Quasi-fixed points of substitutions

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Example I

Let $\mathcal{A} = \{0, 1\}$ and consider a substitution of length 2

$$\varphi(0) = 01 \quad \varphi(1) = 00$$

with its 2-automatic fixed point

$$\mathrm{x} = 01000101\ldots$$

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In other words

$$X_\varphi = \overline{\mathcal{O}(x)} = \overline{\{T^n(x) \mid n \in \mathbb{N}\}},$$

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Put a total order on $\mathcal{A} = \{0 < 1\}$ and ask how the lexicographically minimal sequence $z \in X_\varphi$ looks like.

Example I & theorem

One can show (Currie, Rampersad, Saari, Zamboni, 2013) that lexicographically minimal $z \in X_\varphi$ is the unique solution to

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Let $\varphi\colon \mathcal{A} \to \mathcal{A}^*$ be a substitution and $\tau\colon \mathcal{A} \to \mathcal{B}$ a coding. Fix some total order on $\mathcal{B}$ and let $b \in \mathcal{B}$. Let $z \in \tau(X_\varphi)$ be the lexicographically smallest among sequences starting with b.

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As far as I know open in general.

Example II

Let $\mathcal{A} = \{0, 1, 2, 3\}$ and let $\tau \colon \mathcal{A} \to \mathcal{A}^*$ be the substitution given by

$$\tau(0) = 01023, \quad \tau(1) = 12, \quad \tau(2) = 22, \quad \tau(3) = 33.$$

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and for any n consider the suffix $z_{[n,\infty)} = z_n z_{n+1}\cdots$ of z and $Z_n = \overline{\mathcal{O}(z_{[n,\infty)})} \subset X_\tau$. We have

$$Z_n = \mathcal{O}(z_{[n,\infty)}) \cup \{3^k 2^\omega \mid k \geq 0\} \cup \{2^k 3^\omega \mid k \geq 0\}.$$

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Note that $z = T^2\tau(z)$.

A definition

Let $\varphi\colon \mathcal{A} \to \mathcal{A}^*$ be a substitution (for simplicity assumed to be growing, i.e. $|\varphi^n(a)| \to \infty$ for all $a \in \mathcal{A}$.)

Definition

A sequence z in $\mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$ is called a quasi-fixed point of φ if there exist $k > 0$ and $m \in \mathbb{Z}$ such that

$$T^m(\varphi^k(z)) = z.$$

If $x \in \mathcal{A}^{\mathbb{N}}$ and $m < 0$ this means

$$\varphi^k(z) = T^{-m}z$$

Simple closure properties of quasi-fixed points

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In particular, any transitive subsystem of a substitutive (resp. k-automatic) system is again substitutive (resp. k-automatic).

Quasi-fixed points of substitutions

Theorem (Shallit, Wang & Béal, Perrin, Restivo)

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A one-sided sequence is a quasi-fixed point of φ if and only if it is a suffix of two-sided quasi-fixed point.

Some problems

Problem

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If $y \in Y$ is substitutive, then it is k-automatic.

Some older results

Theorem (Holton, Zamboni, 2001)

Let $\varphi\colon \mathcal{A} \to \mathcal{A}^*$ be a primitive substitution, let $\tau\colon \mathcal{A} \to \mathcal{B}$ be a coding, and let $Y = \tau(X_\varphi)$. Assume that Y is infinite. Then, $\pi(x) \in Y$ is substitutive if and only if x is a quasi-fixed point of φ.

Some questions

Conjecture

Let φ be a general substitution. Let $\pi\colon X_\varphi \to Y$ be a factor map onto some subshift Y. The following hold.

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Quasi-fixed points as rational points

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- $\kappa(\tau(\varphi\text{-periodic points})) = 0$,
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- $\kappa(\tau(\varphi(x))) = k \cdot \kappa(x)$ for any $x \in X$.

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An alternative description for minimal automatic systems

Under the above assumptions, automatic sequences in Y correspond to rational numbers in $\mathbb{Z}_k$ via the map κ, i.e.

$$\{z \in Y \mid z \text{ is k-automatic}\} = \kappa^{-1}(\mathbb{Z}_k \cap \mathbb{Q}).$$

Something about the proof

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We need to show all automatic sequences in Y are of the required form.

We can assume that the base is the same

Theorem (Fagnot, 1997)

Let X be a k-automatic system. If $z \in X$ is automatic, then it is k-automatic.

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Let X be a k-automatic system. If $z \in X$ is automatic, then it is k-automatic.

From my work with Müllner it will follow that if z is substitutive, then it is, in fact, k-automatic.

