Monochromatic arithmetic progressions in Sturmian sequences

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The Open University

One World Combinatorics on Words seminar

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Joint work with Dan Rust

Baudet's conjecture (1921)

Consider a random, indexed string of binary digits:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15 . . .
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Let $k, N \in \mathbb{N}$. For every k, there exists N such that any string of length $n \geq N$ must admit a MAP of length k.

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$$N = W(2,k)_{k\geq 1} = 1, 3, 9, 35, 178, \underbrace{1132}_{\text{SAT solver (2008)}}, ?$$

Questions

Question (1)

Does a given sequence admit an infinite MAP for some d?

Question (2)

If not, can we formulate the longest length of MAPs for any d?

Constant-length substitutive sequences

- Let $\mathcal{A}$ be an alphabet. A constant-length substitution, say θ is where $\forall a \in \mathcal{A}$, $\exists k \in \mathbb{N}$ such that $|\theta(a)| = k$.
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- The sequences generated are also k-automatic because of Cobham (1972).
- Parshina (2015, 2017), Aedo et al. (2023), Sobolewski (2024) studied the MAPs in constant-length substitutive sequences such as the Thue–Morse, Rudin–Shapiro sequences and their generalisations.
- From Dekking (1978), a sequence admits an infinite MAP if the generating substitution (or its power) has a coincidence. For example, the Period-Doubling substitution

$$0 \mapsto 01$$
$$1 \mapsto 00$$

Few results in connection and `Walnut`

- Previous works only managed to find formulae for $A(d) = \texttt{max}(\text{MAP length})$ for specific families of d values.
- For instance, for the Thue–Morse word, Aedo et al. showed that $A(2^n + 1) = 2^n + 2$.

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- For instance, for the Thue–Morse word, Aedo et al. showed that $A(2^n + 1) = 2^n + 2$.
- They also mentioned that $A(2d) = A(d)$, which meant that they need not consider even values of d for the rest of the analysis. Then, they showed that for all odd $d > 1$ we have $A(d) \geq 3$.
- Their conjecture based on the empirical data that $A(d) \neq 3$ is a simple exercise in Walnut which returns TRUE.

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eval tm_aedo_conj "Ad ($odd(d) & d>1) => ~Ei
(T[i]=T[i+d] & T[i]=T[i+2*d] & T[i]!=T[i+2*d])";
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- Sobolewski found a fantastic search algorithm for the Rudin–Shapiro word, which worked better than `Walnut` (in terms of calculation time) for searching individual values of d.

Deviation to non-constant-length substitutive sequences

- Our obvious choice of example was the Fibonacci word, for which we found the following on the OEIS.

A339949 a(n) is the greatest runlength in all n-sections of the infinite **Fibonacci** word A014675. +30 4

2, 3, 5, 6, 7, 3, 2, 12, 4, 4, 4, 4, 18, 2, 3, 6, 20, 5, 3, 2, 30, 4, 3, 4, 4, 9, 2, 3, 9, 4, 4, 3, 4, 47, 2, 3, 5, 10, 6, 3, 2, 15, 4, 4, 4, 4, 13, 2, 3, 7, 8, 5, 3, 2, 77, 4, 3, 5, 6, 8, 3, 2, 10, 4, 4, 3, 4, 24, 2, 3, 6, 78, 6, 3, 2, 22, 4, 3, 4, 4, 11, 2

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS
Equivalently a(n) is the greatest runlength in all n-sections of the infinite **Fibonacci** word A003849.
From Jeffrey Shallit, Mar 23 2021: (Start)
We know that the **Fibonacci** word has exactly n+1 distinct factors of length n.
So to verify a(n) we simply verify there is a **monochromatic** arithmetic progression of length a(n) and difference n by examining all factors of length (n*a(n) - n + 1) (and we know when we've seen all of them). Next we verify there is no **monochromatic** AP of length a(n)+1 and difference n by examining all factors of length n*a(n) + 1.
Again, we know when we've seen all of them. (End)

- We began with some `Walnut` experiments and investigated some patterns in this sequence which could be proven with `Walnut`.
- For instance, if $A(d) = 2$ in the Fibonacci word, then the first occurrence of such a MAP must begin either at index 0 or 2. We presented our findings at CIRM in February 2024.

