Number Walls of Automatic Sequences

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Contents

Part 1: What is a Number Wall?

Brief History of Number Walls

- First appeared in the literature in John Conway and Richard Guy's 1996 **Book of Numbers**.

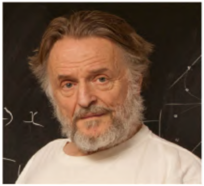


Figure: **Left:** John Conway. **Right:** Richard K. Guy.

- The building blocks of number walls (Toeplitz matrices) have been studied since $\sim$ 1900s.
- Have recently been applied to
  - Diophantine approximation,
  - Discrepancy theory,
  - Dynamics.
- The connection between number walls and automatic sequences has been crucial to each of the above applications.

Definition: Toeplitz Matrices

- **Definition:** An $n \times m$ matrix $(s_{i,j})_{0 \le i \le n, 0 \le j \le m}$ is **Toeplitz** if $s_{i,j} = s_{i+1,j+1}$ for all $0 \le i \le n-1, 0 \le j \le m-1$.
- **Examples:** $\begin{pmatrix} d & e & f & g \\ c & d & e & f \\ b & c & d & e \\ a & b & c & d \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 & 7 & 11 & -5 \\ 3 & 1 & 7 & 11 \\ 0 & 3 & 1 & 7 \end{pmatrix}$.
- Let $\mathcal{S} = (s_i)_{i \in \mathbb{N}}$ be an infinite sequence, and let $m, n \in \mathbb{N}$ with $n \ge m$.
- Define the Toeplitz matrix $T_{\mathcal{S}}(n,m) := (s_{j-i+n})_{0 \le i,j \le m}$, viz.

$$T_{\mathcal{S}}(n,m) := \begin{pmatrix} s_n & s_{n+1} & s_{n+2} & \cdots & s_{n+m-1} & s_{n+m} \\ s_{n-1} & s_n & s_{n+1} & \cdots & \cdots & s_{n+m-1} \\ s_{n-2} & s_{n-1} & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{n-m+1} & \vdots & & \ddots & \ddots & s_{n+1} \\ s_{n-m} & s_{n-m+1} & \cdots & \cdots & s_{n-1} & s_n \end{pmatrix}.$$

Number Walls

- **Examples:**

$$T_{\mathcal{S}}(n,0) = (s_n) \qquad T_{\mathcal{S}}(n,3) = \begin{pmatrix} s_n & s_{n+1} & s_{n+2} & s_{n+3} \\ s_{n-1} & s_n & s_{n+1} & s_{n+2} \\ s_{n-2} & s_{n-1} & s_n & s_{n+1} \\ s_{n-3} & s_{n-2} & s_{n-1} & s_n \end{pmatrix}$$

- Let $\mathcal{S} = (s_i)_{i\in\mathbb{N}}$ be a infinite sequence over an integral domain $\mathcal{R}$.
- The **number wall** of the sequence $\mathcal{S}$ over $\mathcal{R}$ is a two dimensional sequence $W_{\mathcal{R}}(\mathcal{S}) = (W_{\mathcal{R}}(\mathcal{S})[n,m])_{n,m\in\mathbb{Z},\ n\geq m}$ with

$$W_{\mathcal{R}}(\mathcal{S})[n,m] = \begin{cases} \det(T_{\mathcal{S}}(n,m)) & \text{if } m \geq 0 \text{ and } n \geq m \\ 1 & \text{if } m = -1 \\ 0 & \text{if } m < -1. \end{cases}$$

- Above, n is the column and m is the row.
- $W_{\mathcal{R}}(\mathbf{S})[m,n]$ is only defined when $n \geq m$. That is, number walls are triangular.
- When $\mathcal{R} = \mathbb{F}_q$ for some prime power q, abbreviate $W_{\mathcal{R}}(\mathcal{S})$ to $W_q(\mathcal{S})$.

Number Wall Example

- Example: $\mathcal{S} = (s_i)_{0\leq i\leq 4} = \{0, 1, 4, 3, 4\} \in \mathbb{F}_5^5$.
- Row zero: $|T_{\mathcal{S}}(i, 0)| = s_i$: $0, 1, 4, 3, 4$.
- Row one:
  - $|T_{\mathcal{S}}(1,1)| = \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1$,
  - $|T_{\mathcal{S}}(2,1)| = \begin{vmatrix} 4 & 3 \\ 1 & 4 \end{vmatrix} = 3$,
  - $|T_{\mathcal{S}}(3,1)| = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 3$.
- Row two: $|T_{\mathcal{S}}(2,2)| = \begin{vmatrix} 4 & 3 & 4 \\ 1 & 4 & 3 \\ 0 & 1 & 4 \end{vmatrix} = 4$
- So, $W_5(\mathcal{S})$ is equal to

