Balanced rectangles over Sturmian words

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$$
\begin{array}{ccccccccccccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & & \cdots \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & & \cdots \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & & \cdots \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{array}
$$

Joint work with Jeffrey Shallit.

One World Combinatorics on Words Seminar,
3 March 2026

Fibonacci numbers and Fibonacci word

n	0	1	2	3	4	5	6	7	8	9	...
F_n	0	1	1	2	3	5	8	13	21	34	...

0 1

01 2

010 3

010 01 2 5

01001 01001001 ......

2 1

balanced

Balanced rectangles

$2 \times 3 \rightsquigarrow$ bal. $m \times n$?

$2 \times 4 \rightsquigarrow$ not bal.

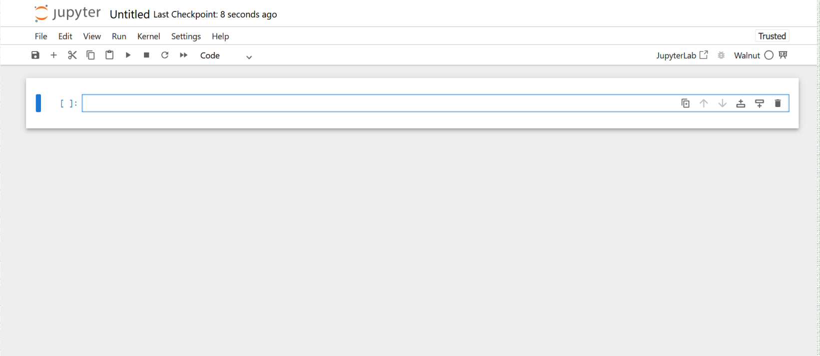
$$
\begin{matrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \cdots \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{matrix}
$$

2002 Berthé & Tijdeman

2025 Anselmi et al.:

$m \le n$, $n = F_k \rightsquigarrow$ bal.

Proof by Walnut magic



Proof by Walnut magic

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Code JupyterLab Walnut

[ ]:

```
reg shift {0,1} {0,1} "([0,0]|[0,1][1,1]*[1,0])*":
def phin "?msd_fib (n=0 & y=0) | Ez $shift(n-1,z) & y=z+1":
def alpha "?msd_fib (n=0&z=0) | (n>0 & Ex $phin(n,x) & z+1+x=2*n)":
def diff "?msd_fib Ea,b,c,d $alpha(i,a) & $alpha(i+m,b) & $alpha(i+n,c) & $alpha(i+m+n,d) & z+b+c=a+d+1":
def bal "?msd_fib m<=n & ~Ei,j i>0 & i<j &
    (($diff(i,m,n,2) & (Ak (i<k & k<j) => $diff(k,m,n,1)) & $diff(j,m,n,2)) |($diff(i,m,n,0) & (Ak (i<k & k<j) => $diff(k,m,n,1)) & $diff(j,m,n,0)))":
```

Proof by Walnut magic

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Code JupyterLab Walnut

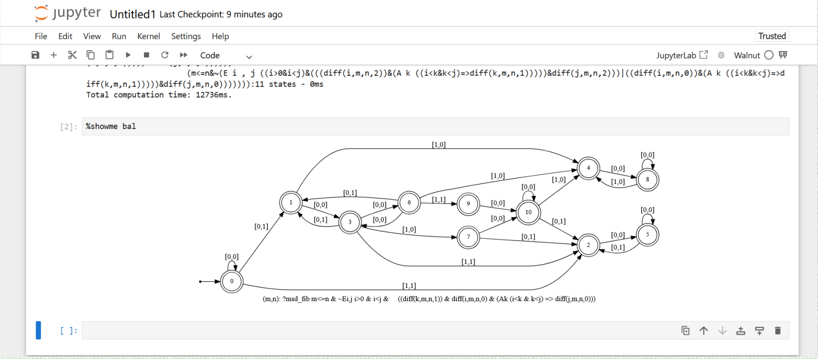
[1]:
```
reg shift {0,1} {0,1} "([0,0]|[0,1][1,1]*[1,0])*":
def phin "?msd_fib (n=0 & y=0) | Ez $shift(n-1,z) & y=z+1":
def alpha "?msd_fib (n=0&z=0) | (n>0 & Ex $phin(n,x) & z+1+x=2*n)":
def diff "?msd_fib Ea,b,c,d $alpha(i,a) & $alpha(i+m,b) & $alpha(i+n,c) & $alpha(i+m+n,d) & z+b+c=a+d+1":
def bal "?msd_fib m<=n & ~Ei,j i>0 & i<j &
    (($diff(i,m,n,2) & (Ak (i<k & k<j) => $diff(k,m,n,1)) & $diff(j,m,n,2)) |($diff(i,m,n,0) & (Ak (i<k & k<j) => $diff(k,m,n,1)) & $diff(j,m,n,0)))":
```

```
computed ~:2 states - 93ms

computed ~:1 states - 2ms
n=0:1 states - 16ms
 y=0:1 states - 0ms
  (n=0&y=0):1 states - 1ms
computed ~:1 states - 1ms
computed ~:2 states - 64ms
   y=(z+1):4 states - 1ms
    (shift((n-1),z))&y=(z+1)):9 states - 1ms
     (E z (shift((n-1),z))&y=(z+1))):7 states - 1ms
      ((n=0&y=0)|(E z (shift((n-1),z))&y=(z+1)))):7 states - 1ms
Total computation time: 104ms.

n=0:1 states - 0ms
 z=0:1 states - 0ms
  (n=0&z=0):1 states - 0ms
```

