Net occurrences in Fibonacci and Thue-Morse Words

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Fibonacci words

Definition

The k-th Fibonacci words F_k over $\Sigma = \{a, b\}$ is defined as follows:

- $F_1 = b$, $F_2 = a$,
- $F_k = F_{k-1} \cdot F_{k-2}$ $(k \geq 3)$

Let $f_k = |F_k|$.   (k-th Fibonacci number)

e.g. $F_3 = a \ b$,

$F_4 = a \ b \ a$,

$F_5 = a \ b \ a \ a \ b$,

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Thue-Morse words

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The k-th Thue-Morse word TM_k over $\Sigma = \{a, b\}$ is defined as follows:

- $\mathrm{TM}_0 = a$,
- $\mathrm{TM}_k = \mathrm{TM}_{k-1} \cdot \overline{\mathrm{TM}_{k-1}}$ $(k \geq 1)$

Let $\tau_k = |\mathrm{TM}_k|$.

We get $\overline{\mathrm{TM}_j}$ by replacing all a and b in TM_j

e.g. $\mathrm{TM}_1 = a\ b$,

$\mathrm{TM}_2 = a\ b\ b\ a$,

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Net occurrence

Definition
In a string T, an occurrence of a substring $S = T[i \ldots j]$ is a net occurrence if S is repeated, while both left extension $T[i-1 \ldots j]$ and right extension $T[i \ldots j+1]$ are unique in T.

e.g. T = the_theoretical_theme

There are three occurrences of "the" in T.

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the left extension _the occurs twice in T.

These two occurrences of "the" are not net occurrences in T.

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The only net occurrence of "the", since both left extension and right extension are unique.

Note. In this talk, we assume that occurrence T[0 ... j] and T[i ... |T| + 1] to be unique in T.

Net Frequency

Definition

In a string T, Net Frequency(NF) of a substring S = T[i ... j] is the number of net occurrence of in T.

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There are three occurrence of "the" in T, but NF of "the" is one.

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Only the is used as a definite article, while the other the are prefixes of other words, which are “theoretical” and “theme”.

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Originally, NF is motivated by Chinese natural language processing tasks. [Lin and Yu, 2001]

Recent works

- There are some algorithms which compute the NF.

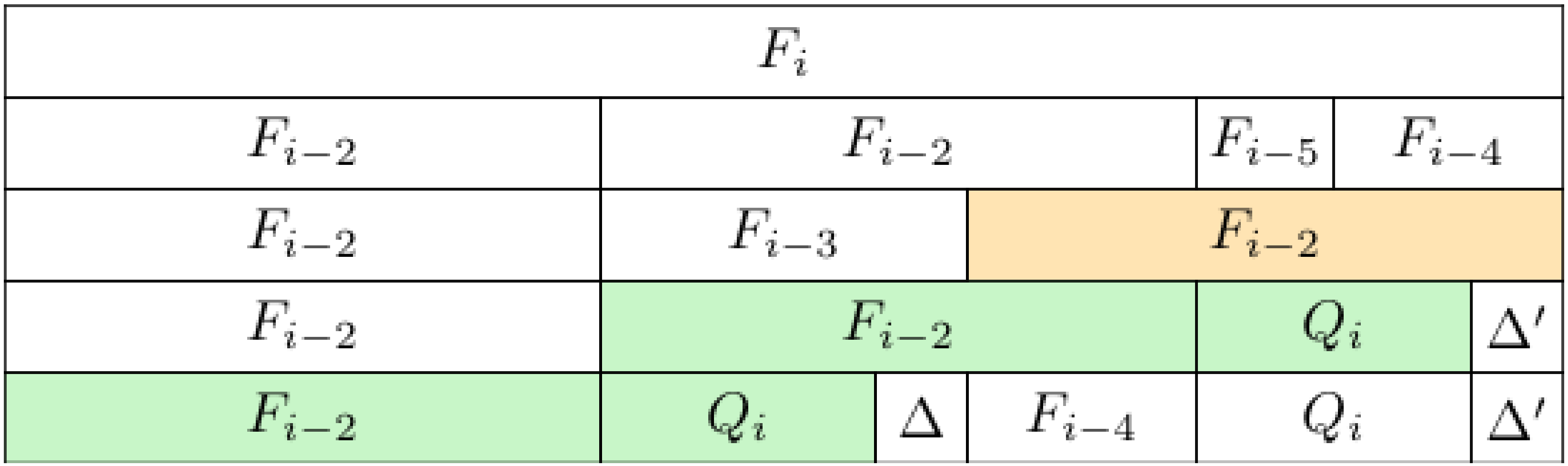
Offline settings: [Guo et al., CPM 2024], [Ohlebusch et al., SPIRE 2024]

Online settings: [Guo et al., SPIRE 2024] [Inenaga, ArXiv 2024]

[Mieno and Inenaga, CPM 2025]

- Also, Mieno and Inenaga (2025) characterized net occurrences in terms of the minimal unique substrings (MUSs).

Net occurrence in Fibonacci words



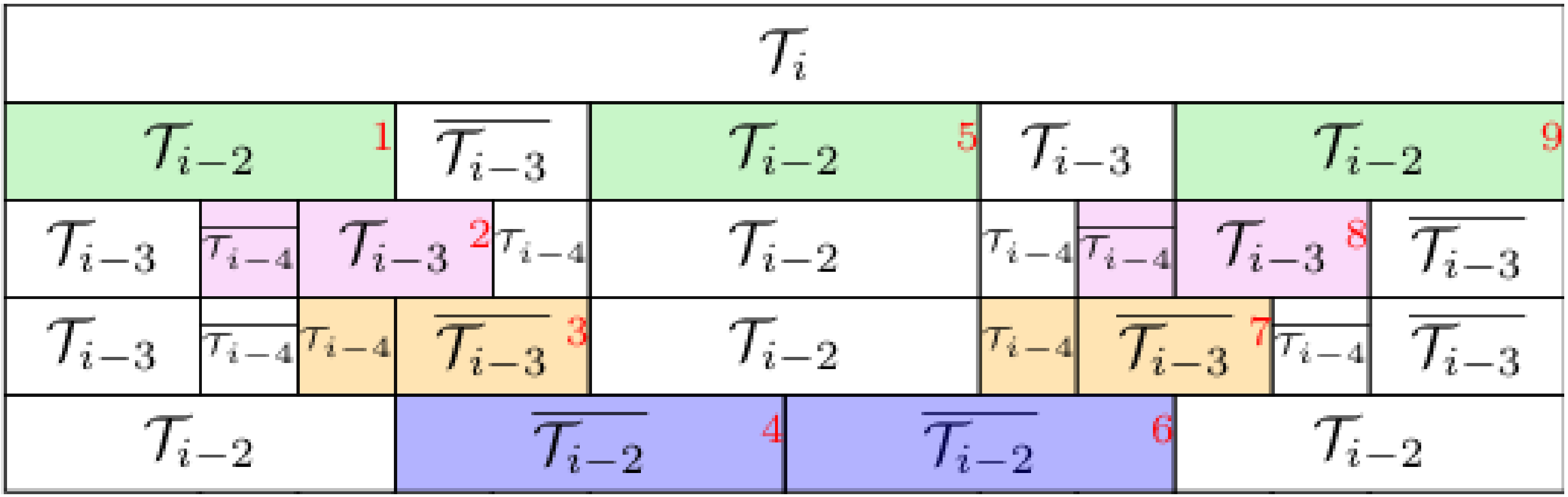
$Q_i := F_{i\text{-}3}[1 \,..\, |F_{i\text{-}3}| - 2]$

We proved:

(1) All three colored occurrences are Net occurrences.

