Escape of Mass of the Sequences

(joint works with Erez Nesharim and Uri Shapira and with Steven Robertson)

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York Conference

Continued Fractions

Given a quadratic irrational $\Theta \in \mathbb{R}$, we write

$$\Theta = b_0 + \cfrac{1}{b_1 + \cfrac{1}{\ddots + \cfrac{1}{b_k + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_\ell + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}}}}}} := [b_0; b_1, \ldots, b_k, \overline{a_1, \ldots, a_\ell}].$$

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Given p prime, how does the continued fraction expansion of $p^k\Theta$ change as $k \to \infty$?

Some Known Results

Given a prime p, write the c.f.e. of $p^k\Theta$ as

$$p^k\Theta = [b_{k,0}; b_{k,1}, \dots, b_{k,m_k}, \overline{a_{k,1}, \dots, a_{k,\ell_k}}].$$

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Theorem (Aka-Shapira 2017)

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2. For every Θ, we have $\lim_{k\to\infty} \max_{i=1,\ldots\ell_k} |a_{k,i}| = \infty$.
3. For every Θ, there is no escape of mass of $p^k\Theta$, that is

$$\lim_{k\to\infty} \frac{\max_{i=1,\ldots,\ell_k} \log |a_{k,i}|}{\sum_{i=1}^{\ell_k} \log |a_{k,i}|} = 0.$$

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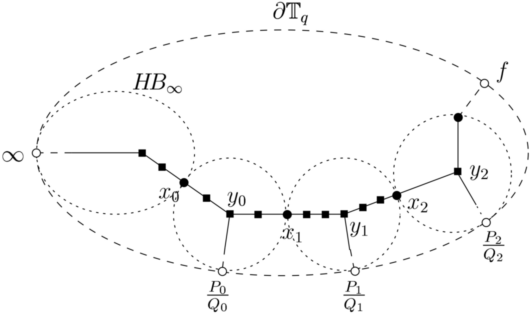
1. Utilize the connection between the Poincare section in the upper half plane and the c.f.e.
2. Then, connect between the upper half plane model and lattices in $\mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$.
3. Then, escape of mass of the c.f.e. corresponds to escape of mass of Haar measures supported on a sequence of compact, nested diagonal orbits.



Hecke Friends

Setting

$$G_\infty = \mathrm{PGL}_2(\mathbb{R}), \Gamma_\infty = \mathrm{PGL}_2(\mathbb{Z}), X_\infty = G_\infty/\Gamma_\infty,$$

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Definition (Hecke Friends)

Let p be a prime. We say that two lattices $\Lambda_1, \Lambda_2 \subseteq X_\infty$ are Hecke neighbors if $\Lambda_2 \subseteq \Lambda_1$ and $\Lambda_1/\Lambda_2 \cong \mathbb{Z}/p\mathbb{Z}$.

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Remark

If Λ_0 is a lattice then, the set of Hecke friends $\Lambda \subseteq \Lambda_0$ forms a $p+1$ regular tree $T_p(\Lambda_0)$.

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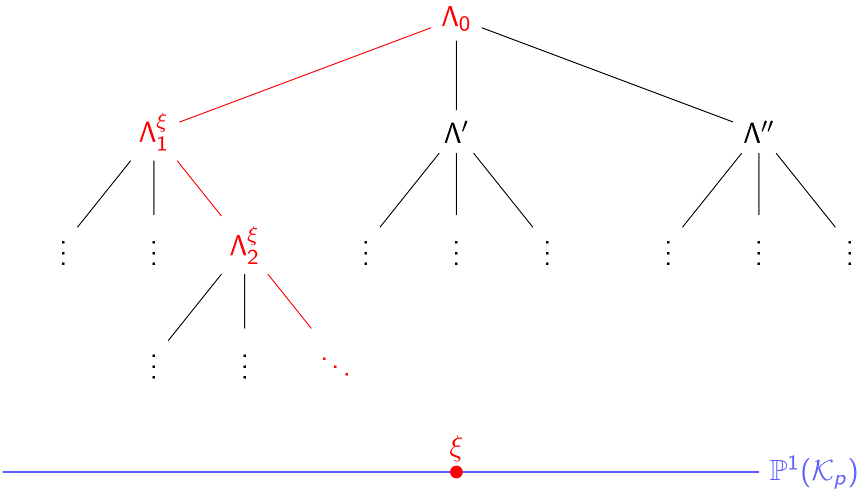
4. Thus, every node in the Hecke tree Λ supports an A invariant probability measure $\mu_{A\Lambda}$.