Proof by Walnut magic



Characterisation for Fibonacci rectangles

n	$\cancel{0}$	$\cancel{1}$	2	3	4	5	6	7	8	9	...
F_n	$\cancel{0}$	$\cancel{1}$	$\underline{1}$	2	3	$\underline{5}$	8	$\underline{13}$	21	34	...

$$19 = 13 + 5 + 1 = F_7 + F_5 + F_2$$

Characterisation for Fibonacci rectangles

For $2 \leq m \leq n$, the $m \times n$ rectangles are balanced if and only if we are in at least one of the following cases:

$$
\begin{aligned}
&(a) \quad m = b_2F_2 + \cdots + b_MF_M \\
&\qquad\; n = \hphantom{b_2F_2 + \cdots + b_MF_M} \; b_{M+1}F_{M+1} + \cdots + b_NF_N;
\end{aligned}
$$

$$
\begin{aligned}
&(b) \quad m = b_2F_2 + \cdots + F_M \\
&\qquad\; n = \hphantom{b_2F_2 + \cdots +} F_M + F_{M+2t+1} + \cdots + b_NF_N;
\end{aligned}
$$

$$
\begin{aligned}
&(c) \quad m = \hphantom{b_2F_2 + \cdots + b_{M-1}F_{M-1} +} \boxed{F_M} \\
&\qquad\; n = b_2F_2 + \cdots + b_{M-1}F_{M-1} + F_{M+2t} + \cdots + b_NF_N.
\end{aligned}
$$

Alternative definition & best approximations

$$\varphi \approx \frac{F_{k+1}}{F_k} \quad \longrightarrow \quad \underbrace{\varphi \cdot F_k}_{\text{very close to an integer}} \approx F_{k+1}$$

$a_1 a_2 a_3 \ldots\ldots$

$$a_n = 1 \quad :\Longleftrightarrow \quad \{n\alpha\} \in [1-\varphi', 1)$$

or :

$$a_n = \lfloor (n+1)\varphi' \rfloor - \lfloor n\varphi' \rfloor$$

Balancedness in 1 dimension

$$Q_i + Q_{i+1} + \dots + Q_{i+n-1} =$$

$$= \lfloor \cancel{(i+1)\varphi'} \rfloor - \lfloor i\varphi' \rfloor$$

$$+ \lfloor \cancel{(i+2)\varphi'} \rfloor - \lfloor \cancel{(i+1)\varphi'} \rfloor$$

$$+ \dots$$

$$+ \lfloor (i+n)\varphi' \rfloor - \lfloor \cancel{(i+n-1)\varphi'} \rfloor$$

$$= \lfloor (i+n)\varphi' \rfloor - \lfloor i\varphi' \rfloor = \lfloor n\varphi' \rfloor + \begin{cases} 1 & \text{if} \\ 0 & \text{else} \end{cases}$$

$$\{i\varphi'\} + \{n\varphi'\} \geq 1$$

Rectangles, intervals, and discrepancy

$$
\begin{matrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \cdots \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{matrix}
$$

$$
\sum \begin{pmatrix} a_i & \cdots & a_{i+n-1} \\ \vdots & & \vdots \\ a_{i+m-1} & \cdots & a_{i+m+n-2} \end{pmatrix} =
$$

$$
= \sum_{\ell=0}^{m-1} \underbrace{\lfloor n\varphi' \rfloor} + \mathbb{1}_{[1-\{n\varphi'\},1)}\left(\{(i+\ell)\varphi'\}\right)
$$

$$
= m \cdot \lfloor n\varphi' \rfloor + \sum_{\ell=0}^{m-1} \mathbb{1}_{[1-\{n\varphi'\}-\{i\varphi'\},\,1-\{i\varphi'\})}\left(\{\ell\varphi'\}\right)
$$

Fibonacci represenation

$0.257\ldots \in [0.257, 0.238]$

n	0	1	2	3	4	5	6	7	8	9	...
F_n	0	1	1	2	3	5	8	13	21	34	...