(2) Other occurrences are not Net occurrences.

Net occurrence in Thue-Morse words



$\mathcal{T}_i := \mathrm{TM}_i$

We proved:

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Basic properties of Net occurrences 1/2

Lemma 1
In a string T, if an occurrence of string S' is a proper **super-**occurrence of a net occurrence of string S in T, then occurrence of S' is not a net occurrence.

e.g. T = the_theoretical_theme

↑ net occurrence in T

An occurrence of "the_" is not a net occurrence since it must be unique in T.

Basic properties of Net occurrences 2/2

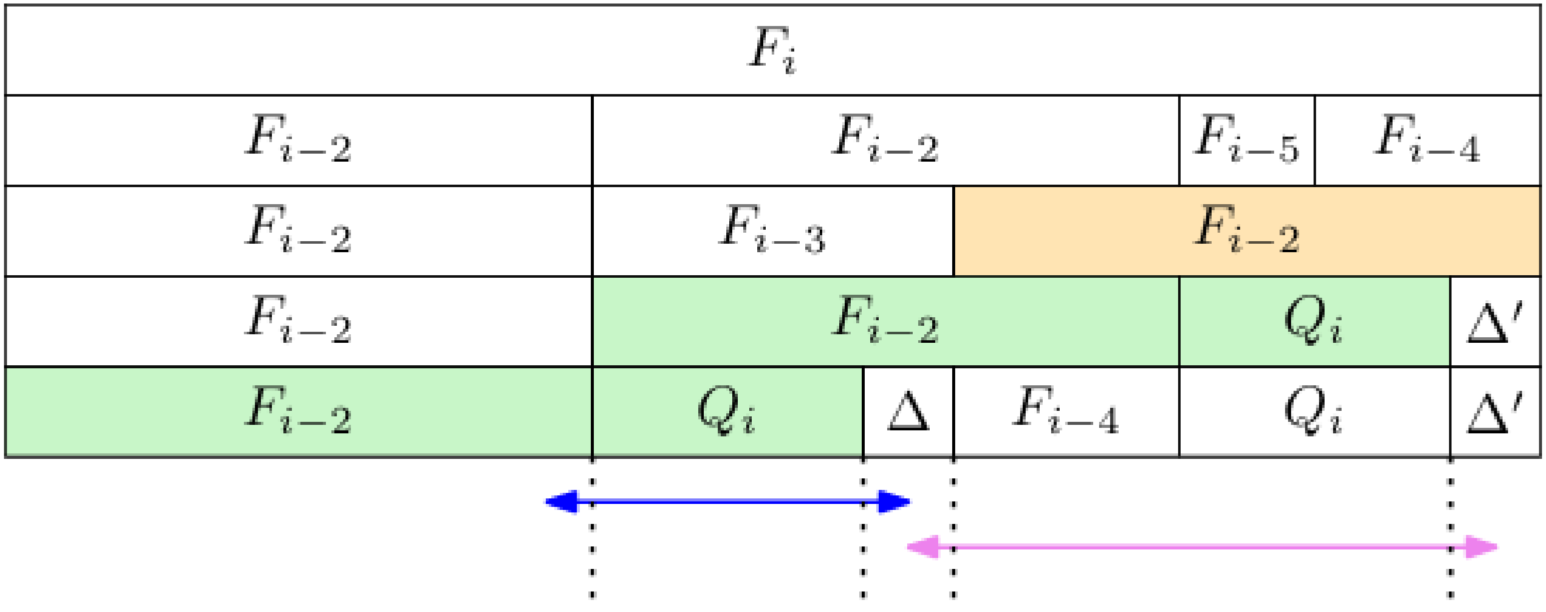
Lemma 2
In a string T, if an occurrence of string S’ is a proper **sub-**occurrence of a net occurrence of string S in T, then occurrence of S’ is not a net occurrence.

e.g. T = the_theoretical_theme

↑ net occurrence in T

An occurrence of “he” is not a net occurrence since it must be repeated in T, but left or right extension is also repeated in T.

Net occurrence in Fibonacci words (restate)



$Q_i := F_{i\text{-}3}[1 .. |F_{i\text{-}3}| - 2]$

We proved:

(1) All three colored occurrences are Net occurrences.

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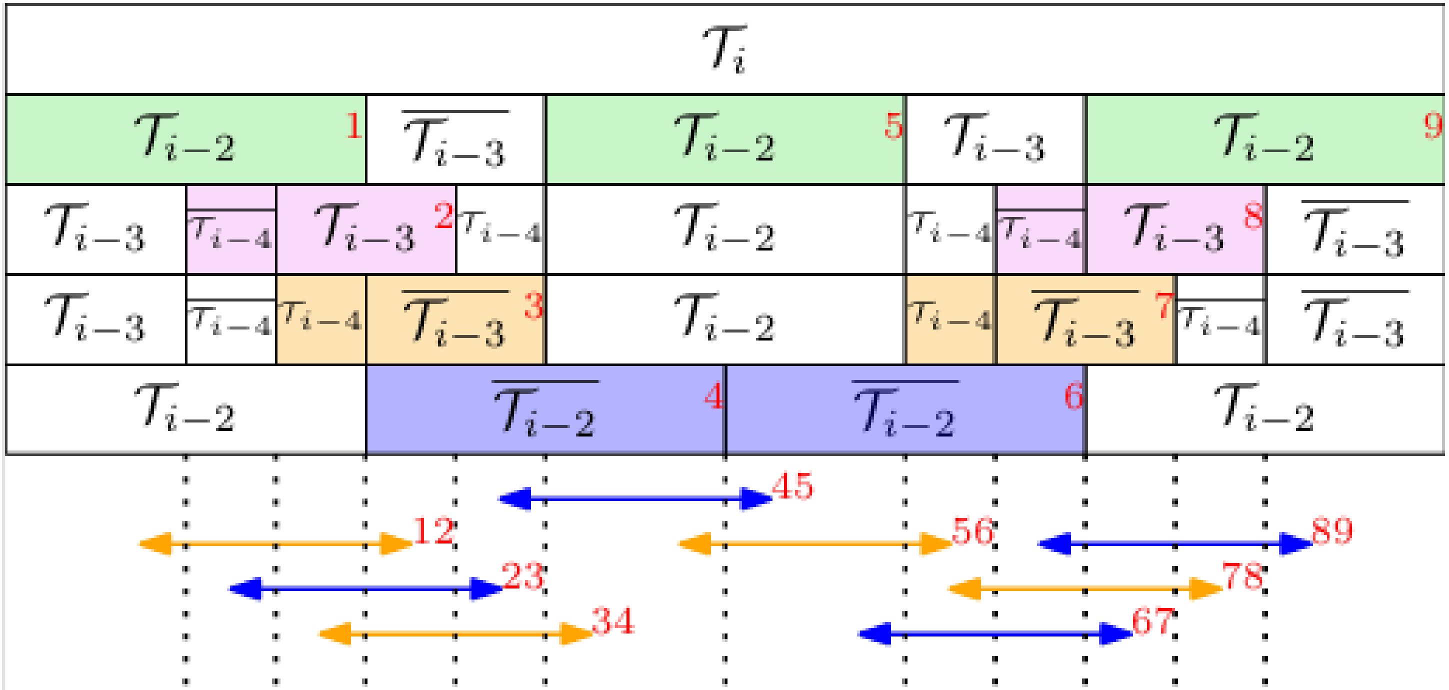
Net occurrence in Thue-Morse words(restate)