Two audience members at CIRM

- First one exposed us to

Theorem (Durand+Goyeheneche, 2019)

Let $\mathbf{x}$ *be a Sturmian sequence. Then it admits no constant arithmetic subsequence.*

- However, this implication stemmed from a more technical and general result about uniformly recurrent morphic sequences, and it did not directly lead us to a method to find the longest MAPs.
- The second nudged us to look at the irrational rotation characterisation of Sturmians to “somehow” study MAPs.
- As remote as it sounded at that time (to me), within a month, we had a systematic way to achieve something really nice.

Definition of Sturmian sequences

A Sturmian sequence is a binary infinite sequence defined as coding of an irrational rotation around a circle with two intervals $[0, 1 - \theta]$ and $[1 - \theta, 1]$. (In literature, θ is usually known as slope.)

Sturmian sequences have multiple equivalent definitions.

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The Golden Example

The Fibonacci word **f** is a Sturmian sequence with $\theta = \tau^{-2}$, where $\tau = 1.618034\ldots$. For $z \in \mathbb{R}$, let $\langle z \rangle = z \bmod 1$.

Then plot $\langle n\tau^{-2} \rangle$ for $n \geq 1$ as follows:

f : 0 1 0 0 1 0 1 0 0 1 0 0 1 0 1 0 $\cdots$

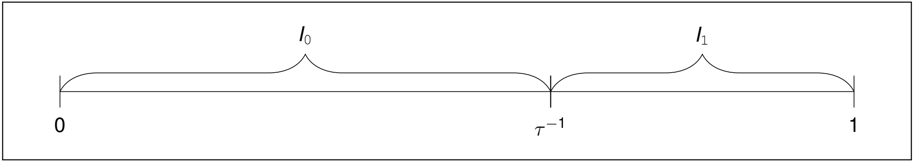

Figure: Generating the Fibonacci word

Proposition

$\langle (n+1)\tau^{-2} \rangle$ codes the n^{th} term of **f** *for all $n \geq 0$.*

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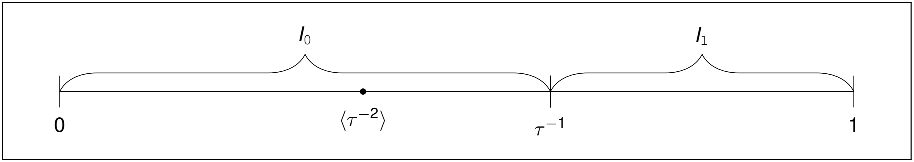

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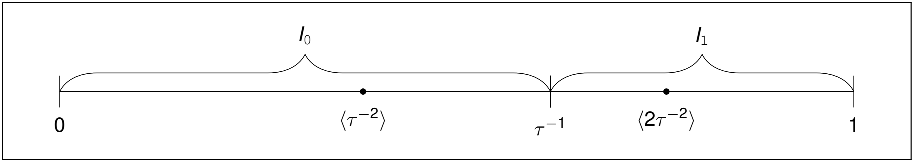

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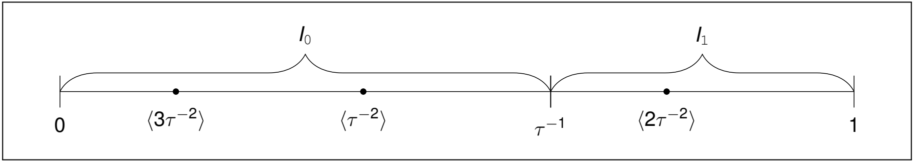

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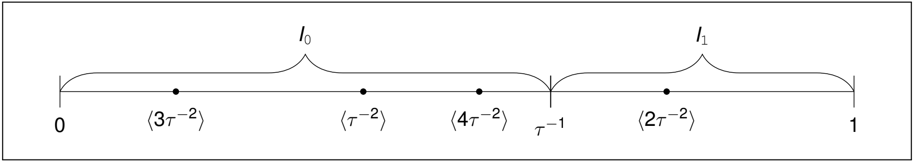