Row 0: 0 1 4 3 4
Row 1: 1 3 3
Row 2: 4

A Bigger Example

- **Example:** $\mathcal{S}$ a sequence of length 100 over $\mathbb{F}_5$.

```
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
1111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
4224104440133112342221144102243203220313021101424221021220012323012410312043444120210040103440101230
 114111011111341101024222134101444344424444141432121342240011004210412404213402341410011124411114114
  102341420010312134221030343121034221313020002341404013342130020134322433004311430141404413342221
   241314400141030023331213302343104304311100012403424333120414141131443320041442431431411420102
    23243342102220014011102134322243232012300030431021202022204032123123131240232024201030233234
     040333112423333222033300122013400111444444443222332312112431122204444004210243211134442401
      3230444011431321444020044243020030014241011233434314422204122032300200443120342230334123
       43120134110443414243421444431414001121414124111403241023232222210014142022130134213244
        001240324213430124413230030222243114202030340441431342000402012323221124103330000433
         0341112220041240241101001340040432244414311413110302200031243101022120232304000021
          44303102004111111222222223003121344014121331234443120001200033312124310431200001
           141434213420000120303001111132002444021014021433102431240004214410043230310000
            0013200204000012223200300312200123134342222123423200103000201211002131243421
             04410033300001101431240040423234333130101124202320011124314121111104000332
              143232410000132310402222431141140233332320134202431132442034404431200032
               2110432134212024211114410440432433013004440022220332121414214140310003
                33234144412333201301114211111341342300311002001343212202302204143333
                 314102034140414142103431000102303234212431200304232144412412411231
                  2043111302320003212430100011142102344000321341440141311024012124
                   31343114131000132101410001413124342100043401233230122141134320
                    302034421300021131231414104200140040002412412204314232121234
                     3134420041414343313401314130042001111132044431214210423442
                      13321002204432411014124012431141424143414020121403232344
                       224342143112334134130141041212243403141431114344400021
                        2034044321243342314144134132310443130020442431320003
                         13421121200430434012342241013421444001413300331000
                          003111140023231141440244222030240232341420042342
                           0230023421023132012324212431244313112033232412
                            34004414312422134411301001004121143141441321
                             232340402041402414411100200421443020041314
                              2244444414032310224314144440424134004133
                               31040113431014314004021414212133312442
                                444442033222332300134200133401042423
                                 4023431113332101414104004323122231 3