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We proved:

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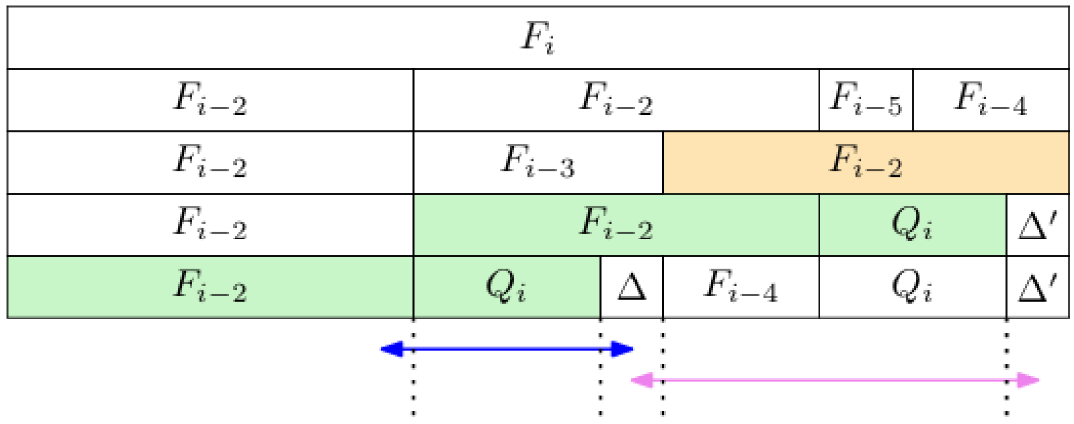
How to find Net occurrences?

To confirm an occurrence of substring T[i ... j] is a net occurrences in T, we have to prove

1. T[i ... j] is repeated in T.
2. T[i − 1 ... j] is unique in T.
3. T[i ... j + 1] is unique in T.

If we can count the number of occurrences of all of the substrings T[i ... j] in T, we can prove the above and find all of the Net occurrences in T.

Sketch of the proof of the Net occurrence in Fibonacci and Thue-Morse words



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We proved the above by counting the number of occurrences of substrings in F_i and TM_i carefully.

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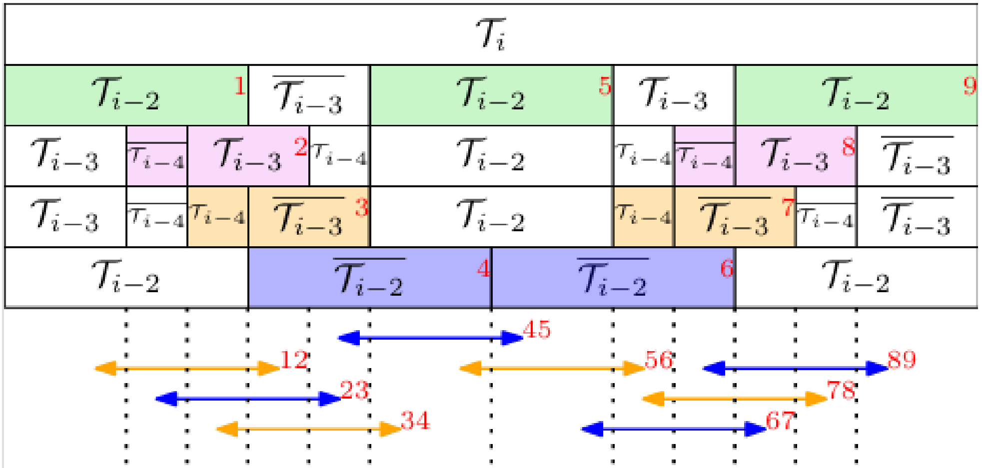

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We proved the above by counting the number of occurrences of substrings in F_i and TM_i carefully.

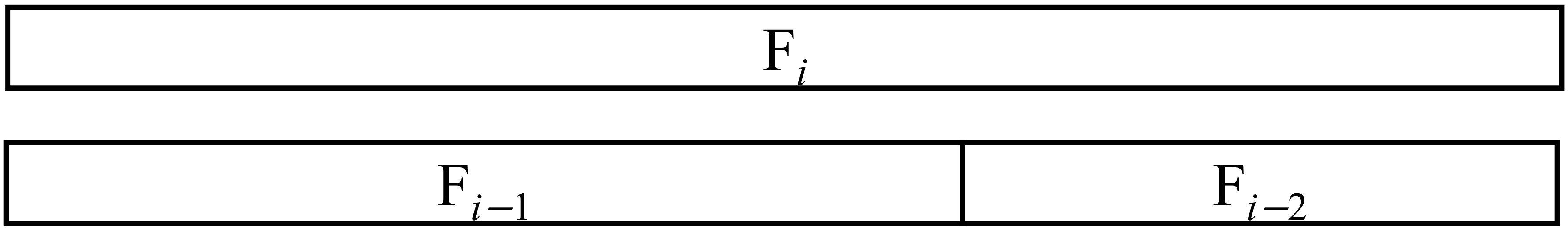
To count these substrings, we characterized the occurrences of in F_j and TM_j in F_i and TM_i where $j \leqq i$.

Factorization of F_i with smaller order

F_i

There is a factorization of F_i , where all factors are F_{j+1} or F_j when $j \leqq i$.