Therefore, we can study the evolution of nested number fields or c.f.e's by studying the distribution of these measures $\mu_{A\Lambda}$ when Λ is taken along a branch in the tree $T_p(\Lambda_0)$.

The Hecke Tree $T_p(\Lambda_0)$



Escape of Mass in Hecke Trees

Definition

We say that probability measures $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X_\infty)$ exhibit c escape of mass if every accumulation point $\mu = \lim_{k\to\infty} \mu_{n_k}$ satisfies $\mu(X_\infty) \leq 1 - c$.

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Question

What happens when we replace $\mathbb{R}$ with $\mathbb{F}_q((t^{-1}))$?

Function Field Setting

Let

$$\mathbb{F}_q[t] = \left\{ \sum_{n=0}^{N} a_n t^n : a_n \in \mathbb{F}_q, N \in \mathbb{N} \right\},$$

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Given Θ in $\mathbb{F}_q((t^{-1}))$, we can write its continued fraction expansion $\Theta = [A_0; A_1, \dots]$, where $A_i \in \mathbb{F}_q[t]$.

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Lemma

$\Theta \in \mathrm{QI}$ if and only if the c.f.e. of Θ is periodic.

Question

How does the c.f.e. of $t^k\Theta$ distribute?

Dynamics of CFE's in Function Fields

Let the c.f.e. of $t^k\Theta$ be

$$t^k\Theta = [b_{k,0}; b_{k,1}, \ldots, b_{k,m_k}, \overline{a_{k,1}, \ldots, a_{k,\ell_k}}].$$

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1. $\lim_{k\to\infty} \max_{i=1,\dots,\ell_k} \deg(a_{k,i}) = \infty$ (de Mathan-Teulie 2004).

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3. (Paulin-Shapira 2018) Let $\Theta \in \mathrm{QI}$, let

$$e_{k,N}(\Theta) := \frac{\sum_{i=1}^{\ell_n} \max\{\deg(a_{k,i}) - N, 0\}}{\sum_{i=1}^{\ell_n} \deg(a_{k,i})}.$$

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Then, there exists $c = c(\Theta) > 0$, such that

$$\lim_{N\to\infty} \liminf_{k\to\infty} e_{k,N} \geq c \text{ (c-escape of mass).}$$

Idea of Proof

Method

- Again use Hecke trees $\mathbb{T}_t(\Lambda_\Theta)$ in $\mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))/\mathrm{PGL}_2(\mathbb{F}_q[t])$ and study the escape of mass of measures $\mu_{A\Lambda_n}$ taken along the branches.

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- Use Paulin's results to connect c escape of mass of measures $\mu_{A\Lambda_n}$ in the Hecke tree to escape of mass of the c.f.e.

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Conjecture (Kemarsky-Paulin-Shapira/Paulin-Shapira 2018)

For every $\Theta \in \mathbf{QI}$, Θ exhibits full escape of mass, that is we can take $c = 1$.

We Disprove This Conjecture!

Theorem (A.-Nesharim-Shapira, A.-Robertson)

For every prime p, there exists $\Theta^{(p)} = \sum_{n=1}^{\infty} \Theta_n^{(p)} t^{-n} \in \mathbb{F}_p((t^{-1}))$, s.t.

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1. (Partial escape of mass)

$$\lim_{N\to\infty} \liminf_{k\to\infty} e_{k,N}(\Theta^{(p)}) = \begin{cases} \frac{2}{p} & p \text{ odd (A.-Robertson)} \\ \frac{2}{3} & p = 2 \text{ (A.-Nesharim-Shapira)} \end{cases}.$$

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2. (Full Maximal Escape of Mass) $\lim_{N\to\infty} \limsup_{k\to\infty} e_{k,N}(\Theta^{(p)}) = 1$.