                                  31411113042222230243421203420213
                                   203003312303024142224332302320
                                    1400423423222332404024231202
                                     22224104113423332333414422
                                      124224344004102021120433
                                       0141324100321333331243
                                        40234141411131001000
                                         342400124031200200
                                          0140010134044440
                                           34141241243103
                                            223334104043
                                             3201213431
                                              31220233
                                               022132
                                                2141
                                                 40
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Number Wall Example... In Colour!

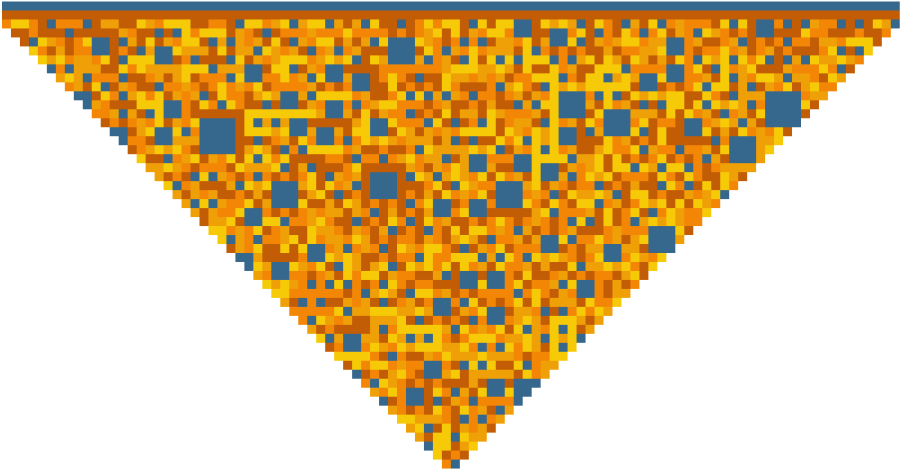


Figure: Number wall of a randomly generated sequence over $\mathbb{F}_5$. The zero entries are in blue.

Number Wall Example... In Colour!

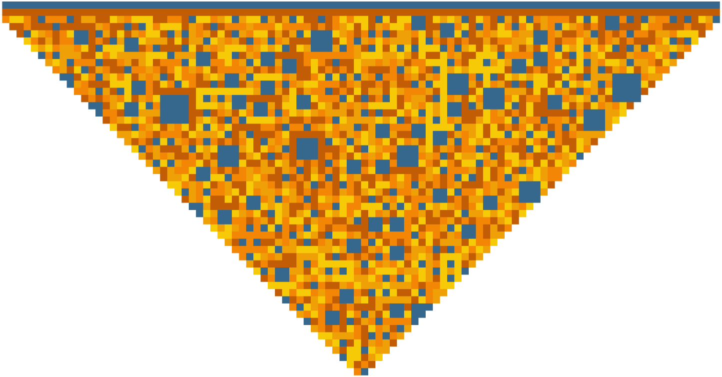


Figure: Number wall of a randomly generated sequence over $\mathbb{F}_5$. The zero entries are in blue.

Square Window Theorem, [Lunnon, 2001]: Zero entries in a Number Wall can only occur within **windows**; that is, within square regions with horizontal and vertical edges.

Inner and Outer Frames

Square Window Theorem, [Lunnon, 2001]: Zero entries in a Number Wall can only occur within **windows**; that is, within square regions with horizontal and vertical edges.

- The entries of a number wall surrounding a window are called the **inner frame**. The entries surrounding the inner frame are known as the **outer frame**.

	E_0	E_1	$\cdots$	E_l	E_{l+1}	
F_0	$B_0 = A_0$	A_1	$\cdots$	A_l	$A_{l+1} = C_{l+1}$	G_{l+1}
F_1	B_1	0	$\cdots$	0	C_l	G_l
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
F_l	B_l	0	$\cdots$	0	C_1	G_1
F_{l+1}	$B_{l+1} = D_{l+1}$	D_l	$\cdots$	D_1	$D_0 = C_0$	G_0
	H_{l+1}	H_l	$\cdots$	H_1	H_0	

The Structure of the Inner Frame

	E_0	E_1	$\cdots$	E_l	E_{l+1}	
F_0	$B_0 = A_0$	A_1	$\cdots$	A_l	$A_{l+1} = C_{l+1}$	G_{l+1}
F_1	B_1	0	$\cdots$	0	C_l	G_l
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
F_l	B_l	0	$\cdots$	0	C_1	G_1
F_{l+1}	$B_{l+1} = D_{l+1}$	D_l	$\cdots$	D_1	$D_0 = C_0$	G_0
	H_{l+1}	H_l	$\cdots$	H_1	H_0	

- **Theorem**, [Lunnon, 2001]: The inner frame of a window of size $\ell \geq 1$ is comprised of 4 geometric sequences.
- These are along the top, left, right and bottom edges and they have ratios r_A, r_B, r_C and r_D respectively with origins at the top left and bottom right. Furthermore, these rations satisfy the relation

$$\frac{r_A r_D}{r_B r_C} = (-1)^\ell.$$

Forgetting Determinants

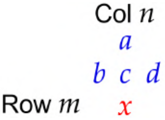
- **Theorem**, [Lunnon, 2001]: Given a doubly infinite sequence $\mathcal{S} = (s_i)_{i \in \mathbb{Z}}$ over an integral domain $\mathcal{R}$, the number wall $W_{\mathcal{R}}(\mathcal{S}) := (W_{n,m})_{n,m \in \mathbb{Z}}$ can be generated by the following recurrence relations in the rows.

