On some 2-binomial coefficients of binary words: geometrical interpretation, partitions of integers, and fair words

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One World Combinatorics on Words Seminar, March 17th 2026

Contents of the talk

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  - [Černý 2009]
    - Fair word $w : \forall a, b$ letters $\binom{w}{ab} = \binom{w}{ba}$
    - Question about the number of fair words

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- About the results
  - Already partially presented at the workshop on "Combinatorics on words and subwords", Liège, May 12-16th 2025
  - For more details see Theoret. Comput. Sci. 1066 (2026) (or arXiv (2025))

Contents

Binomial coefficient of words

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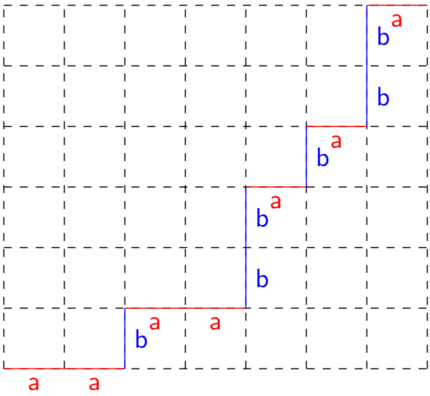
Example

- $\binom{aabba}{ab} = 4 \quad (w = aabba = aabba = aabba = aabba)$
- for any word u and letter α,
  $\binom{u}{\alpha}$ = number of occurrences of the letter α also denoted $|u|_\alpha$

Geometrical representation of $\binom{w}{ab}$, $w \in \{a, b\}^*$

- A usual representation of a word w as a path in the grid:



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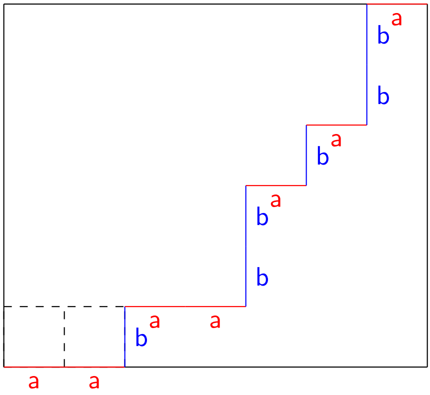

- $\binom{w}{ab} :=$ number of occurrences of ab as a (scattered) subword in w
- One way to compute it: $\displaystyle\binom{w}{ab} = \sum_{pb \text{ prefix of } w} |p|_a$

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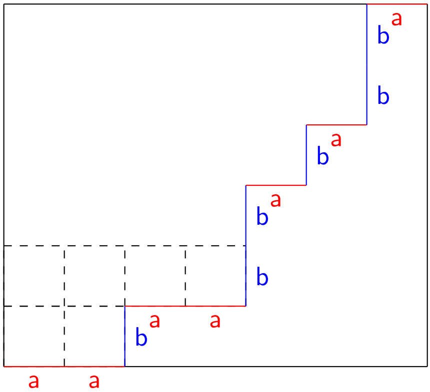

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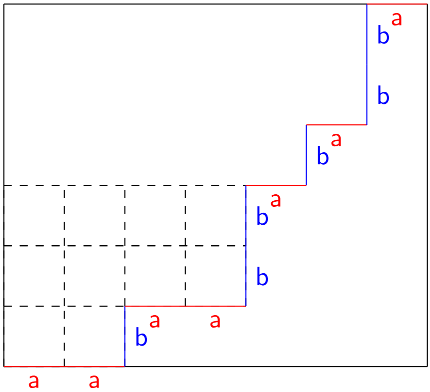

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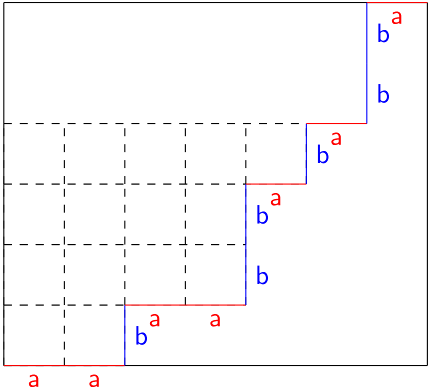