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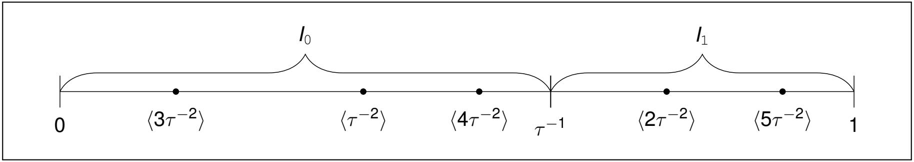

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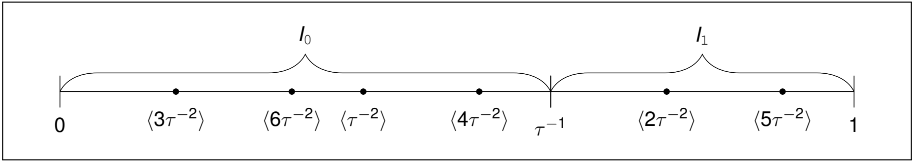

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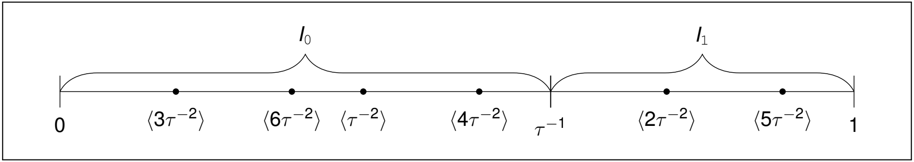

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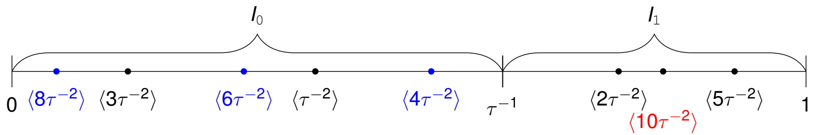

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Finiteness of MAPs in Sturmians

Lemma

Fix $d \geq 1$. There exists a length-ℓ MAP of difference d on the symbol j in a Sturmian sequence $\mathbf{s}$ starting at position $i-1$ if and only if all of the points $\langle i\theta \rangle$, $\langle (i+d)\theta \rangle$, $\langle (i+2d)\theta \rangle$,..., $\langle (i+(\ell-1)d)\theta \rangle$ belong to the interval I_j.

Theorem (Weyl's equidistribution criterion, 1909)

For an irrational θ, the sequence $(\langle n\theta \rangle)_{n \geq 1}$ is uniformly distributed over $[0, 1]$.

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Onto question #2 for the Fibonacci word

- We define step distance as

$$g(d) := \min\{\langle d\tau^{-2}\rangle, 1 - \langle d\tau^{-2}\rangle\}.$$

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For case (1),

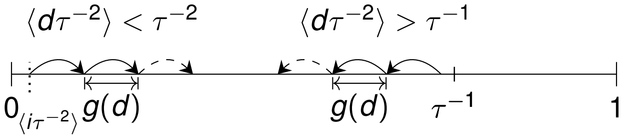


Figure: Illustrating how consecutive points appear in I_0 with Case 1

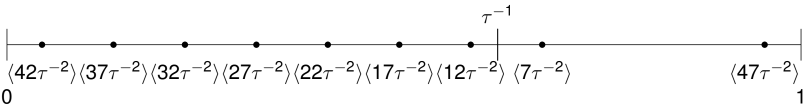
Case 1 with example

If $g(d) \leq \tau^{-2}$,

$$A(d) = \left\lceil \frac{|I_0|}{g(d)} \right\rceil .$$

Take $d = 5$. Then $g(5) \approx 0.09 \leq \tau^{-2} \approx 0.382$,

$$A(5) = \left\lceil \frac{\tau^{-1}}{0.034} \right\rceil = \left\lceil \frac{\approx 0.618}{0.034} \right\rceil = 7.$$



And if $g(d) > \tau^{-2}$,

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For example, take $d = 17$. Then $g(17) \approx 0.507 > \tau^{-2}$. We may suitably choose $i = 71$. The points $\langle (i + nd)\tau^{-2} \rangle$ are denoted only by n for convenience.