$$W_{m,n} = \begin{cases} 0 & \text{if } m < -1 \text{ or } (m,n) \text{ is in a window} \\ 1 & \text{if } m = -1 \\ s_n & \text{if } m = 0 \\ \dfrac{W_{m-1,n}^2 - W_{m-1,n-1} W_{m-1n+1}}{W_{m-2,n}} & \text{if } m > 0 \text{ and } W_{m-2,n} \neq 0 \\ D_k = \dfrac{(-1)^{\ell k} B_k C_k}{A_k} & \text{if } m > 0 \text{ and } W_{m-2,n} = 0 = W_{m-1,n} \\ H_k = \dfrac{\frac{r_B E_k}{A_k} + (-1)^k \frac{r_A F_k}{B_k} - (-1)^k \frac{r_D G_k}{C_k}}{r_C / D} & \text{if } m > 0 \text{ and } W_{n,m-2} = 0 \neq W_{n,m-1}. \end{cases}$$

Entries in Number Walls are Calculated Locally

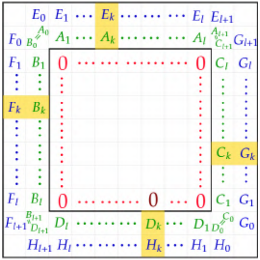
Given a sequence $\mathcal{S}$ over an integral domain $\mathcal{R}$, there are two cases when calculating $W_{\mathcal{R}}(\mathcal{S})[m, n]$:

1. $W_{\mathcal{R}}(\mathcal{S})[m-2, n] \neq 0$:



2. $W_{\mathcal{R}}(\mathcal{S})[m-2, n] = 0$:

   Below, $H_k = W_{\mathcal{R}}(\mathcal{S})[m, n]$ and $W_{\mathcal{R}}(\mathcal{S})[m-2, n]$ is in dark red .



Part 2: Automatic Sequences in One and Two Dimensions

Automatic Sequences

- An alphabet is a non-empty finite set.
- **Example:** $\Sigma = \{A, B, C\}$.
- For $k \in \mathbb{N}$, a uniform k-morphism is a function $\phi : \Sigma \to \Sigma^k$.
- **Example:** A 3-morphism $\phi : \Sigma \to \Sigma^3$ could be given by

$$\phi(A) = ABC \qquad \phi(B) = BCA \qquad \phi(C) = ACB$$

- A uniform k-morphism can be applied to a letter multiple times to create a sequence
- **Example:**

$$\phi : A \mapsto ABC$$

$$\phi : ABC \mapsto ABCBCAACB.$$

Codings and Sequences

- Let Σ and Δ be alphabets. For $d \in \mathbb{N}$, a d-coding is a function $\tau : \Sigma \to \Delta^d$.
- **Example:** Let $\Delta = \{0, 1\}$. Then a 1-coding is a function $\tau : \Sigma \to \Delta$:

$$\tau(A) = 1 \qquad \tau(B) = 0 \qquad \tau(C) = 1$$

and

$$\tau(ABC\ldots) = 010\ldots$$

- A sequence $\mathbf{S}$ is called k-automatic over an alphabet $\Sigma = \{A, \ldots\}$ if $\mathbf{S} = \tau(\phi^{\infty}(A))$.

Two Dimensional Automatic Sequences

- An alphabet is a non-empty finite set.
- **Example:** $\Sigma = \{A, B, C\}$.
- For $k, l \in \mathbb{N}$, a uniform $[k, l]$-morphism is a function $\phi : \Sigma \to \Sigma^{k \times l}$.
- **Example:** A $[2, 3]$-morphism $\phi : \Sigma \to \Sigma^{2\times 3}$ could be given by

$$\phi(A) = \begin{matrix} A & B & C \\ C & B & A \end{matrix} \quad \phi(B) = \begin{matrix} B & C & A \\ B & C & A \end{matrix} \quad \phi(C) = \begin{matrix} C & B & A \\ C & C & C \end{matrix}$$

- A uniform $[k, l]$-morphism can be applied multiple times to create a two-dimensional sequence
- **Example:**

$$\phi : A \mapsto \begin{matrix} A & B & C \\ C & B & A \end{matrix}$$

$$\phi : \begin{matrix} A & B & C \\ C & B & A \end{matrix} \mapsto \begin{matrix} A & B & C & B & C & A & C & B & A \\ C & B & A & B & C & A & C & C & C \\ C & B & A & B & C & A & A & B & C \\ C & C & C & B & C & A & C & B & A \end{matrix}$$

Two Dimensional Codings and Sequences