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2. (Full Maximal Escape of Mass) $\lim_{N\to\infty} \limsup_{k\to\infty} e_{k,N}(\Theta^{(p)}) = 1$.
3. (Full Generic Escape of Mass) Moreover, for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \overline{d}(k \in \mathbb{N} : e_{k,N}(\Theta^{(p)}) < 1 - \varepsilon) = 0.$$

Hankel Matrices and DANW's

Given $\{\Theta_n\}_{n\in\mathbb{N}} \subseteq \mathbb{F}_q$, we define

$$H_\Theta(k;m) = \begin{pmatrix} \Theta_k & \Theta_{k+1} & \cdots & \Theta_{k+m} \\ \Theta_{k+1} & \Theta_{k+2} & \cdots & \Theta_{k+m+1} \\ \vdots & \ddots & \cdots & \vdots \\ \Theta_{k+m} & \Theta_{k+m+1} & \cdots & \Theta_{k+2m} \end{pmatrix}.$$

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Define the diagonally aligned number wall (DANW) of Θ as

$$\mathfrak{F}_{m,n}(\Theta) = \begin{cases} 0 & n > m+1 \\ 1 & n = m+1 \\ 0 & m = n \mod 2 \\ \det H_\Theta\left(\frac{m-n}{2}, \frac{m+n}{2}\right) & m > n \text{ and } m \neq n \mod 2 \end{cases}.$$

Diophantine Properties of Hankel Matrices I

Let $\Theta = \sum_{n=1}^{\infty} \Theta_n t^{-n} = [0; a_1, a_2, \dots]$ and let $N = N_0 + N_1 t + \cdots + N_h t^h$. Then, $|\langle N\Theta \rangle| < q^{-\ell}$ if and only if for every $i = 1, \dots, \ell$,

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Equivalently,

$$\Delta_\Theta(\ell, h+1)\mathbf{N} := \begin{pmatrix} \Theta_1 & \Theta_2 & \dots & \Theta_{h+1} \\ \Theta_2 & \Theta_3 & \dots & \Theta_{h+2} \\ \vdots & \dots & \ddots & \vdots \\ \Theta_\ell & \dots & \dots & \Theta_{h+\ell} \end{pmatrix} \begin{pmatrix} N_0 \\ \vdots \\ N_h \end{pmatrix} = 0.$$

Diophantine Properties of Hankel Matrices II

Fact

If $\frac{P_n}{Q_n} = [0; a_1, \dots, a_n]$ is a convergent of Θ, then,

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Hankel Matrix Interpretation

Let $m = \deg(Q_n)$. Then, there exists ℓ such that

1. $\Delta_\Theta(\ell, m+1)\mathbf{v} = 0$ for some $\mathbf{v}$, and
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Lemma

Assume that $t^k\Theta = [0; a_{1,k}, a_{2,k}, \ldots]$ and let $\frac{P_{n,k}}{Q_{n,k}} = [0; a_{1,k}, \ldots, a_{n,k}]$.

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Lemma

Assume that $t^k\Theta = [0; a_{1,k}, a_{2,k}, \dots]$ and let $\frac{P_{n,k}}{Q_{n,k}} = [0; a_{1,k}, \dots, a_{n,k}]$.
Then, $\mathfrak{F}_{m,k} \neq 0$ if and only if there exists some n such that $\deg(Q_{n,k}) = m$.

Escape of Mass in DANW's

Corollary

Let $\Theta = \sum_{n=1}^{\infty} \Theta_n t^{-n} \in \mathbf{QI}$ and let $t^k\Theta = [0; \overline{a_{k,1}, \ldots, a_{k,\ell_k}}]$. Let $h_{k,j}$ be the j-th non-zero coordinate in the k-th column of $\mathfrak{F}$. Then,

$$e_{k,N}(\Theta) = \frac{\sum_{i=1}^{\ell_n} \max\{\deg(a_{k,i}) - N, 0\}}{\sum_{i=1}^{\ell_n} \deg(a_{k,i})}$$

$$= \frac{\sum_{j=0,\ldots,\ell_k-1} \max\{h_{k,j+1} - h_{k,j} - N, 0\}}{\sum_{j=0}^{\ell_k-1}(h_{k,j+1} - h_{k,j})}.$$

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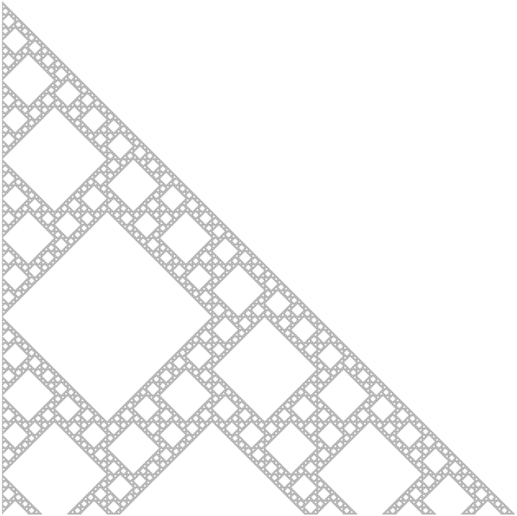
In other words, escape of mass of the c.f.e. corresponds to the proportion that the large distances between non-zero coordinates in the column takes up in the period of the column.