$30\varphi' \bmod 1$

$\varphi' = 2 - \varphi \approx 0.38$

$30 = 21 + 8 + 1$

$$30\varphi' = \underbrace{(F_8 + F_6)\varphi'}_{\approx 0{,}1 \bmod 1} + \underline{1\varphi'}$$

$\alpha \quad q_0, q_1, q_2, q_3, \ldots\ldots$

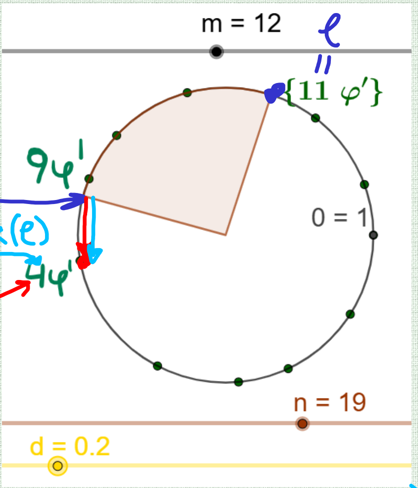
$$n = \sum_{k=0}^{N} b_k q_k$$

. . . special cases in characterisation!

And general Ostrowski representations.

Proof strategy (gain – lose – gain – lose . . .)

$m \le n$



$\ell =$

$9\varphi'$

$\{(\ell+n)\varphi\}$

$-x(\ell)$

$4\varphi'$

$f(11)$

$f(\ell)$

$0, 1\varphi', \dots, (m-1)\varphi'$

$f: \{0, 1, \dots, m-1\} \circlearrowleft$

Lem: f is bij.

$\iff$ intervals are balanced

$x(\ell) \in$ in range(m, n, ℓ)

$f(\ell) = \ell + n - x(\ell)$

$\boxed{x(\ell) - \ell \text{ bij}}$

Characterisation in the general case

Theorem 2.1. *Let $\alpha \in (0,1)$ be irrational and $2 \leq m \leq n$. Then the $m \times n$ rectangles of the Sturmian words with slope α are balanced if and only if the Ostrowski representations of m, n with respect to α are of at least one of the following four shapes:*

They have "split representations" in the following sense:

(i) $m = \sum_{k=0}^{M} b_k q_k$ *and* $n = \sum_{k=M+1}^{N} b_k q_k$;

(ii) $m = \sum_{k=0}^{M} b_k q_k$ *with* $b_M \neq 0$, *and* $n = q_M + \sum_{k=M+1+2t}^{N} b_k q_k$ *with* $t \geq 0$ *and* $b_{M+1+2t} \neq 0$.

The smaller number m is the denominator of a (semi-)convergent, and we have certain parity restrictions on the large digits in n:

(iii) $m = q_M$ *and* $n = \sum_{k=0}^{M-1} b_k q_k + \sum_{k=M+2t}^{N} b_k q_k$ *with* $t \geq 0$ *and* $b_{M+2t} \neq 0$;

(iv) $m = q_{M-1} + a q_M$ *with* $1 \leq a < a_{M+1}$ *and* $n = \sum_{k=0}^{M-1} b_k q_k + \sum_{k=M+1+2t}^{N} b_k q_k$ *with* $t \geq 0$ *and* $b_{M+1+2t} \neq 0$.

$\alpha = \pi/4$

$$\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & \cdots \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \cdots \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

$\alpha = [0; 1, 3, 1, 1, 1, 15, 2, 72, \ldots]$

$\alpha = \pi/4$

$$\begin{array}{ccccccccccccccccc}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & & \cdots \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & & \cdots \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{array}$$

$\alpha = [0; 1, 3, 1, 1, 1, 15, 2, 72, \ldots]$

$\cancel{q_0 = 1}$
$q_1 = 1$
$q_2 = 4$
$q_3 = 5$
$q_4 = 9$
$\ldots$

2x10 bol?

$m = 2 = 2 \cdot q_1$

$n = 10 = 9 + 1 = \underline{q_4} + \underline{q_1}$

$m = 5 = q_3$

Thank you!

$$\begin{array}{ccccccccccccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & & \cdots \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & & \cdots \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & & \cdots \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{array}$$