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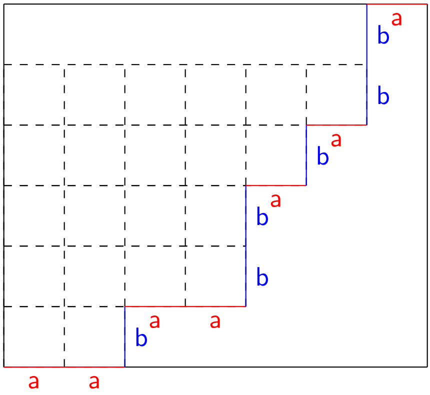

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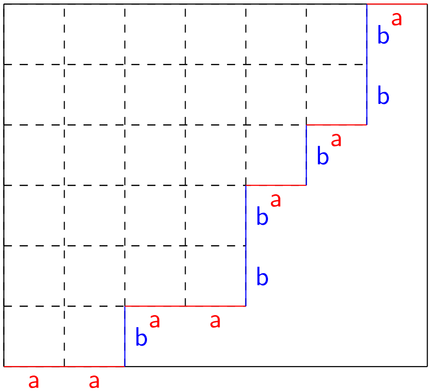

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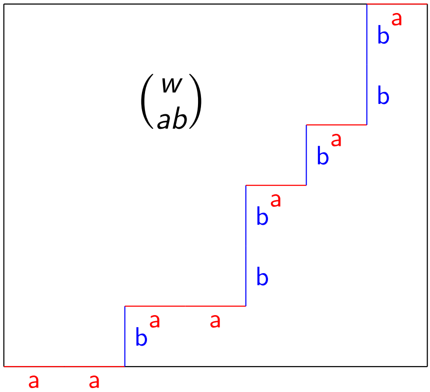

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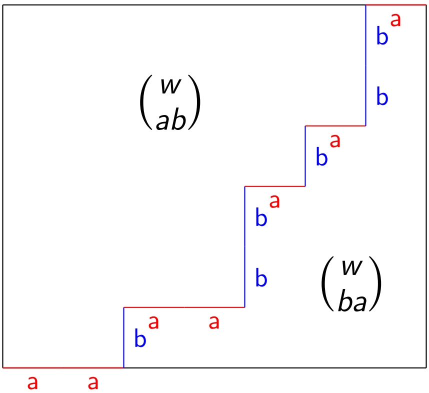
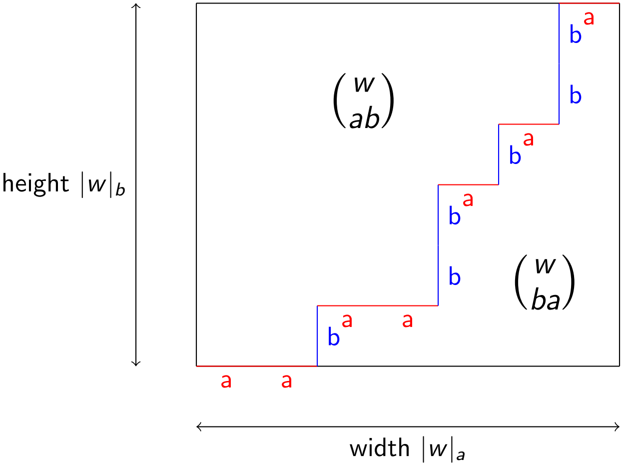
Geometrical representation of $\binom{w}{ba}$

$$\binom{w}{ba} = \sum_{bs \text{ suffix of } w} |s|_a$$



A basic formula

$$\binom{w}{ab} + \binom{w}{ba} = |w|_a \times |w|_b$$



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The 2-binomial equivalence

Definition

Two words u, v are 2-binomial equivalent, denoted $u \sim_2 v$, if

$$\binom{u}{x} = \binom{v}{x} \text{ for all } x \text{ with } |x| \leq 2$$

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Remark on the binary case

On $\{a, b\}^*$, $u \sim_2 v$ if and only if

$$|u|_a = |v|_a, \qquad |u|_b = |v|_b \qquad \text{and} \qquad \binom{u}{ab} = \binom{v}{ab}$$

