Maximal 2-dimensional binary words of bounded degree

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January 6, 2026

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- $(h, w) \in \mathbb{Z}^2_{>0}$.

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- For $W[i, j] = \blacksquare$, its *degree* is the number of $\blacksquare$'s in its neighborhood.
- For $d \in \{0, 1, 2, 3, 4\}$, $\mathcal{W}^{\leq d}_{h\times w}(\{\square, \blacksquare\})$ ($\mathcal{W}^{\leq d}_{h\times w}$ for short) the set of all words in $\mathcal{W}_{h\times w}$ such that no $\blacksquare$ entry has degree greater than d.

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- $|W|_{\blacksquare}$ (resp. $|W|_{\square}$) is the number of $\blacksquare$ (resp. $\square$) entries in W. We also refer to $|W|_{\blacksquare}$ as the *area* of the word.

The goal

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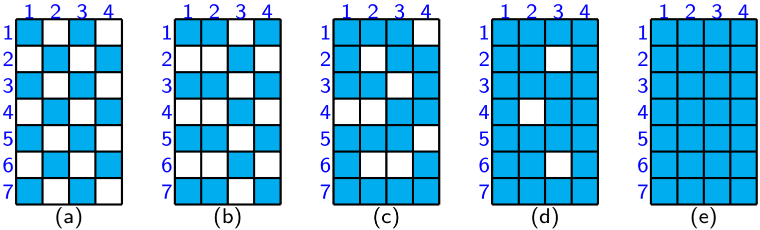

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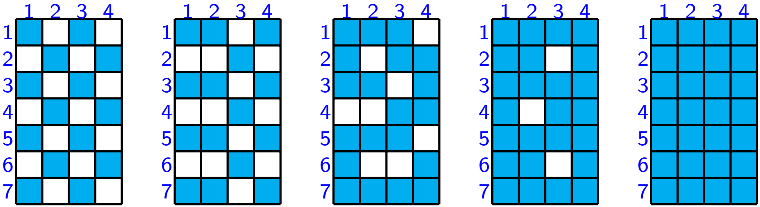

(a) (b) (c) (d) (e)

$W \in \mathcal{W}^{\leq d}_{h\times w}$ is d-full: $|W|_{\blacksquare} = \max_{\leq d}(h, w)$.

Why words?

This problem can be framed as a problem in graph theory, combinatorics (polyominoes) etc.

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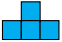
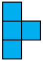
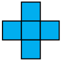
We found word theory to be the most natural framing and used a lot of it's tools.
It can also be seen as a pattern avoidance problem.

$d = 0 :$ { ,  }.

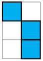
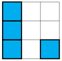
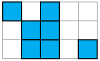
$d = 1 :$ { , , ... }.

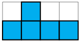
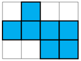
$d = 2 :$ { , , ... }.

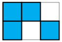
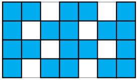
$d = 3 :$ {  }.

$d = 4 : \varnothing$.

Additional notation

- Horizontal concatenation:  ⦶  = .

- Vertical concatenation:  $\ominus$  = .

- Exponential: $W =$ , $W^{2\times 7/3} =$ .

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From now on we'll assume $h \geq w$.

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Lemma ($d = 0$)

$\max_{\leq 0}(h, w) = \lceil hw/2 \rceil$ *for any* $(h, w) \in \mathbb{Z}^2_{>0}$.

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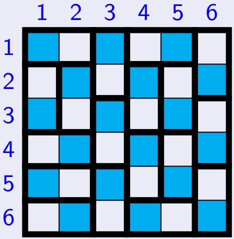
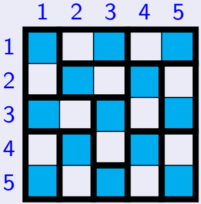
$\max_{\leq 0}(h, w) = \lceil hw/2 \rceil$ for any $(h, w) \in \mathbb{Z}^2_{>0}$. By pigeonhole principle



(a) 18 *dominoes,* 0 *monomino*



(b) 12 *dominoes,* 1 *monomino*

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Lemma ($d = 1$)