DANW of the Thue Morse Sequence



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Adiceam-Nesharim-Lunnon proved that Θ satisfies t-adic Littlewood if and only if $\mathfrak{F}(\Theta)$ has zero windows of unbounded size.

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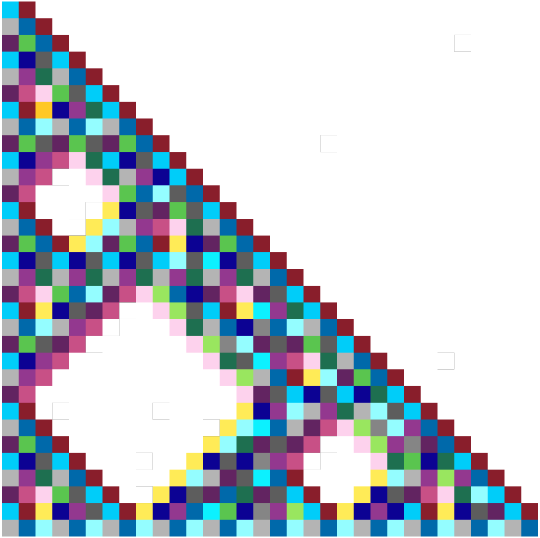
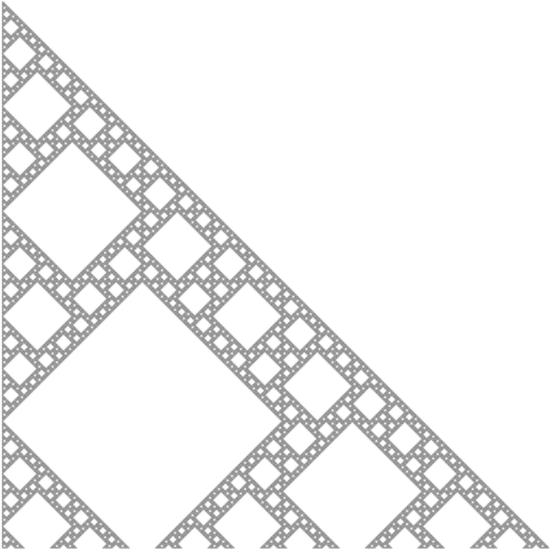
6. For generic escape of mass use Markov chains to describe the tiling and show that the density of sequences not exhibiting full escape of mass is 0.

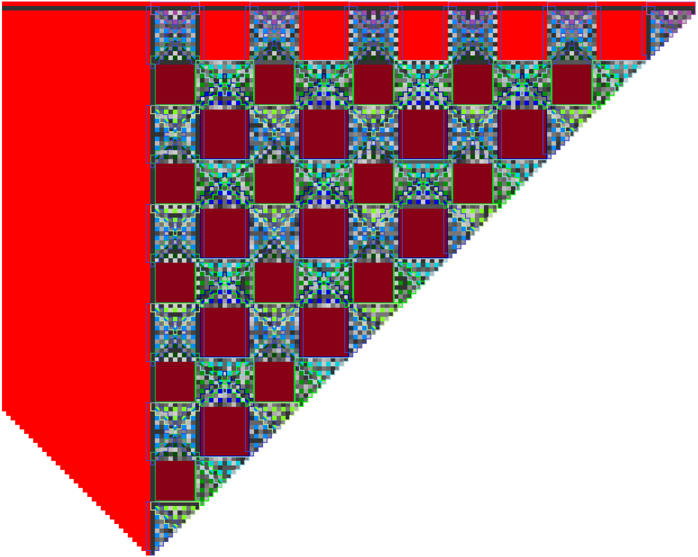
The Tiling for the Thue Morse Sequence



The Coding for the Thue Morse Sequence



The Tiling for the 11-Cantor Sequence



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4. What happens in the building of PGL_3?

Thank you very much