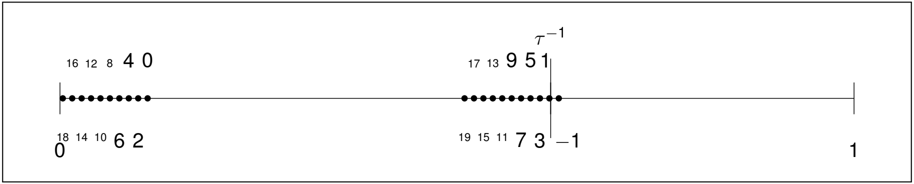

Figure: Counting the repetition of 0 to find $A(17)$

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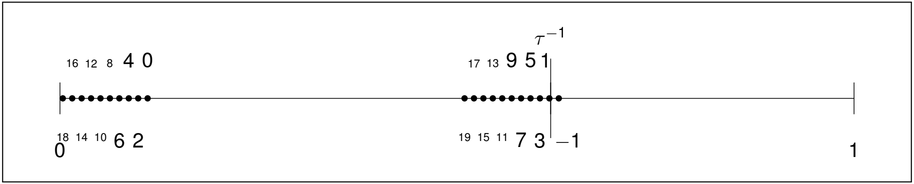


Figure: Counting the repetition of 0 to find $A(17)$

$$A(17) = 2\left\lceil \frac{\tau^{-1} - g(17)}{g(34)} \right\rceil = 20.$$

$A(d)$ for Sturmians

Theorem

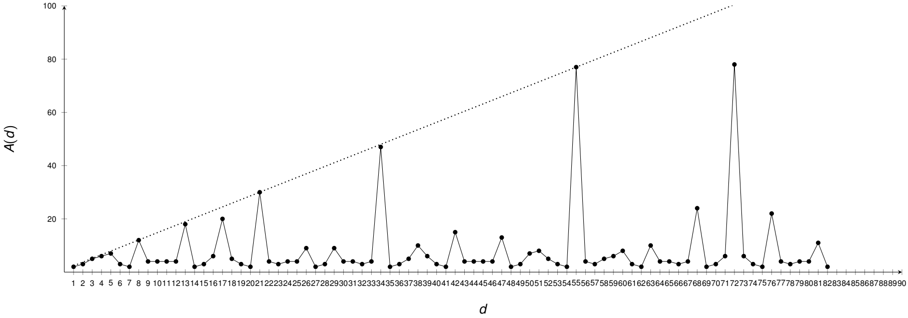
Let $d \in \mathbb{N}$. The following formulae evaluate $A(d)$ for any d, for any Sturmian of slope $0 < \theta < 0.5$.

$$A(d) = \begin{cases} \left\lceil \dfrac{1-\theta}{g(d)} \right\rceil & \text{if } g(d) \leq \theta, \\ k \left\lceil \dfrac{1-\theta-(k-1)g(d)}{1-kg(d)} \right\rceil & \text{if } g(d) > \theta \text{ and } g(d) < 1/k, \\ k \left\lceil \dfrac{g(d)-\theta}{kg(d)-1} \right\rceil + k - 2 & \text{if } g(d) > \theta \text{ and } g(d) > 1/k. \end{cases}$$

where

$$k = \left\lfloor \frac{1-\theta}{g(d)} \right\rfloor .$$

The plot $A(d)$ versus d

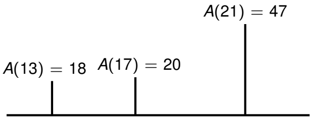


Conjecture (J-Rust, 2025)

For all d,

$$\frac{A(d) - 1}{d} < 1 + \tau^{-2} \approx 1.382$$

Approachable information



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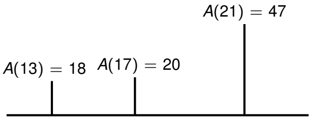

Theorem (τ-modified three-gap theorem (Steinhaus conjecture) - Slater, 1950)

Given $0 \leq j \leq d$, all the points $\langle j\tau \rangle$ are plotted on the interval $[0, 1]$ with at most 3 distinct pairwise gaps. Furthermore, if $F_{n-1} \leq d < F_n$ then the smallest gap length is $|F_n - F_{n-1}\tau|$, immediately followed by $|F_{n-1} - F_{n-2}\tau|$.