- Let Σ and Δ be alphabets. For $d, g \in \mathbb{N}$, a $[d, g]$-coding is a function $\tau : \Sigma \to \Delta^{d \times g}$.
- **Example:** Let $\Delta = \{0, 1\}$. Then a $[1, 2]$-coding is a function $\tau : \Sigma \to \Delta^{1 \times 2}$:

$$\tau(A) = 1\ 1 \qquad \tau(B) = 1\ 0 \qquad \tau(C) = 0\ 0$$

and

$$\tau \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{matrix}$$

- A two-dimensional sequence $\mathbf{S}$ is called $[k, \ell]$-automatic over an alphabet $\Sigma = \{A, \dots\}$ if $\mathbf{S} = \tau(\phi^{\infty}(A))$.

The Big Idea

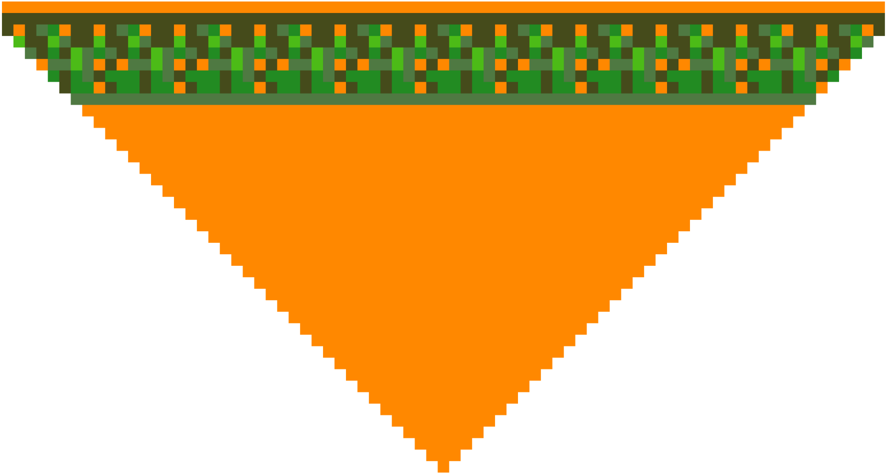
- An automatic sequence has 3 key components:
  - The alphabet, $\Sigma = \{\sigma_1, \ldots, \sigma_{|\Sigma|}\}$,
  - The morphism, $\phi : \Sigma \to \Sigma^k$ for some $k \in \mathbb{N}$,
  - The coding, $\tau : \Sigma \to \Delta^d$ for some $d \in \mathbb{N}$.

- **Big Question**: What can we say about number walls generated by automatic sequences?

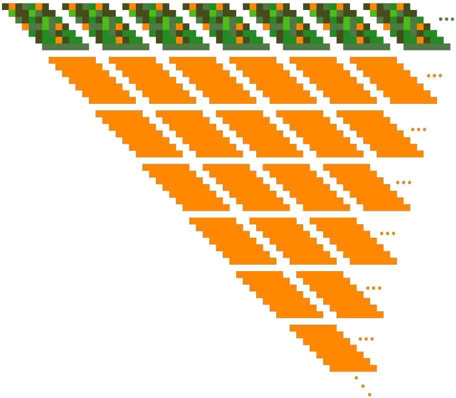
Part 3: Number Walls of Automatic Sequences

Simplest Case: Number Walls of Periodic Sequences

$\mathcal{S} = (s_n)_{n\in\mathbb{Z}} = (\overline{1, 0, 1, 2, 3, 0, 1})$ be a periodic sequence with period $p = 7$ over $\mathbb{F}_5$. Zeros are drawn in orange.



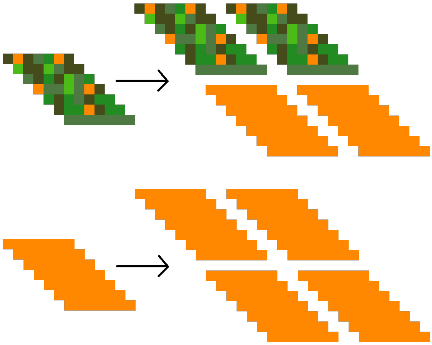
Simplest Case: Number Walls of Periodic Sequences



Then...

- ...for $0 \leq m < p$, $(W_{\mathbb{K}}(\mathbf{S})[m,n])_{n \geq m}$ is periodic with period p,
- ...$W_{\mathbb{K}}(\mathbf{S})[m,n] = 0$ for all $n \in \mathbb{Z}$ and $m \geq p$.

Simplest Case: Number Walls of Periodic Sequences



And hence $W_{\mathbb{K}}(\mathbf{S})$ is defined...

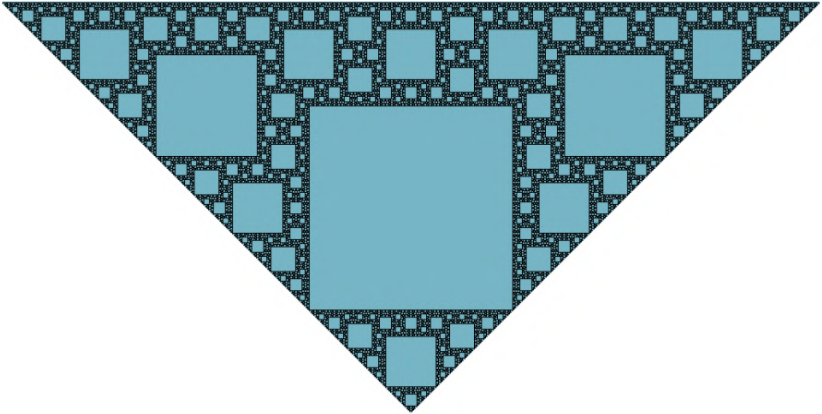
- ...by a $[2, 2]$-morphism,
- ...under a $[p, p]$-coding.

Next Simplest Case: the Thue-Morse Sequence

The **Thue-Morse** sequence is given by $\phi^{\infty}(0)$, where

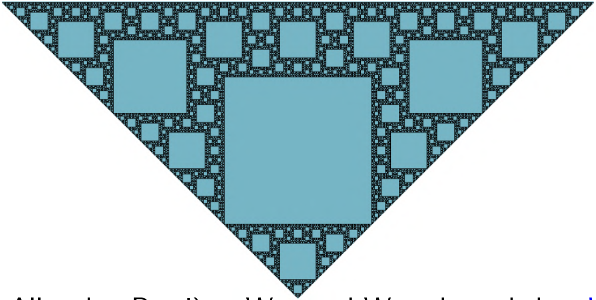
$$\phi : 0 \mapsto 01 \quad \text{and} \quad \phi : 1 \mapsto 10.$$

The number wall of the Thue-Morse sequence $\phi^{\infty}(0)$ over $\mathbb{F}_2$:



Number Wall of the Thue Morse Sequence over $\mathbb{F}_2$

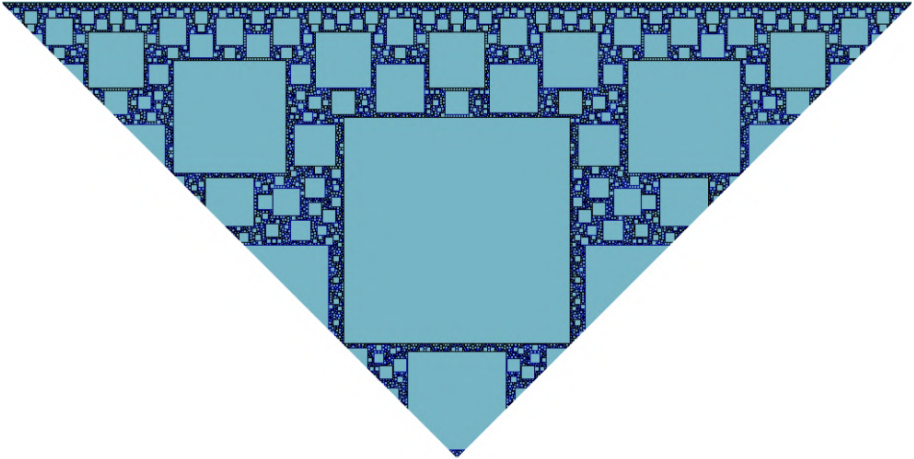
Let $\mathbf{T}$ be the Thue-Morse sequence.



- In 1998, Allouche, Peyrière, Wen and Wen showed that $W_2(\mathbf{T})$ is $[2,2]$-automatic.
- (Nesharim, Shapira and Soffer-Aranov, 2025): By studying the symmetries of the so-called **diagonally aligned number wall**, $W_2(\mathbf{T})$ is...
  - ...$[2,2]$-automatic,
  - ...over an alphabet of size 15,
  - ...and under a $[5,5]$-coding.

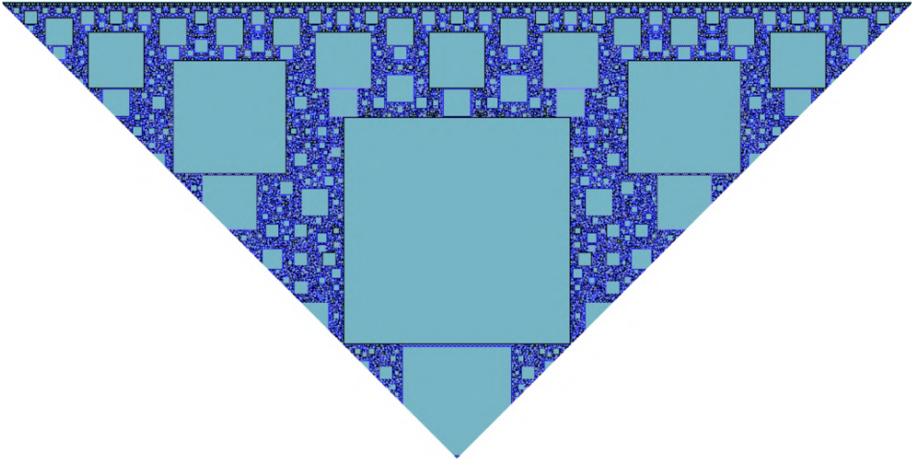
Number Wall of the Thue Morse Sequence over $\mathbb{F}_3$

The number wall of the first 1024 digits of the Thue-Morse sequence over $\mathbb{F}_3$:



Number Wall of the Thue Morse Sequence over $\mathbb{F}_5$

The number wall of the first 1024 digits of the Thue-Morse sequence over $\mathbb{F}_5$:



The Paperfolding Sequence

- Let $\Sigma = \{A, B, C, D\}$ be an alphabet, and let $\phi : \Sigma \to \Sigma$ be a 2-morphism defined as

$$\phi(A) = AB \qquad \phi(B) = CB \qquad \phi(C) = AD \qquad \phi(D) = CD.$$

- Let $\Delta = \{0, 1\}$ and define the coding $\pi : \Sigma \to \Delta$ as