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For any $(h, w) \in [\![1, h]\!] \times [\![1, w]\!]$, *where* $h \geq w$,

$$\max_{\leq 1}(h, w) = \begin{cases} hw/2, & \text{if } (h, w) \equiv_2 (0, 0); \\ (h-1)w/2 + \lceil 2w/3 \rceil, & \text{if } (h, w) \equiv_2 (1, 0); \\ h(w-1)/2 + \lceil 2h/3 \rceil, & \text{otherwise.} \end{cases}$$

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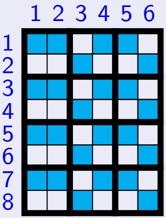
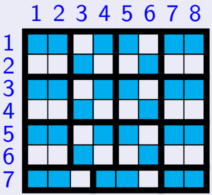


$(8, 6)$



$(7, 8)$

A detour on dominating sets

Definition

Dominating set of a graph Let $G = (V, E)$ be a graph. $S \subset V$ is a *dominating set* of G if for each $v \in V$, either $v \in S$ or there exist $s \in S$ such that $(v, s) \in E$.

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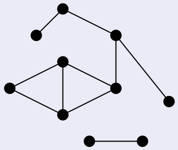

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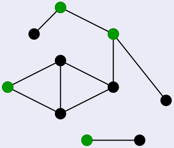

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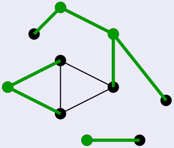

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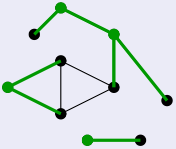
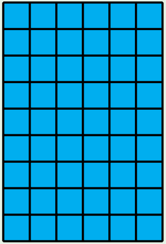


Figure: Dominating set

We say a dominating set S is *minimal* if there exists no dominating set of cardinality less than S.
The domination number $\gamma(G)$ of a graph G is the cardinality of a minimal dominating set.

$d = 3$ with dominating sets

Example



Constructing a 3-full word.

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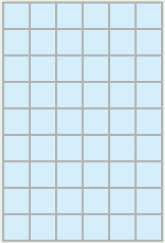

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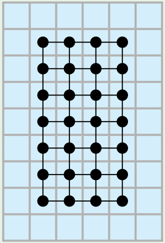

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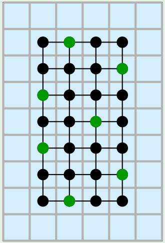

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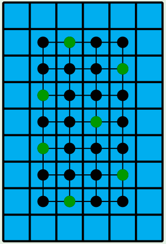

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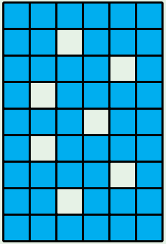

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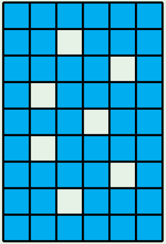

Constructing a 3-full word.

Lemma

$$\max_{\leq 3}(h, w) = \begin{cases} hw, & \text{if } 1 \leq w \leq 2; \\ hw - \gamma(G_{h-2,w-2}), & \text{otherwise,} \end{cases}$$

where $G_{k,l}$ is the grid graph of dimensions $k \times l$.

Domination number of grid graphs

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- This feat was achieved mainly through the use of computer programs based on dynamic programming.
- This suggests that uniform and elegant proofs are hard to find for this kind of problem.

$d = 2$ and excess

h \ w	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	5	6	8	9	11	12	14
3	3	5	8	10	12	14	16	18	20
4	4	6	10	12	14	17	20	22	25
5	5	8	12	14	18	21	24	28	31
6	6	9	14	17	21	26	29	33	38
7	7	11	16	20	24	29	34	38	43
8	8	12	18	22	28	33	38	44	49
9	9	14	20	25	31	38	43	49	56

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3	3	5	8	10	12	14	16	18	20
4	4	6	10	12	14	17	20	22	25
5	5	8	12	14	18	21	24	28	31
6	6	9	14	17	21	26	29	33	38
7	7	11	16	20	24	29	34	38	43
8	8	12	18	22	28	33	38	44	49
9	9	14	20	25	31	38	43	49	56

Definition (Excess)

Let $W \in \mathcal{W}_{h\times w}^{\leq 2}$. The *excess* of W (noted $e(W)$) is defined by $e(W) = |W|_{■} - 2hw/3$. $e_{max}(h, w) = \max\{e(W) : W \in \mathcal{W}_{h\times w}^{\leq 2}\}$.

