenvalues may not be enough:** In principle, it is possible for there to exist a pair (w, u) such that (w, u) satisfies *Property* Q but $M(w, u)$ has an eigenvalue of absolute value 2. Our method cannot detect such a case.

Generalizing to other bases

Fix an integer $b \geq 2$. For $n \geq 0$ and a word $w \in \{0, \ldots, b-1\}^*$, let $s_w^b(n)$ be the number of occurrences of w as a scattered subword of the base-b expansion of n. When do we have

$$\sum_{n=0}^{N} (-1)^{s_w^b(n)} = O\left(N^{1-\epsilon}\right)$$

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Our linear-algebraic framework readily generalizes to explore the above question (now, instead of just 0- and 1-splits, we'll have i-splits for all $i \in \{0, \ldots, b-1\}$). However, we have not tried to extend our results to this case in order to avoid distracting from the essential ideas.

Generalizing even further

Fix a positive integer d. Let $\zeta_d = e^{\frac{2\pi i}{d}}$. When do we have

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Remark: For $d = 3$, we can loosely interpret (5) analogously to the $d = 2$ case — the probability that $s_w^b(n) \equiv r \pmod 3$ (for 'randomly chosen' n and r) is $\frac{1}{3}$.

Thank you!