$$\pi(A) = 0 \qquad \pi(B) = 0 \qquad \pi(C) = 1 \qquad \pi(D) = 1.$$

- Then, define the **Paperfolding Sequence** as

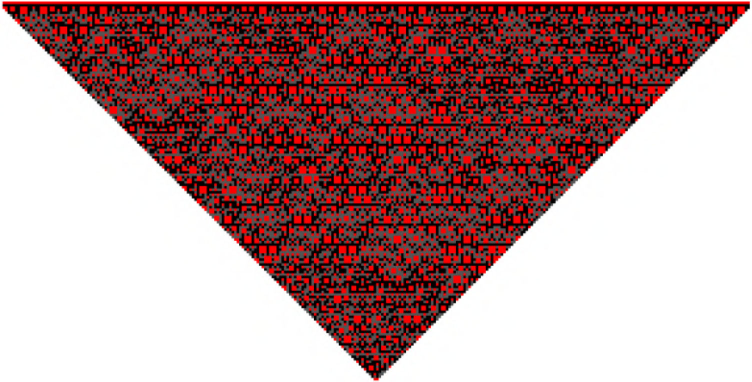
$$\mathbf{P} = (p_n)_{n \geq 0} = \pi(\phi^{\infty}(A)).$$

Number Wall of the Paperfolding Sequence

The number wall of $(p_n)_{0 \leq n < 300}$ over $\mathbb{F}_3$.

Adiceam, Nesharim, Lunnon, 2021: $W_3(\mathbf{P})$ is...

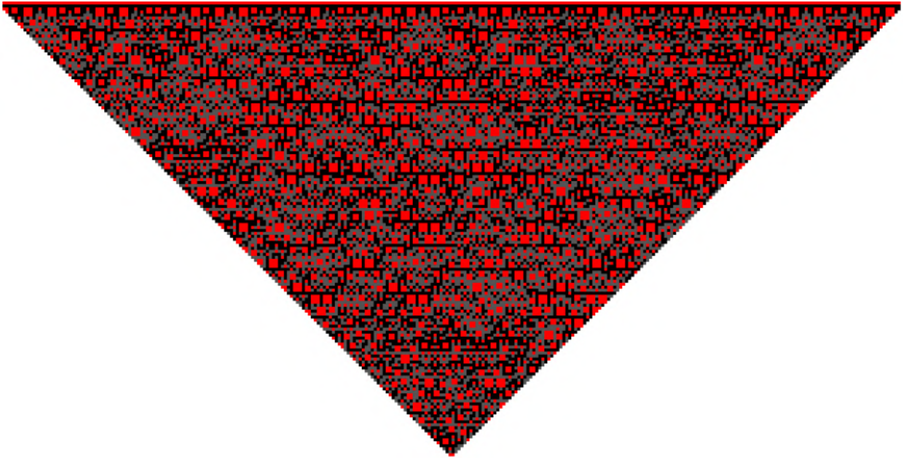
- ...$[2, 2]$-automatic,
- ... over an alphabet of size 2353,
- ... under a $[13, 13]$-coding.



Number Wall of the Paperfolding Sequence

(Garrett, R., 2024): $W_3(\mathbf{P})$ is

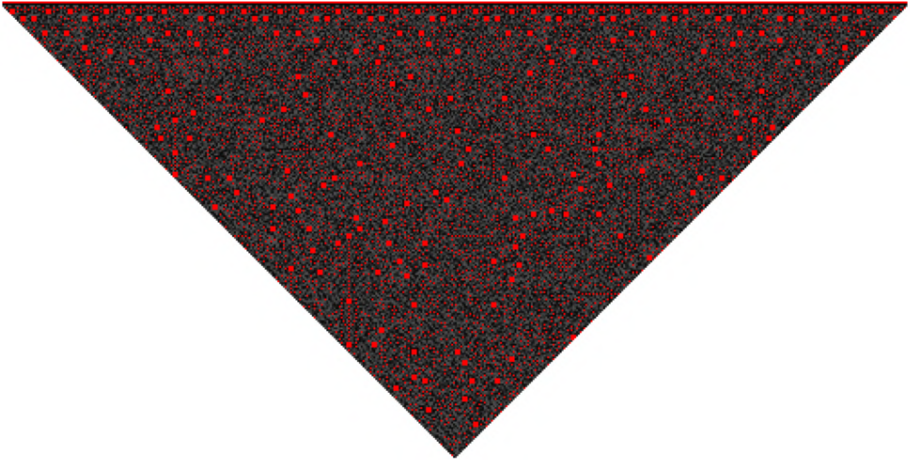
- ... $[2, 2]$-automatic
- ... over an alphabet of size 390,
- ... under a $[8, 16]$-coding.



Number Wall of the Paperfolding Sequence

(Garrett, R., 2024): $W_7(\mathbf{P})$ is

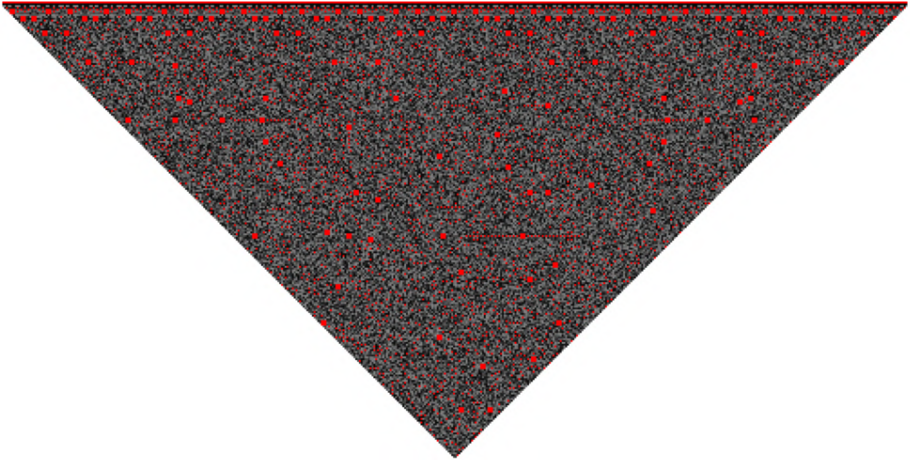
- ... $[2, 2]$-automatic
- ... over an alphabet of size 1778011,
- ... under a $[8, 16]$-coding.



Number Wall of the Paperfolding Sequence

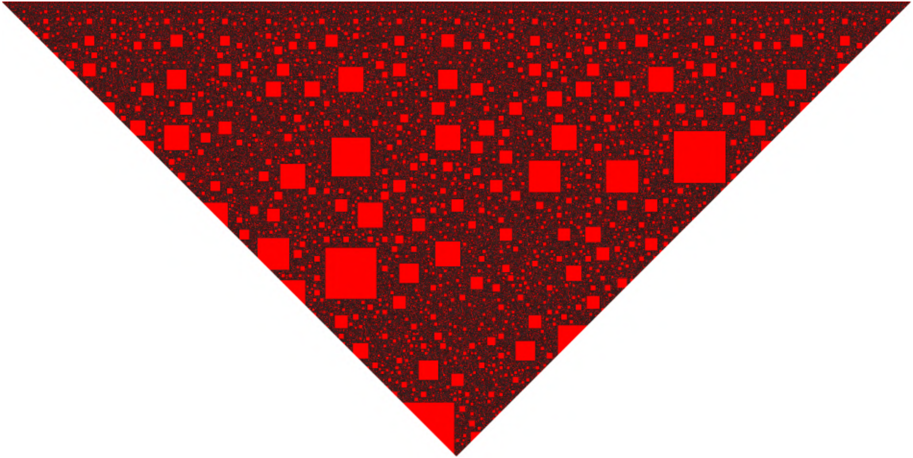
(Garrett, R., 2024): $W_{11}(\mathbf{P})$ is

- ... $[2, 2]$-automatic
- ... over an alphabet of size 70360006,
- ... under a $[8, 16]$-coding.



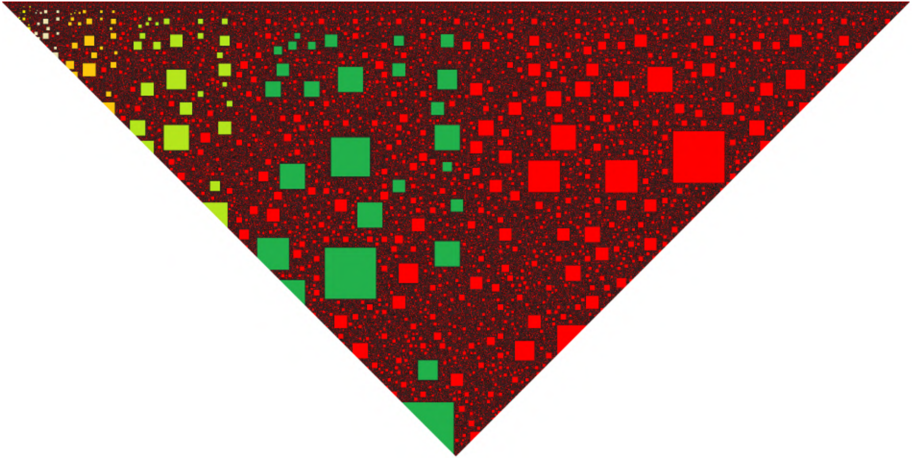
Number Wall of the Paperfolding Sequence

The number wall of the Paperfolding Sequence over $\mathbb{F}_5$:



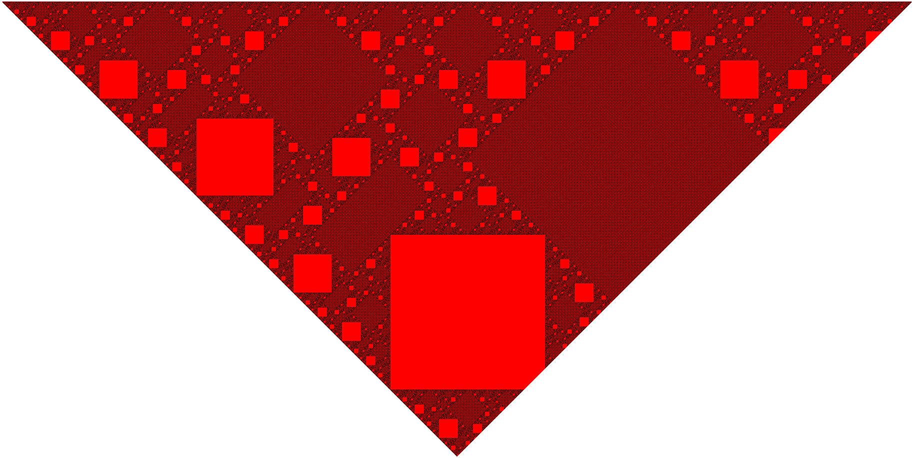
Number Wall of the Paperfolding Sequence

The number wall of the Paperfolding Sequence over $\mathbb{F}_5$:



Number Wall of the Paperfolding Sequence

The number wall of the Paperfolding Sequence over $\mathbb{F}_2$:



The p-Cantor Sequence

- Let p be an odd prime and let $p_2 := \frac{p-1}{2}$.
- Let $\Sigma = \{0, 1, \ldots, p-1\}$
- Let $\phi : \Sigma \to \Sigma^p$ be a p-morphism, and let $\phi_i(n)$ be the i^{th} letter of $\phi(n)$.
- The p-Cantor sequence is the p-automatic sequence defined as $\phi^\infty(1)$, where

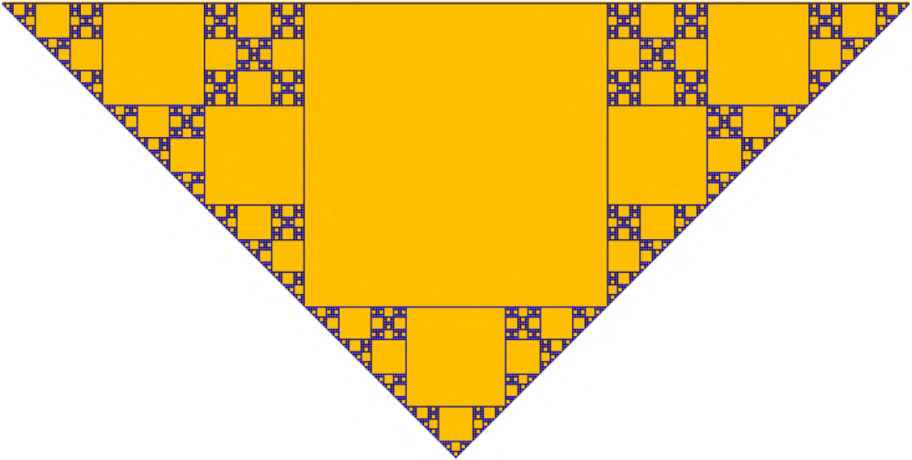
$$\phi_i(n) = n \cdot \binom{p_2}{i/2} \mod p$$

- For example, the 7-Cantor sequence is defined by the morphism

$$\phi(0) = 0000000 \qquad \phi(1) = 1030301 \qquad \phi(2) = 2060602$$
$$\phi(3) = 3020203 \qquad \phi(4) = 4050504 \qquad \phi(1) = 5010105$$
$$\phi(6) = 6040406.$$

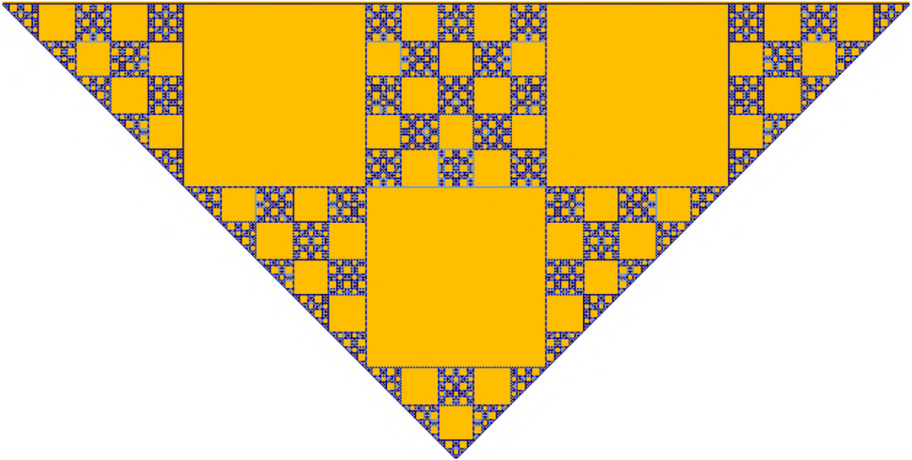
Number Wall of the *p*-Cantor Sequence

The number wall of the 3-Cantor Sequence over $\mathbb{F}_3$:



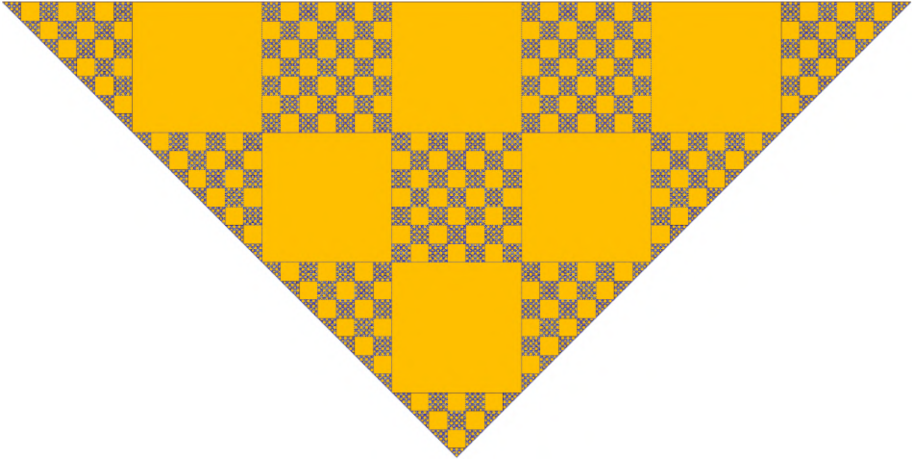
Number Wall of the p-Cantor Sequence

The number wall of the 5-Cantor Sequence over $\mathbb{F}_5$:



Number Wall of the *p*-Cantor Sequence

The number wall of the 7-Cantor Sequence over $\mathbb{F}_7$:



Number Wall of the p-Cantor Sequence

- (**Theorem**: Wen, Wu 2014:) The number wall $W_3(\mathbf{C}(3))$ is 3-automatic.
- The **profile** of a number wall $W_p(\mathbf{S})$, denoted $\chi(W_p(\mathbf{S}))$, is the two-dimensional sequence $(\chi(W_p(\mathbf{S}))[m,n])_{m,n\in\mathbb{Z}}$ such that

$$\chi(W_p(\mathbf{S}))[m,n] = \begin{cases} 0 & \text{if } W_p(\mathbf{S})[m,n] = 0, \\ 1 & \text{otherwise.} \end{cases}$$