Excess

h \ w	1	2	3	4	5	6	7	8	9
1	$1/3$	$2/3$	1	$4/3$	$5/3$	2	$7/3$	$8/3$	3
2	$2/3$	$4/3$	1	$2/3$	$4/3$	1	$5/3$	$4/3$	2
3	1	1	2	2	2	2	2	2	2
4	$4/3$	$2/3$	2	$4/3$	$2/3$	1	$4/3$	$2/3$	1
5	$5/3$	$4/3$	2	$2/3$	$4/3$	1	$2/3$	$4/3$	1
6	2	1	2	1	1	2	1	1	2
7	$7/3$	$5/3$	2	$4/3$	$2/3$	1	$4/3$	$2/3$	1
8	$8/3$	$4/3$	2	$2/3$	$4/3$	1	$2/3$	$4/3$	1
9	3	2	2	1	1	2	1	1	2

Excess

h \ w	1	2	3	4	5	6	7	8	9
1	1/3	2/3	1	4/3	5/3	2	7/3	8/3	3
2	2/3	4/3	1	2/3	4/3	1	5/3	4/3	2
3	1	1	2	2	2	2	2	2	2
4	4/3	2/3	2	4/3	2/3	1	4/3	2/3	1
5	5/3	4/3	2	2/3	4/3	1	2/3	4/3	1
6	2	1	2	1	1	2	1	1	2
7	7/3	5/3	2	4/3	2/3	1	4/3	2/3	1
8	8/3	4/3	2	2/3	4/3	1	2/3	4/3	1
9	3	2	2	1	1	2	1	1	2

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6	2	1	2	1	1	2	1	1	2
7	7/3	5/3	2	4/3	2/3	1	4/3	2/3	1
8	8/3	4/3	2	2/3	4/3	1	2/3	4/3	1
9	3	2	2	1	1	2	1	1	2

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2	2/3	4/3	1	2/3	4/3	1	5/3	4/3	2
3	1	1	2	2	2	2	2	2	2
4	4/3	2/3	2	4/3	2/3	1	4/3	2/3	1
5	5/3	4/3	2	2/3	4/3	1	2/3	4/3	1
6	2	1	2	1	1	2	1	1	2
7	7/3	5/3	2	4/3	2/3	1	4/3	2/3	1
8	8/3	4/3	2	2/3	4/3	1	2/3	4/3	1
9	3	2	2	1	1	2	1	1	2

Periodicity of 6 cases up to symmetry.

Excess

h \ w	1	2	3	4	5	6	7	8	9
1	1/3	2/3	1	4/3	5/3	2	7/3	8/3	3
2	2/3	4/3	1	2/3	4/3	1	5/3	4/3	2
3	1	1	2	2	2	2	2	2	2
4	4/3	2/3	2	4/3	2/3	1	4/3	2/3	1
5	5/3	4/3	2	2/3	4/3	1	2/3	4/3	1
6	2	1	2	1	1	2	1	1	2
7	7/3	5/3	2	4/3	2/3	1	4/3	2/3	1
8	8/3	4/3	2	2/3	4/3	1	2/3	4/3	1
9	3	2	2	1	1	2	1	1	2

Periodicity of 6 cases up to symmetry.

Note: excess is additive with respect to concatenation.

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h \ w	1	2	3	4	5	6	7	8	9
1	1/3	2/3	1	4/3	5/3	2	7/3	8/3	3
2	2/3	4/3	1	2/3	4/3	1	5/3	4/3	2
3	1	1	2	2	2	2	2	2	2
4	4/3	2/3	2	4/3	2/3	1	4/3	2/3	1
5	5/3	4/3	2	2/3	4/3	1	2/3	4/3	1
6	2	1	2	1	1	2	1	1	2
7	7/3	5/3	2	4/3	2/3	1	4/3	2/3	1
8	8/3	4/3	2	2/3	4/3	1	2/3	4/3	1
9	3	2	2	1	1	2	1	1	2

Periodicity of 6 cases up to symmetry.

Note: excess is additive with respect to concatenation.

Notation: $a \equiv_3 b \iff a \equiv b \mod 3$

Base cases for $d = 2$

The base cases that are taken care of individually:

$$(h, w) \in (\mathbb{Z}_{>0} \times \{1, 2, 3, 4, 5, 6\}) \cup (7, 7)$$

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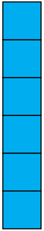
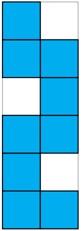


(a) A 6-pillar.