Factorization of F_i with smaller order



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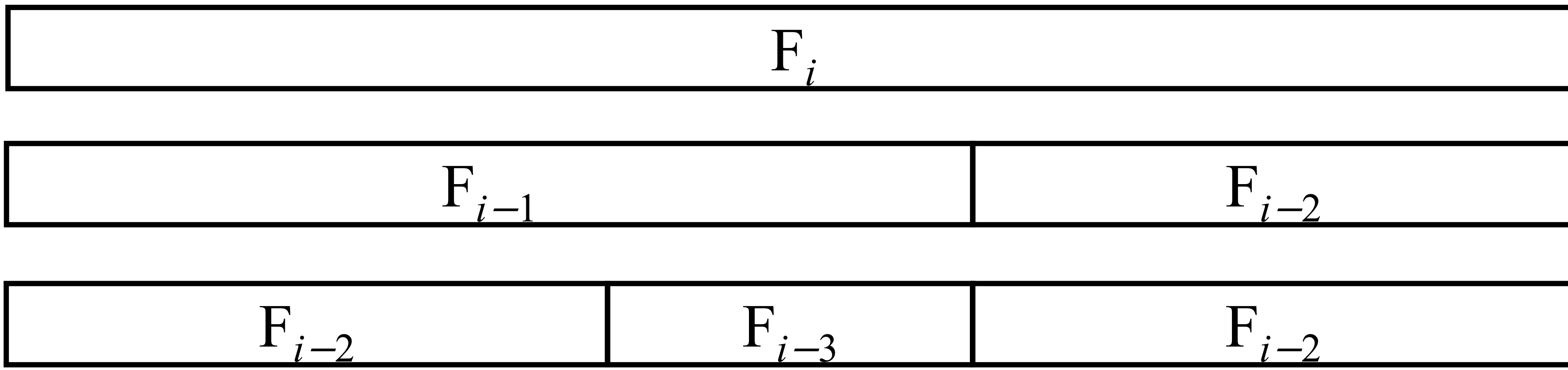

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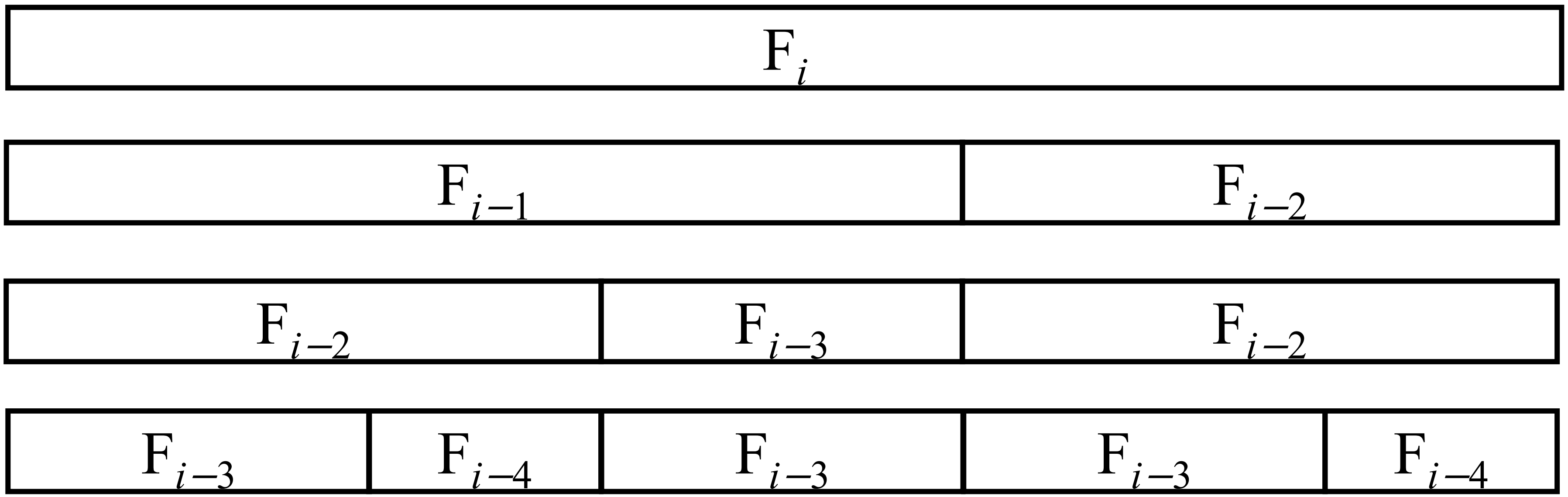

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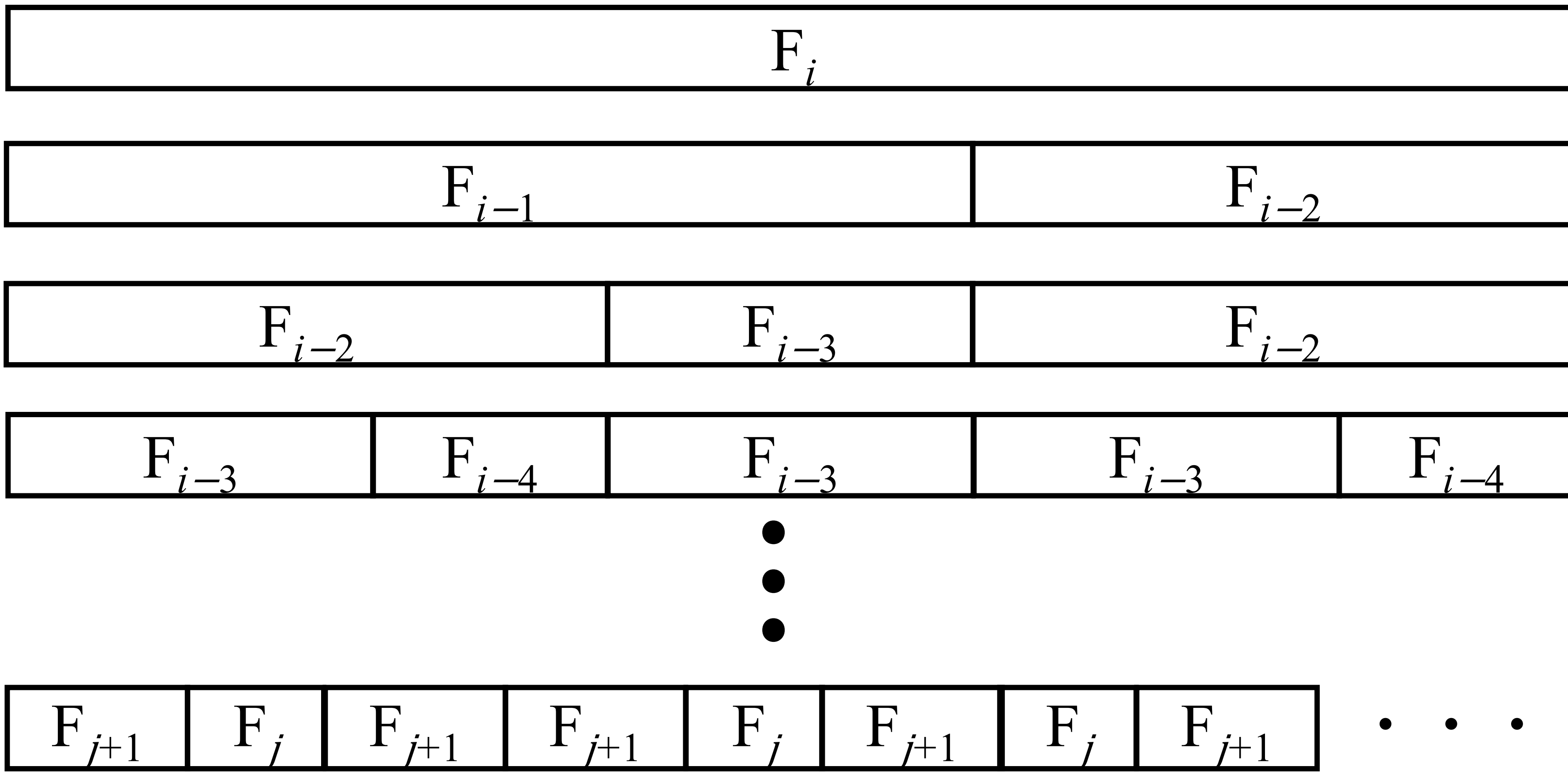