- (**Theorem**: R. and Soffer Aranov, 2025) For any odd prime p ,

$$\chi(W_p(\mathbf{C}(p)) = \Pi(\Phi^\infty(A))$$

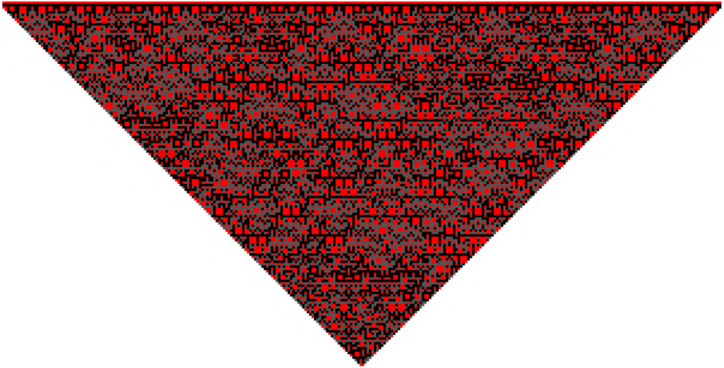
where

- $\Phi : \Sigma \to \Sigma^{p\times p}$ is a $[p,p]$-morphism,
- $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_{12}\}$ is an alphabet,
- $\Pi : \Sigma \to \{0,1\}$ is a $[1,1]$-coding.

Part 4: A Brief Discussion of Methodology

Recall: Number Wall of the Paperfolding Sequence

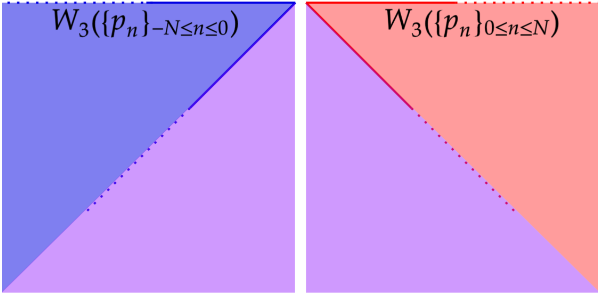
The number wall of the first 300 entries of the Paperfolding Sequence over $\mathbb{F}_3$.



Both Adiceam, Nesharim & Lunnon and Garrett & R. showed that $W_3(\mathbf{P})$ is $[2, 2]$-automatic.

Number Wall of a Doubly Infinite Sequence

- Adiceam, Nesharim and Lunnon, 2021: extend the **Paperfolding Sequence** $\mathbf{P} = \{p_n\}_{n \geq 1}$ into a doubly infinite sequence $\mathcal{P} = \{p_n\}_{n \in \mathbb{Z}}$ by defining $p_{-n} = 1 - p_n$.
- Then, $W_3(\mathcal{P})[m, n]$ is defined for all $m, n \in \mathbb{Z}$.



- Adiceam, Nesharim, Lunnon, 2021: $(W_3(\mathcal{P})[m, n])_{m,n \geq 0}$ is $[2, 2]$-automatic.

Embracing the Triangle

Garrett and R., 2024: can we find a morphism that embraces the triangular shape of a number wall?

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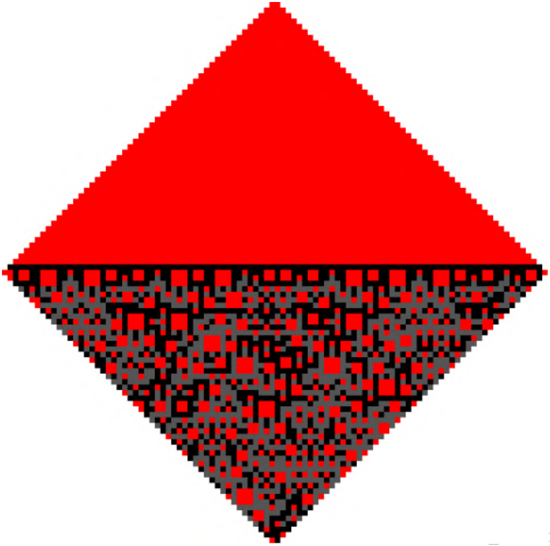

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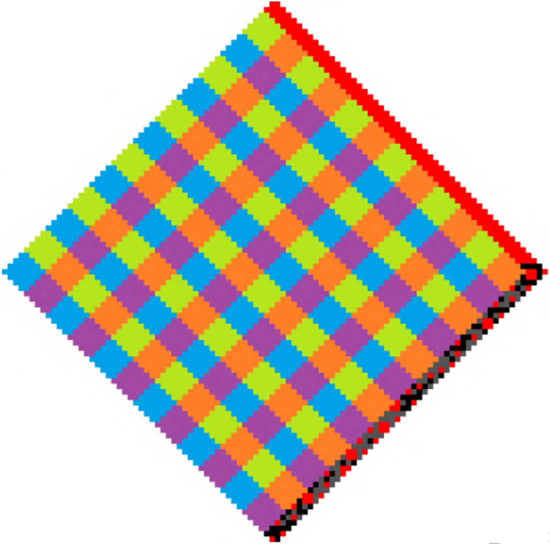

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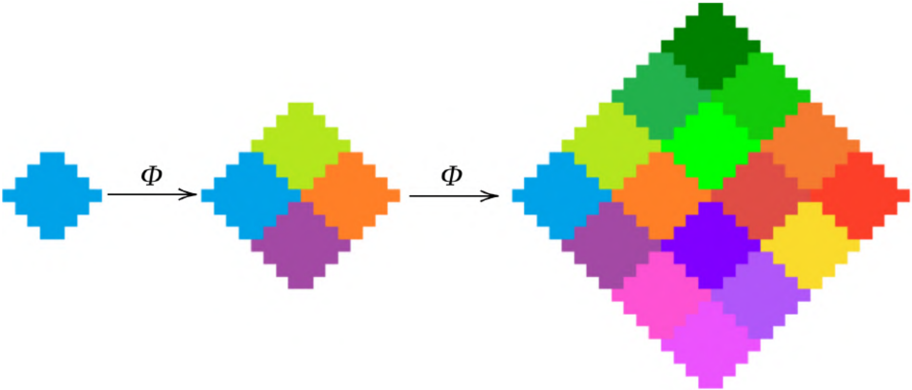
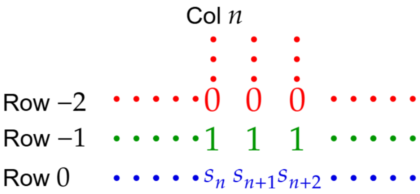


First Rows of a Number Wall

- Recall that for any $n \in \mathbb{N}$, and any sequence $\mathbf{S} = (s_i)_{i \in \mathbb{N}}$ over some integral domain $\mathcal{R}$,

$$W_{\mathcal{R}}(\mathbf{S})[-i, n] = 0 \text{ for } i > 1 \quad W_{\mathcal{R}}(\mathbf{S})[-1, n] = 1 \quad W_{\mathcal{R}}(\mathbf{S})[0, n] = s_n.$$

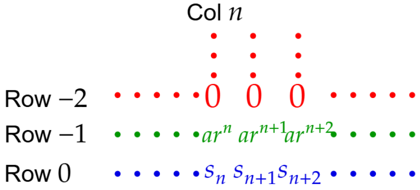


- Recall also that the m^{th} row of $W_{\mathcal{R}}(\mathbf{S})$ is calculated using some recurrence relations known as the Frame Constraints.

Generalising Number Walls

- Let $\mathbb{K}$ be a field and let $r, a \in \mathbb{K}\backslash\{0\}$.
- Define the (r, a)-number wall of $\mathbf{S}$ over $\mathbb{K}$ as the two-dimensional sequence $W_{\mathbb{K}}^{(r,a)}(\mathbf{S})$ where
  - $W_{\mathbb{K}}^{(r,a)}(\mathbf{S})[-i, 0] = 0$ for $i > 1$,
  - $W_{\mathbb{K}}^{(r,a)}(\mathbf{S})[-1, n] = ar^n$,
  - $W_{\mathbb{K}}^{(r,a)}(\mathbf{S})[0, n] = s_n$

  and with $W_{\mathbb{K}}^{(r,a)}(\mathbf{S})[m, n]$ calculated by the Frame Constraints.

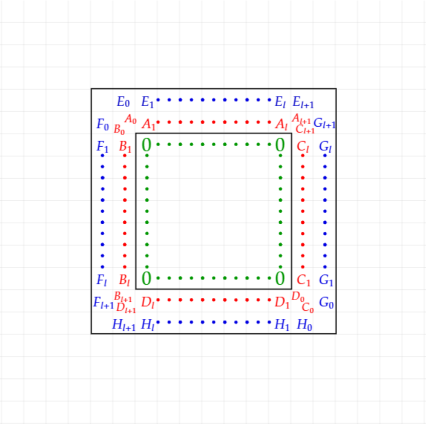


- (**Lemma**, R., Soffer-Aranov, 2025):

$$W_{\mathbb{K}}^{(r,a)}(\mathbf{S})[m, n] = \frac{1}{r^{nm}a^m} W_{\mathbb{K}}(\mathbf{S})[m, n].$$

Number Walls contain Number Walls

- Recall the inner and outer frame of a window.



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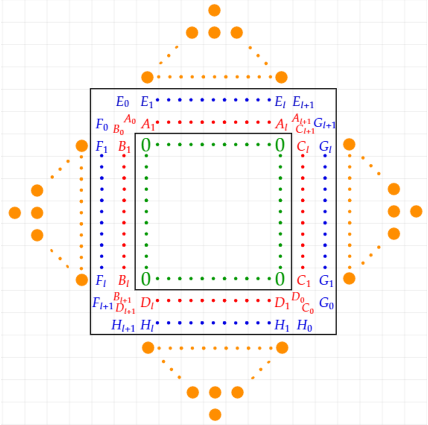

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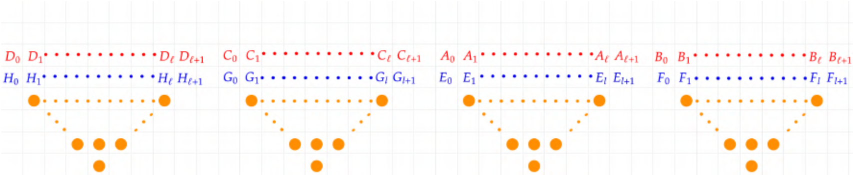

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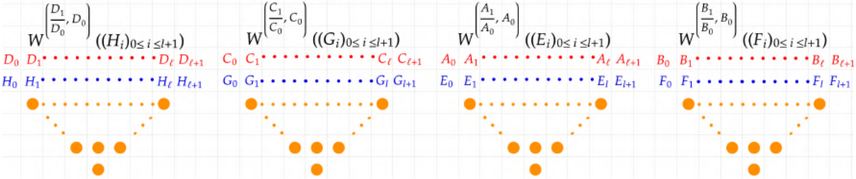