(b) 2-full 6×2 word.

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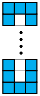
- There exists a word of excess 2 for all values of h: 

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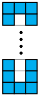
- There exists a word of excess 2 for all values of h: 
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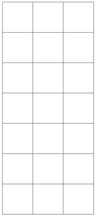

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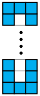
- There exists a word of excess 2 for all values of h: 
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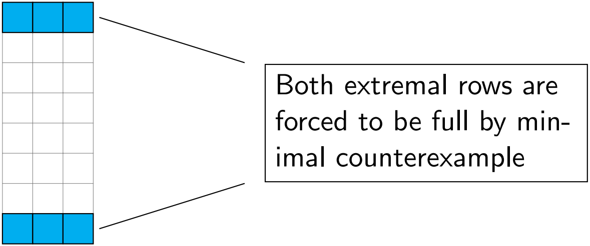

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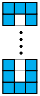
- There exists a word of excess 2 for all values of h: 
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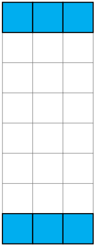

We claim that each internal row is of excess 0.

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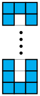
- There exists a word of excess 2 for all values of h: 
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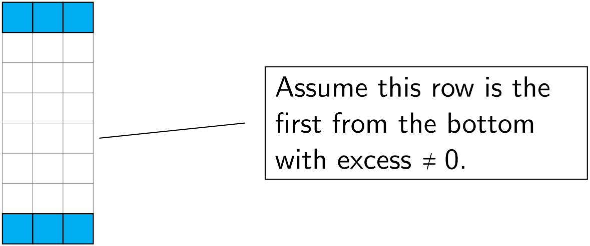

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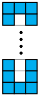
- There exists a word of excess 2 for all values of h: 
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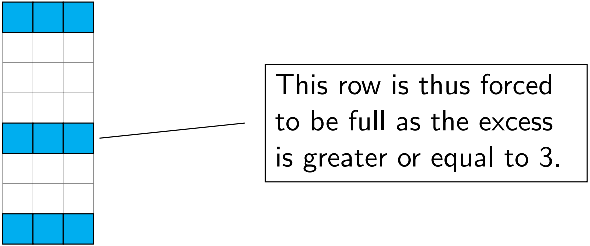

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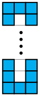
- There exists a word of excess 2 for all values of h: 
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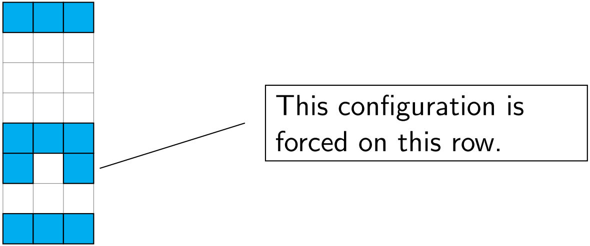

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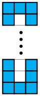
- There exists a word of excess 2 for all values of h: 
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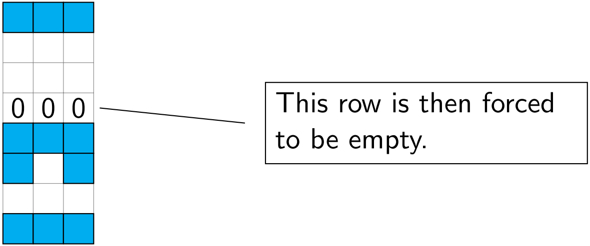

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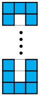
- There exists a word of excess 2 for all values of h: 
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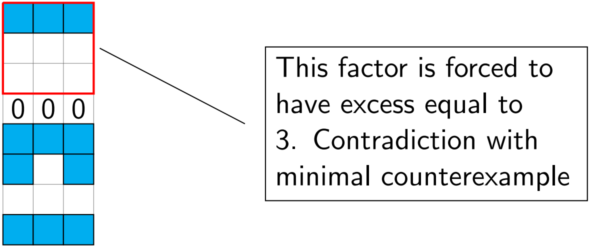
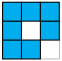
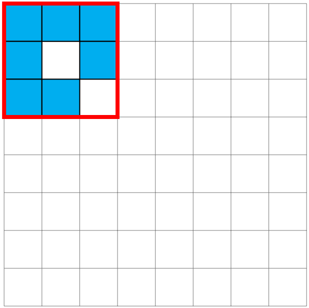


Tiling for the general case

Start in the upper left corner with the tile .