Factor maps are finite-to-one over nonperiodic points

Proposition

Let φ be a substitution of constant length k. There exists $K \geq 1$ such that every factor map $\pi\colon X_\varphi \to Y$ is K-to-1 on nonperiodic points, i.e. $|\pi^{-1}(y)| \leq K$ for all nonperiodic $y \in Y$.

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Handles the factor map/coding π.

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Kernel characterisation of automaticity

A sequence $y = (y_n)_n$ is k-automatic if and only if its k-kernel

$$K_k(y) = \{(y_{i+k^m n})_n \mid m \geq 0, 0 \leq i < k^m\}$$

is finite.

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Using finiteness of the kernel, Critical Factorisation Theorem, and some counting arguments one shows that z has to be a quasi-fixed point of φ.

Critical factorisation theorem/recognizability

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Let $\mathcal{A}$ be an alphabet and let $W \subset \mathcal{A}^*$ be a finite set of nonempty words of size d. Let $x \in \mathcal{A}^{\mathbb{Z}}$ be nonperiodic. Then, x has at most d W-factorisations with pairwise disjoint cuts.

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Related result for substitutive system is recognizability

Theorem (Berthé, Steiner, Thuswaldner, Yassawi, 2017)

Let φ be a substitution. For any nonperiodic point $x \in X_\varphi$ there exist a unique $x' \in X_\varphi$ and a unique $0 \le n < |\varphi(x'_0)|$ such that $x = T^n(\varphi(x'))$.

Primitive substitutions

Theorem (Holton, Zamboni

Let $\varphi\colon \mathcal{A} \to \mathcal{A}^*$ be a primitive substitution, let $\tau\colon \mathcal{A} \to \mathcal{B}$ be a coding, and let $Y = \tau(X_\varphi)$. Then

1. A sequence $y \in Y$ is substitutive if and only if $y = \pi(x)$ for some quasi-fixed point x of φ.
2. If $y \in Y$ is substitutive and nonperiodic, then all points in $\pi^{-1}(y)$ are quasi-fixed points of φ.

One could assume that the base is the same

One could assume that the "base" of the sequence y $\in$ Y is the same as that of the substitutin φ, but it is not needed (more about bases later).

Factor maps are finite-to-one over nonperiodic points

Proposition(Durand)

Let φ be a primitive substitution. There exists $K \geq 1$ such that every factor map $\pi\colon X_\varphi \to Y$ to an infinite system Y is K-to-1, i.e. $|\pi^{-1}(y)| \leq K$ for all $y \in Y$.

Finitary characterisation of primitive substitutivity

Consider the sequence

$$x = 011|010|0110|01|011|010|01|0110|011|010|01\dots$$

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Characterisation of primitive substitutive sequences in terms of derived sequences (Durand, 2000)

Let x be a uniformly recurrent sequence. Then x is primitive substitutive if and only if the set of derived sequences $D_x(u)$ is finite where u runs over all prefixes of x.

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How about general (growing?) substitutions?

Conjecture I

Let $\varphi\colon \mathcal{A} \to \mathcal{A}^*$ be a (growing) substitution, let $\tau\colon \mathcal{A} \to \mathcal{B}$ be a coding, and let $Y = \tau(X_\varphi)$. Then

1. A sequence $y \in Y$ is substitutive if and only if $y = \pi(x)$ for some quasi-fixed point x of φ.
2. If $y \in Y$ is substitutive and nonperiodic, then all points in $\pi^{-1}(y)$ are quasi-fixed points of φ.

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It would be enough to show that any $y \in Y$ which is substitutive and generates Y (i.e. $\mathcal{L}(y) = \mathcal{L}(Y)$) is of required form.

One can reduce to this case using the characterisation of transitive subsystems of X_φ.

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Can we assume that the "base" is the same?

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Let $\alpha, \beta > 1$ be real numbers. Let x be an α-substitutive sequence, and let y be a β-substitutive sequence. Assume L(x) = L(y).

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1. x and y are automatic (Fagnot's theorem),
2. x and y are primitive substitutive (dynamical proof using the structure of ergodic measures on systems generated by x and y).

Conjectures are connected

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Are factor maps finite-to-one over nonperiodic points?

Question

Let φ be a general substitution. Let $\pi\colon X_\varphi \to Y$ be a factor map. Does there exist $K \geq 1$ such that π is K-to-1 on nonperiodic points, i.e. $|\pi^{-1}(y)| \leq K$ for all nonperiodic $y \in Y$?

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Suitable finitary characterisation of substitutivity

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