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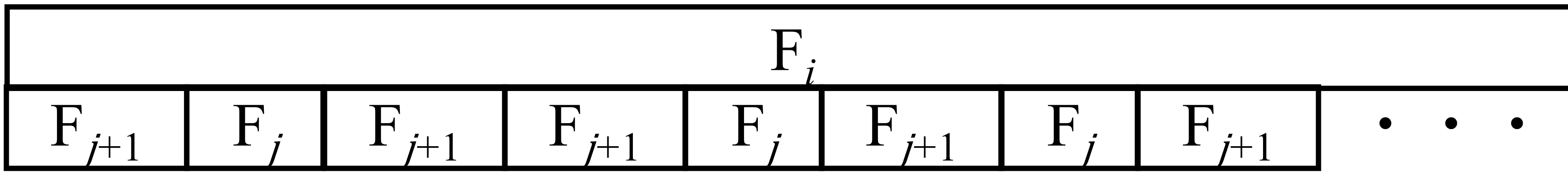
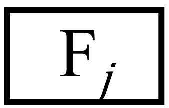
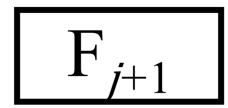
Occurrences of Fibonacci Words of smaller order



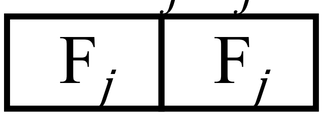
Case (1) F_j occurs in F_j



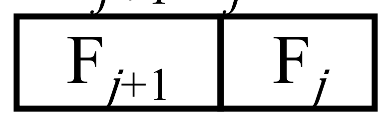
Case (2) F_j occurs in F_{j+1}



Case (3) F_j occurs across the border of F_j F_j

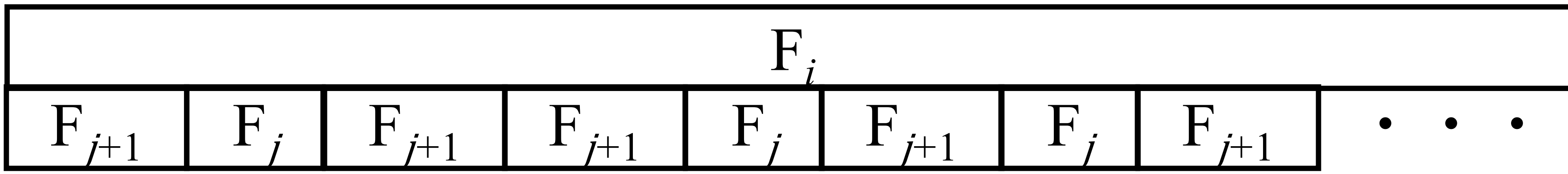


Case (4) F_j occurs across the border of F_{j+1} F_j



Since $f_j < f_{j+1}$, there are only four cases where F_j , occurs in F_i.

Occurrences of Fibonacci Words of smaller order



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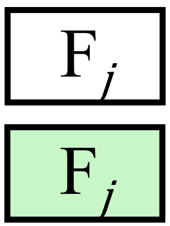
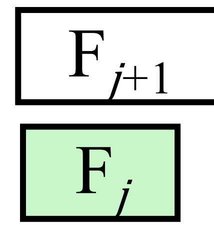


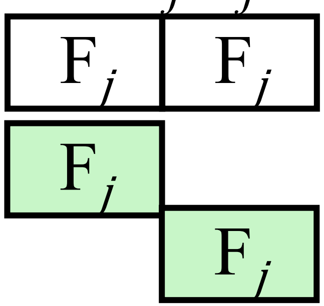
only once

Case (2) F_j occurs in F_{j+1}



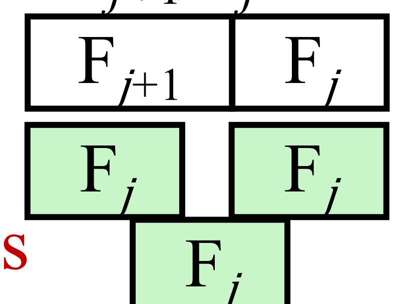
only once [Navarro et al., 2020]

Case (3) F_j occurs across the border of F_j F_j



only twice [Droubay, 1995]

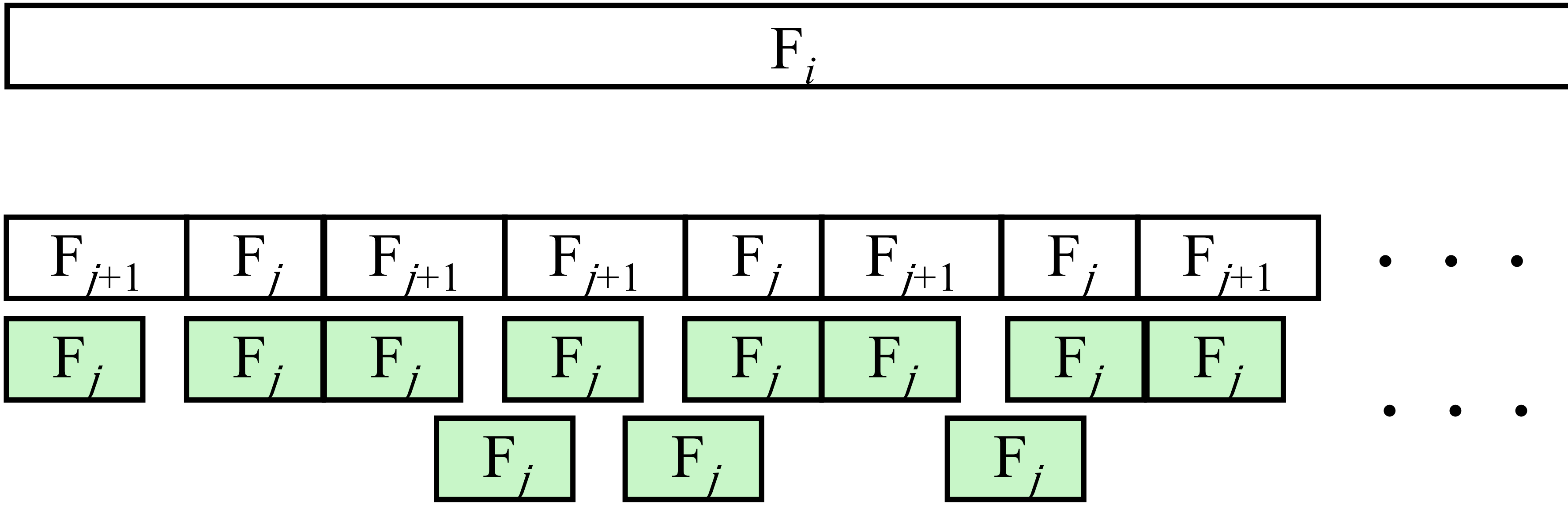
Case (4) F_j occurs across the border of F_{j+1} F_j



only three times [Navarro et al., 2020]

We can characterize the occurrence of F_j in all cases using some theorem about Fibonacci words.