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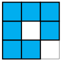
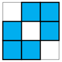
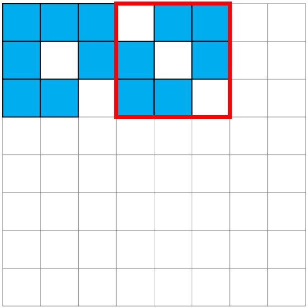
Start in the upper left corner with the tile .

Proceed to tile the rest of the rectangle with the tile , while correcting for the congruence of h and w modulo 3.



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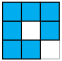
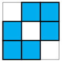
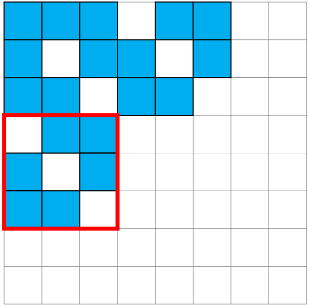
Start in the upper left corner with the tile .

Proceed to tile the rest of the rectangle with the tile , while correcting for the congruence of h and w modulo 3.



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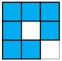
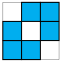
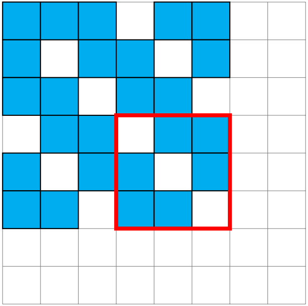
Start in the upper left corner with the tile .

Proceed to tile the rest of the rectangle with the tile , while correcting for the congruence of h and w modulo 3.



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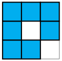
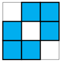
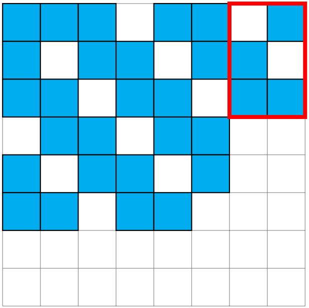
Start in the upper left corner with the tile .

Proceed to tile the rest of the rectangle with the tile , while correcting for the congruence of h and w modulo 3.



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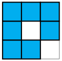
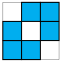
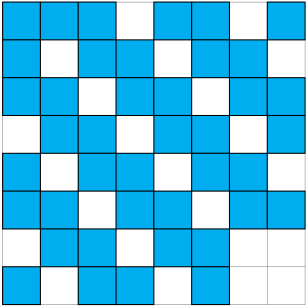
Start in the upper left corner with the tile .

Proceed to tile the rest of the rectangle with the tile , while correcting for the congruence of h and w modulo 3.



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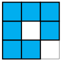
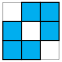
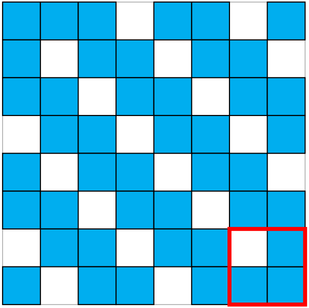
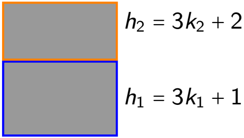
Start in the upper left corner with the tile .

Proceed to tile the rest of the rectangle with the tile , while correcting for the congruence of h and w modulo 3.



The general case: strategic cuts

6 cases:



$(h, w) \equiv_3 (0, 0)$
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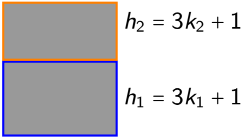

$(h, w) \equiv_3 (2, 2)$
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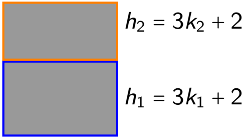

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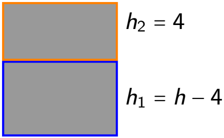

$(h, w) \equiv_3 (0, 1)$
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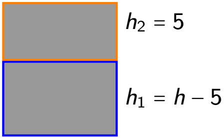

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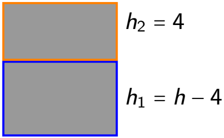

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The general case example

We use minimal counterexample to prove maximal excess of each cases.