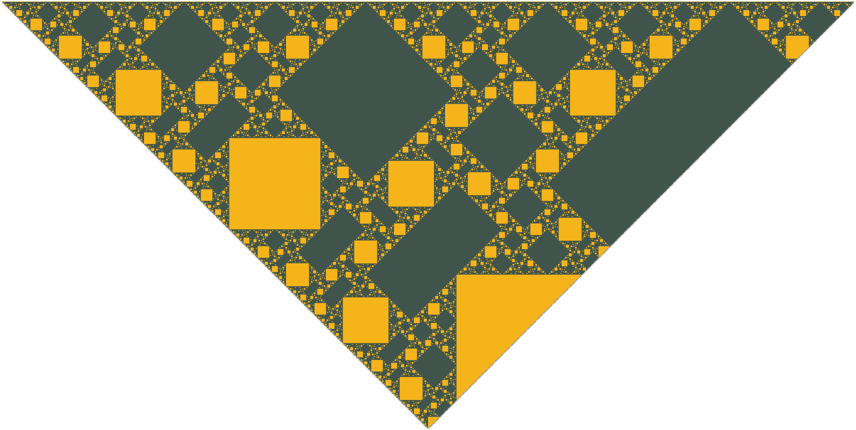


Example

- Let $P_n = f_{n+1} - f_{n-1}$, where $(f_n)_{n\in\mathbb{Z}}$ is the Paperfolding sequence.



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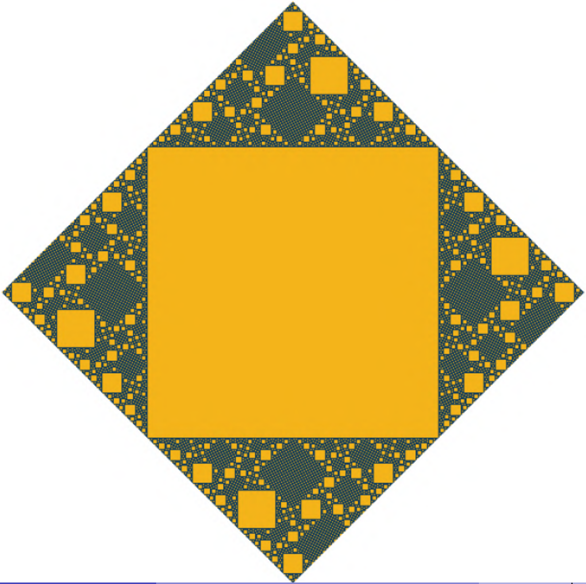

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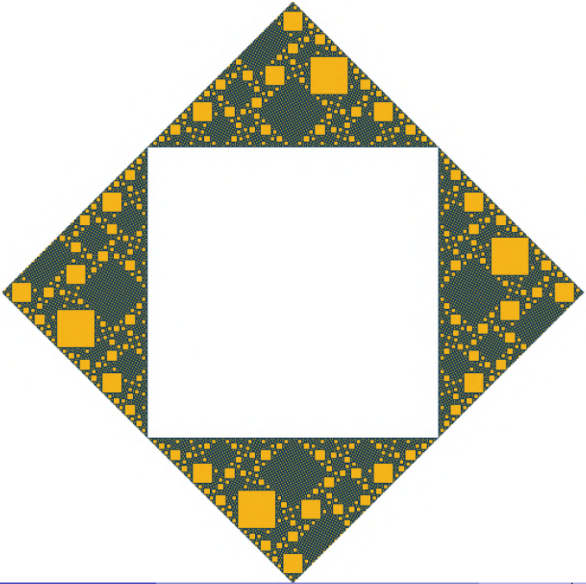

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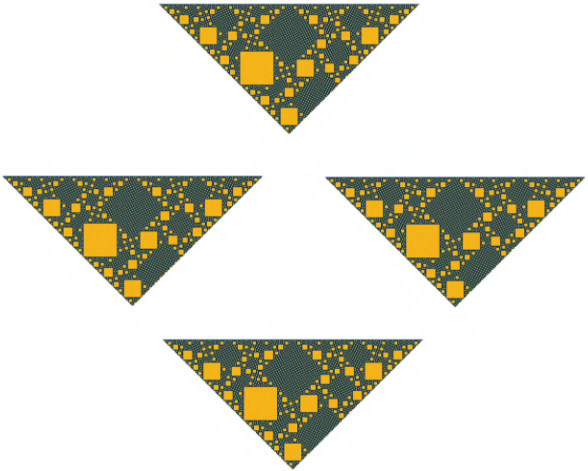
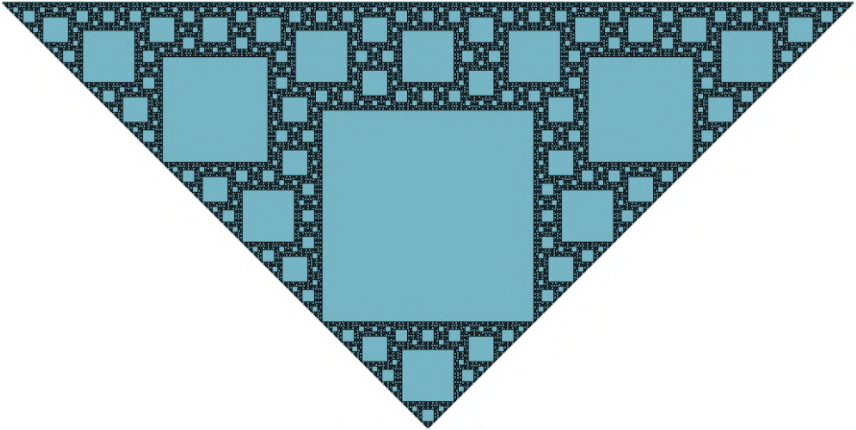


Self-Similar Number Walls and Automaticity

- Let $(t_n)_{n\geq 0}$ be the **Thue-Morse** sequence.
- Let $T(n) = (t_n)_{0\leq i<2^n}$.
- Then, $W_2(T(10))$ appears as



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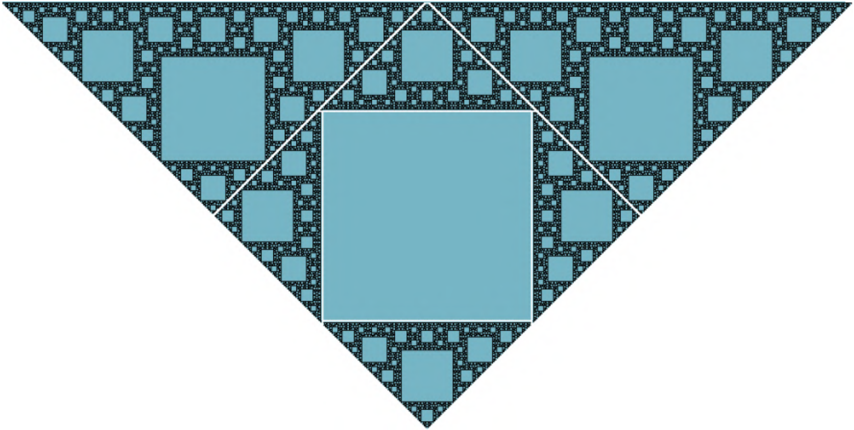

Self-Similar Number Walls and Automaticity

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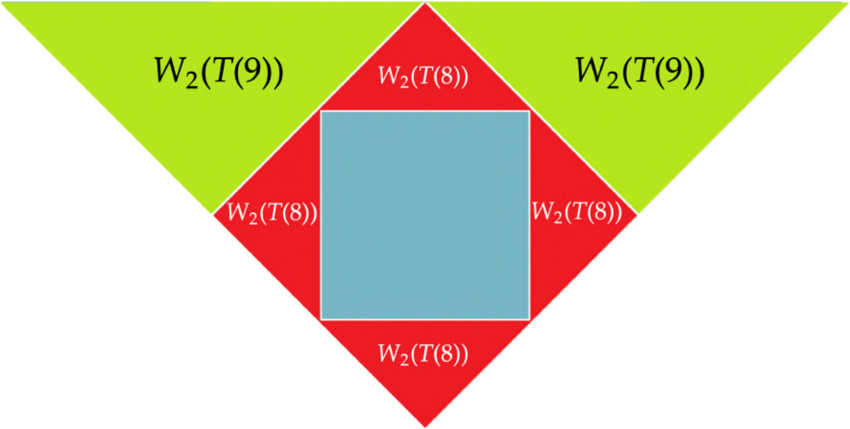

Codings are Not Unique

- Recall the **Thue-Morse** sequence generated by the 2-morphism

$$\phi : A \to AB \qquad \phi : B \to BA$$

and the 1-coding

$$\tau_0 : A \to 0, \qquad \tau_0 : B \to 1.$$

- However, τ_0 is not the only coding that generates the Thue-Morse sequence.
- For $i \in \mathbb{N}$, define the 2^i-coding τ_i as

$$\tau_i : A \to \tau_0(\phi^n(A)) \qquad \tau_i : B \to \tau_0(\phi^n(B)).$$

- For example,

$$\tau_1 : A \to 01 \qquad \tau_1 : B \to 10$$

and

$$\tau_2 : A \to 0110 \qquad \tau_2 : B \to 1001.$$

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, recolour the sequence according to the image of the 2-coding τ_1.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, recolour the sequence according to the image of the 2-coding τ_1.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_2.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_2.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_3.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_3.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_4.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_4.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_5.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_5.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_6.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_6.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_7.

The Sequence Within the Number Wall

- Consider the number wall of the Thue-Morse sequence.
- Next, add in any entries of the number wall that are generated solely by the image of τ_7.

Part 5: Open Problems

Conjectures

- **Conjecture** (Garrett, R., 2024): The number wall of an automatic sequence is itself a 2-dimensional automatic sequence.
- **Conjecture** (R., ?): Let $\Sigma \subset \mathbb{Z}$ be a finite set and let $\mathbf{S}$ be an automatic sequence over Σ. Then, $W_{\mathbb{Z}}(\mathbf{S})$ is a regular sequence.
- **Conjecture** (Garrett, R., 2024): Let $\mathbf{S}$ be a binary sequence. Then, $W_2(\mathbf{S})$ has windows of unbounded size .
- **Conjecture** (Shallit, 2000): There exists a finite $\Sigma \subset \mathbb{Z}$ and a sequence $\mathbf{S}$ over Σ such that $W_{\mathbb{Z}}(\mathbf{S})$ has no zero entries.