Occurrences of Fibonacci Words of smaller order



Therefore, we can characterize all the occurrence of F_j in F_i .

Note : This is also proved in [Iliopoulos et al., 1997] and [Rytter, 2006] in a different way.

Factorization of TM_i with smaller order

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There is a factorization of TM_i, where all factors are TM_j or $\overline{\mathrm{TM}_j}$ when $j \leqq i$.

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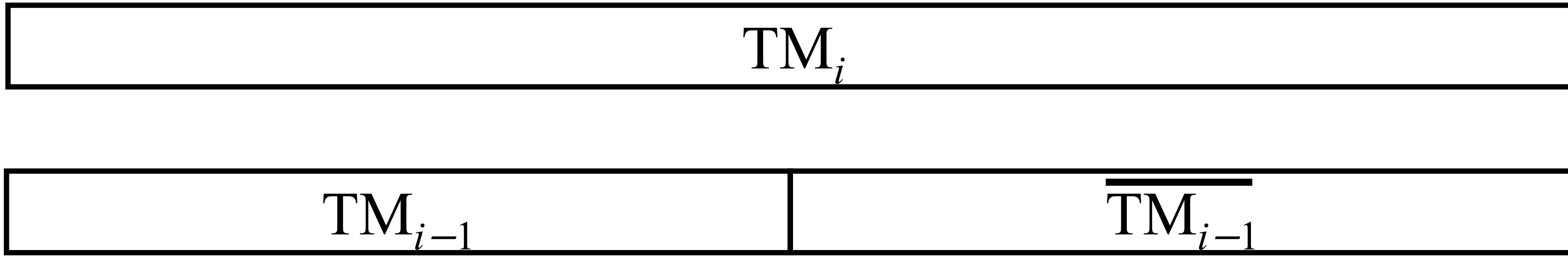

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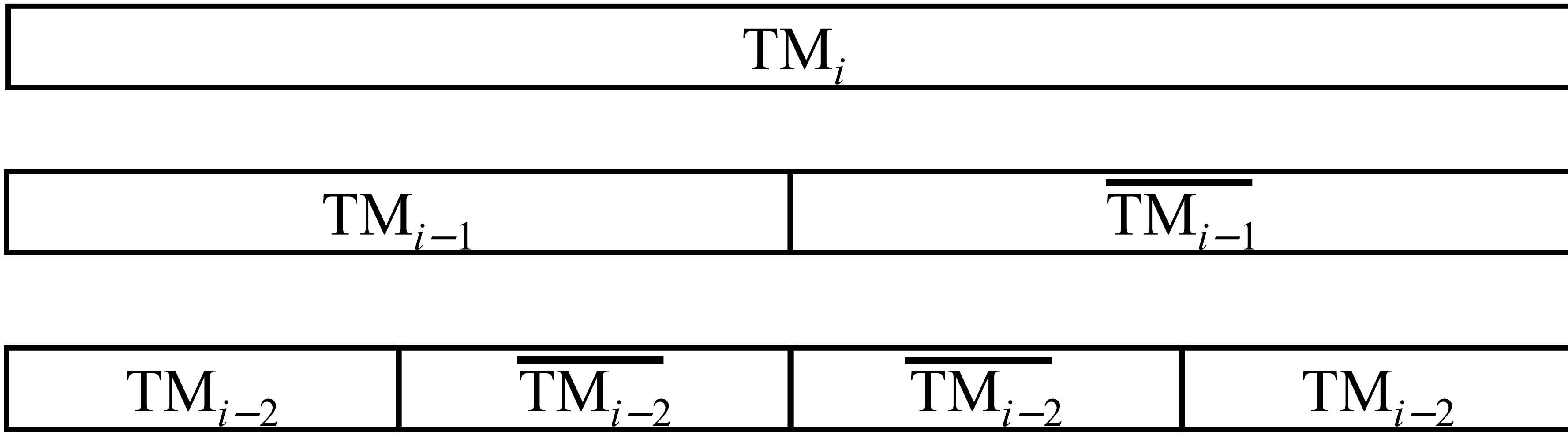

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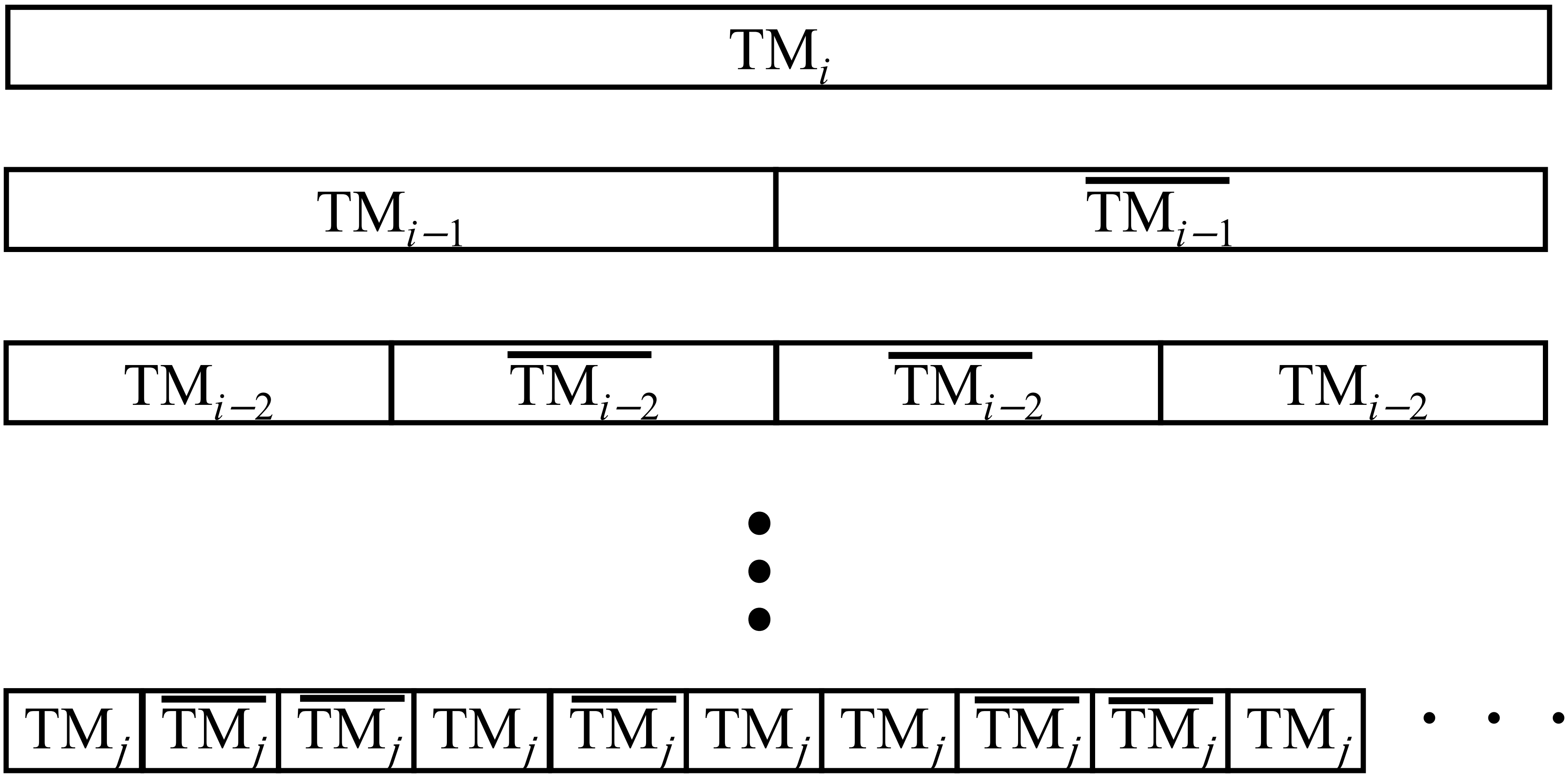