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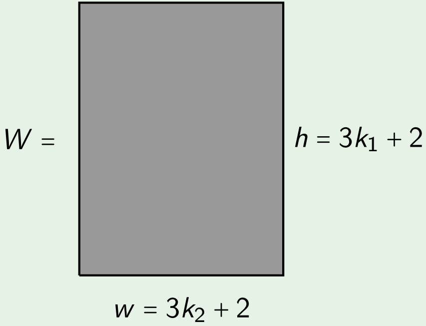

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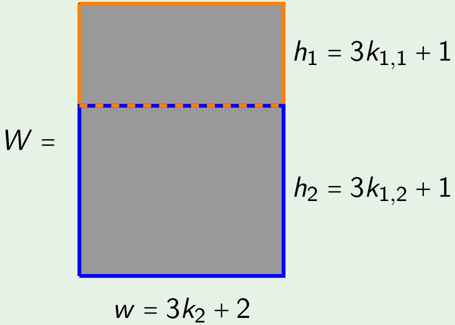

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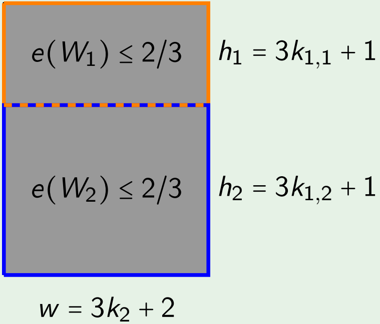
$W =$ 

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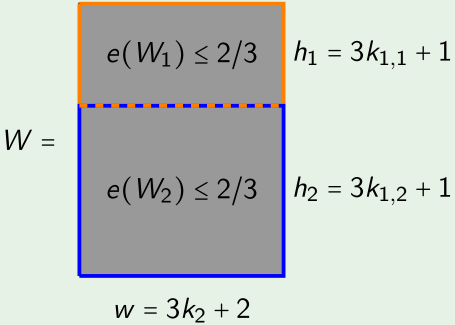

Contradiction!

Theorem (Blondin Massé, Goupil, L'Heureux, M. 2025+)

Let $(h, w) \in \mathbb{Z}^2_{>0}$, *then*

$$
\begin{aligned}
\max_{\le 0}(h,w) &= \lceil hw/2 \rceil, \\
\max_{\le 1}(h,w) &= \begin{cases} hw/2, & \textit{if } h, w \equiv_2 0; \\ (h-1)w/2 + \lceil 2w/3 \rceil, & \textit{if } h \equiv_2 1 \textit{ and } w \equiv_2 0; \\ h(w-1)/2 + \lceil 2h/3 \rceil, & \textit{otherwise,} \end{cases} \\
\max_{\le 2}(h,w) &= \begin{cases} hw, & \textit{if } w = 1 \textit{ or } h = w = 2; \\ 3hw/4 + 1/2, & \textit{if } h \equiv_2 1,\ h \ge 3 \textit{ and } w = 2; \\ 3hw/4, & \textit{if } h \equiv_2 0,\ h \ge 4 \textit{ and } w = 2; \\ 2hw/3 + 2, & \textit{if } w = 3 \textit{ or } h \equiv_3 w \equiv_3 0; \\ 2hw/3 + 4/3, & \textit{if } w \ge 4 \textit{ and } h \equiv_3 w \not\equiv_3 0; \\ 2hw/3 + 1, & \textit{if } w \ge 4,\ w \equiv_3 0 \textit{ and } h \not\equiv_3 w; \\ 2hw/3 + 2/3, & \textit{otherwise,} \end{cases} \\
\max_{\le 3}(h,w) &= \begin{cases} hw, & \textit{if } 1 \le w \le 2; \\ hw - \gamma(G_{h-2,w-2}), & \textit{otherwise,} \end{cases} \\
\max_{\le 4}(h,w) &= hw.
\end{aligned}
$$

Conclusion

- The result in case $d = 2$ can be interpreted as giving the maximal cardinality of an induced subgraph of the grid graph such that every connected component is either a cycle or a path.
- It also provides an upper bound for the snake polyomino of maximal area bounded by a rectangle of any dimensions.
- A future problem to consider is the enumeration of the maximal words of bounded degree
- Another problem would be to find a statistic analogous to excess for other lattices. 3D words?

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Thank You!!!!