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$$\left(\text{Hint: Take } \bmod 1 \text{ on both sides of } F_{n+1} - \tau F_n = \frac{(-1)^n}{\tau^n} \right)$$

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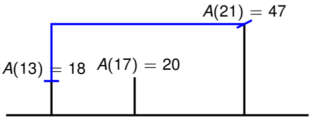

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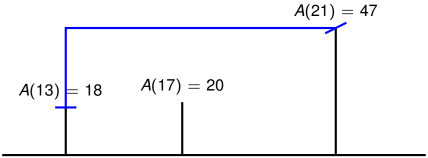
2. A new maxima of $A(d)$ is achieved at $d = F_n$ for every n.
(Step-shaped threshold)

Discussion continued...



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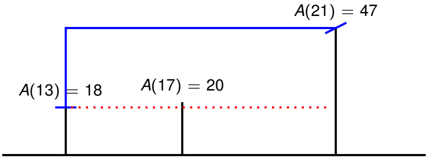

1. Furthermore, if $g(d) \leq \tau^{-2}$, then for all such $d < F_n$, $A(d) \leq A(F_{n-1})$, again from the three-gap theorem.

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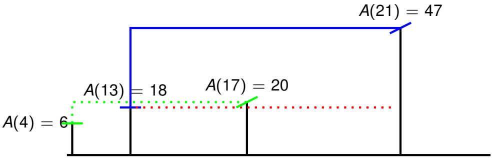

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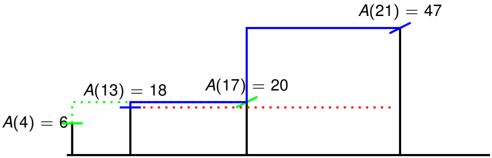

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3. then the blue line can be brought down as shown!

Final step(s)

- For all n, $A(F_n/2) - A(F_{n-2}) = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases}$

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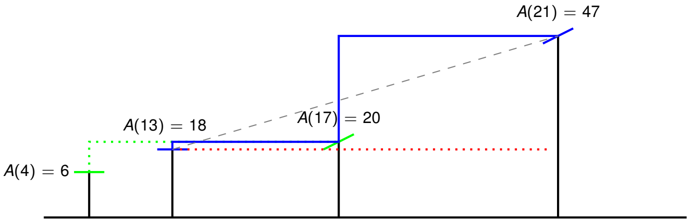

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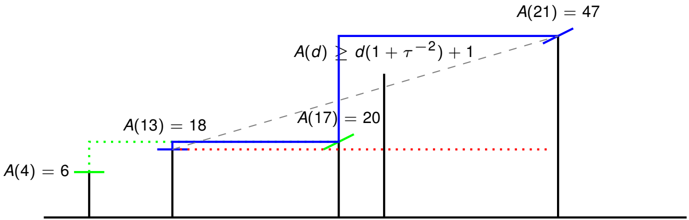

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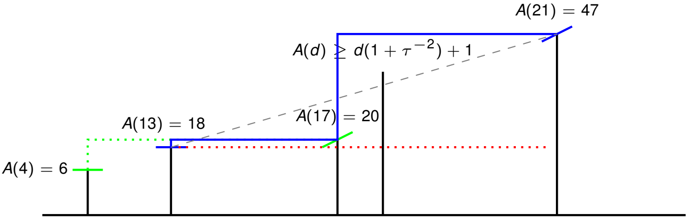

- Then show contradiction. That is,

$$2\left\lceil \frac{\tau^{-1} - g(d)}{g(2d)} \right\rceil < d(1 + \tau^{-2}) + 1.$$

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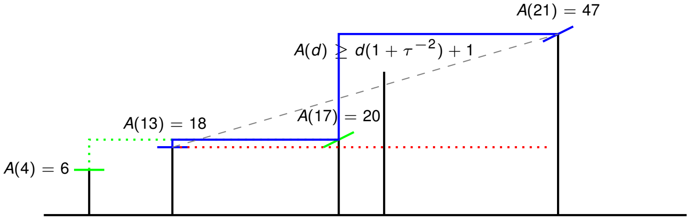

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- The slope value changes with θ for Sturmians but the conjecture remains plausible with empirical data.

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- We now have formulae that count the longest MAP lengths – $A(d)$ for any d for Sturmians with slope θ.
- This technique could potentially work for some other families of sequences based on the irrational rotation definition, say (with much more difficulty) higher-dimensional Billiard sequences.

Thank you!