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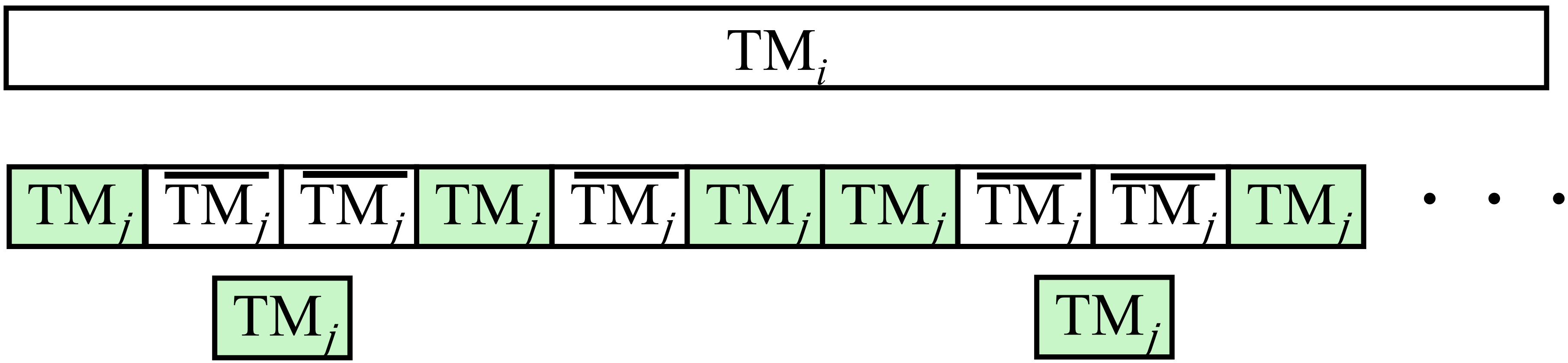
Occurrences of Thue-Morse Words of smaller order

Theorem [Lothaire, 1997]
TM_i has no overlapping occurrences of the same string.



Since Thue-Morse words is overlapping free and TM_j occurs only once in $\overline{TM_j}\,\overline{TM_j}$, we can characterize all of the occurrences of TM_j in TM_i .

Note : This is also proved in [Radoszewski and Rytter, 2012] in a different way.

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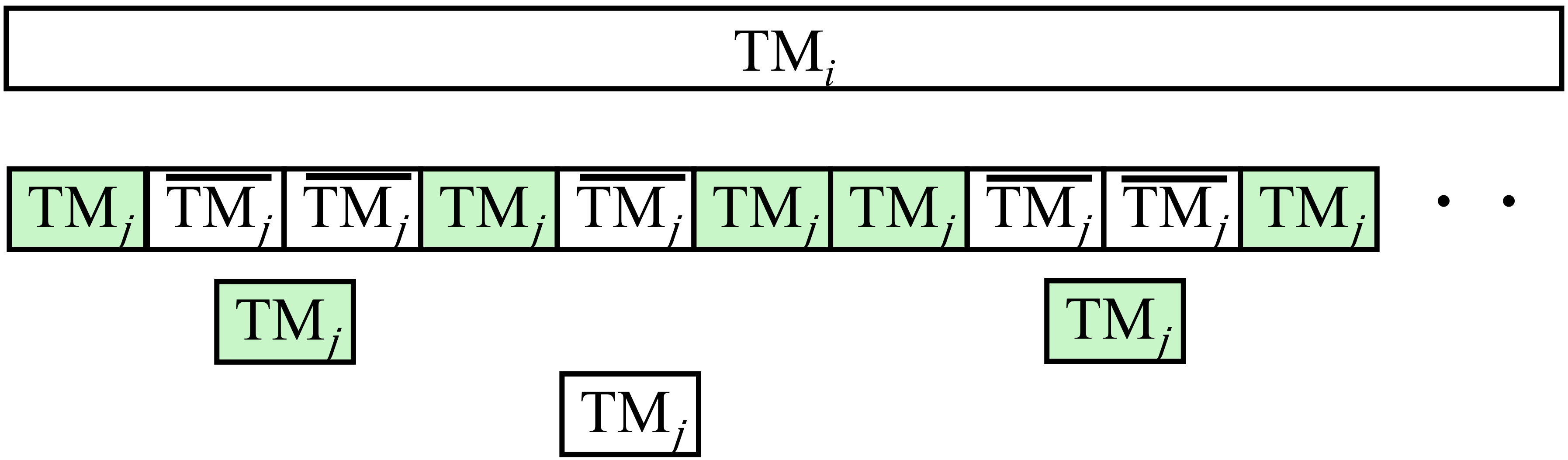

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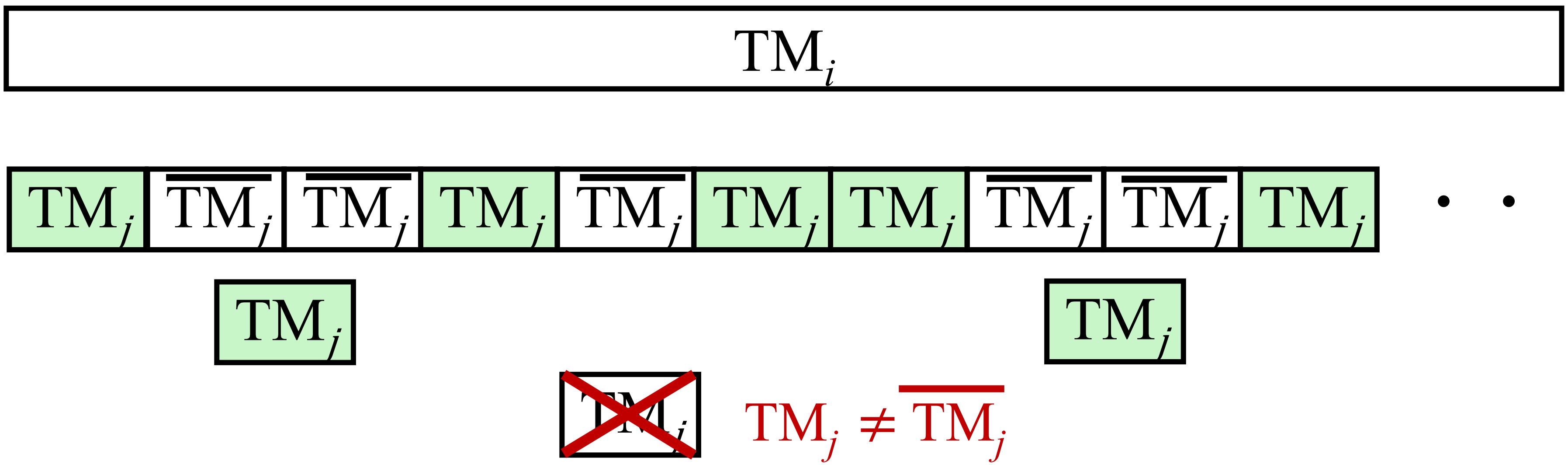

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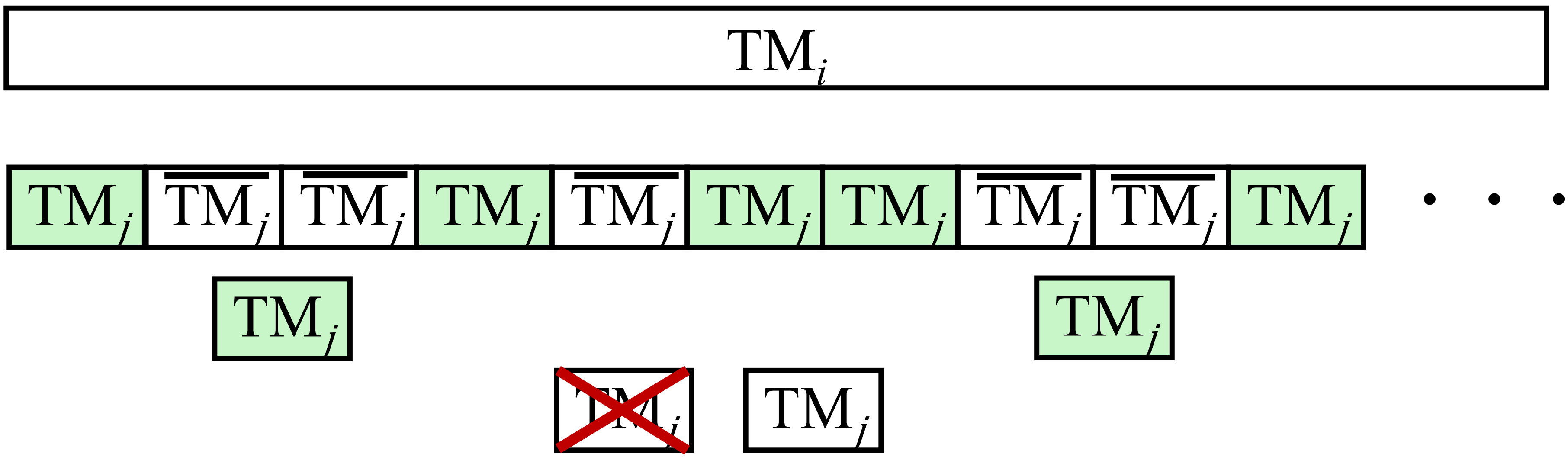

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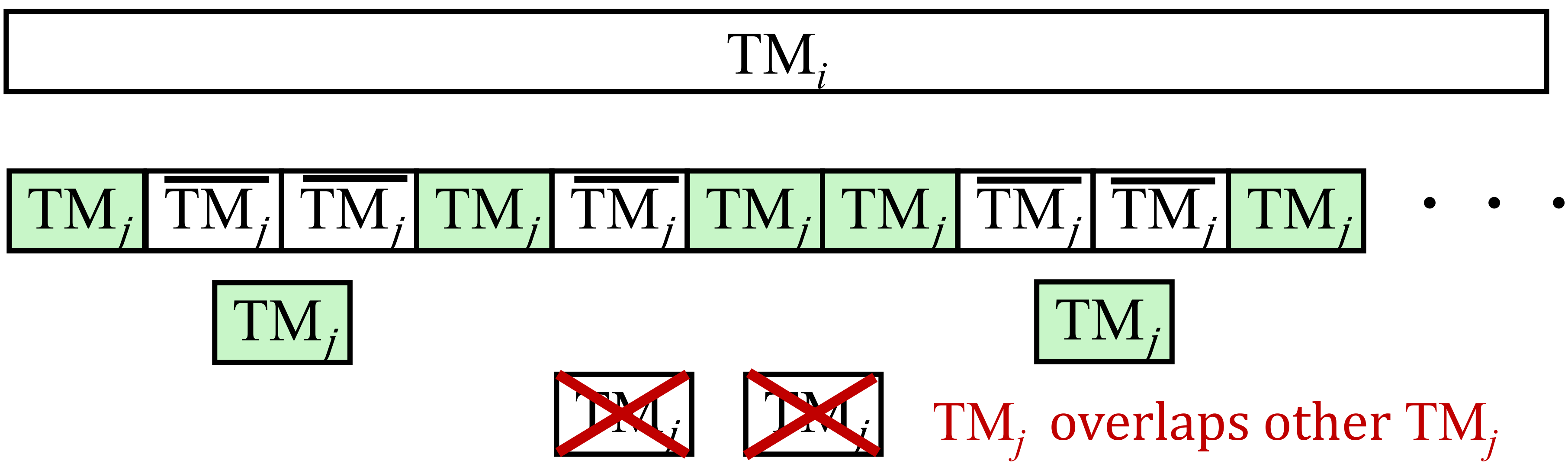

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Conclusion

- We introduced the net occurrences and its some properties.
- We showed characterization of occurrences of Fibonacci words and Thue-Morse Words of smaller order.
- We showed there are only three net occurrences in Fibonacci words and only nine net occurrences in Thue-Morse words.

Walnut and Net Occurrences

- Walnut is a free software that has been used to prove and disprove many theorems in combinatorics on words.
- Jeffrey Shallit provided alternative proofs using Walnut for our results on net occurrences in Fibonacci and Thue-Morse words.
- Jeffrey and we have been studying the net occurrences in other automatic sequences.

Future works

- Characterization of smaller-order occurrences of other words.

- Net occurrences of other morphic words,
  - Such as *k*-bonacci and Thue-Morse-like words.