A consequence of relations

- $\binom{w}{\alpha\alpha} = |w|_\alpha(|w|_\alpha - 1)$ for any word w and letter α
- $\binom{w}{ab} + \binom{w}{ba} = |w|_a \times |w|_b$

In binary case, 2-binomial equivalence = Parikh equivalence

Definition (Mateescu, Salomaa, Salomaa, Yu 2000)

Parikh matrix of u over $\{a, b\}$ = $\begin{bmatrix} 1 & |u|_a & \binom{u}{ab} \\ 0 & 1 & |u|_b \\ 0 & 0 & 1 \end{bmatrix}$

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Reformulation of previous remark

[Mateescu, Salomaa, Salomaa, Yu 2000, Prodinger 1979]
For any words u and v over $\{a, b\}$

$$u \sim_2 v \Leftrightarrow u \text{ and } v \text{ have same Parikh matrix}$$

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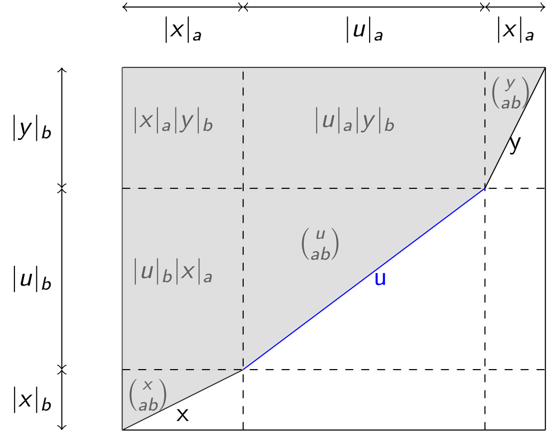
Remark.
The “Parikh matrix” appears in [Prodinger 1979] earlier than its formal definition in [Mateescu, Salomaa, Salomaa, Yu 2000].

$\sim_2$ is a congruence: a geometrical interpretation

$$xuy \sim_2 xvy \Leftrightarrow u \sim_2 v$$



$$\binom{x{\color{blue}u}y}{ab} = \binom{{\color{blue}u}}{ab} + \binom{x}{ab} + \binom{y}{ab} + |{\color{blue}u}|_b|x|_a + |x|_a|y|_b + |{\color{blue}u}|_a|y|_b$$

A rewriting point of view of the 2-binomial equivalence

For $u, v \in \{a, b\}^*$:

- $u \rightarrow v$ (or $v \leftarrow u$) : if $u = xabybaz$ and $v = xbayabz$ for some words x, y and z.

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- $u \to v$ (or $v \leftarrow u$) : if $u = xabybaz$ and $v = xbayabz$ for some words x, y and z.
- Reflexive closure of these relations: $\equiv$ $(\equiv = \leftarrow \cup \to)$.
- Reflexive and transitive closure of this relation: $\equiv^*$.

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Theorem (Fossé Richomme 2004)

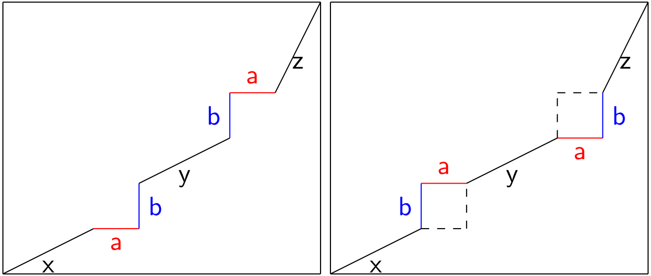
For two binary words u and v,

$$u \equiv^* v \Leftrightarrow u \sim_2 v$$

Next slides = a geometrically guided proof

Switching *ab* to *ba* and *ba* to *ab*

$$\binom{xabybaz}{ab} = \binom{xbayabz}{ab}$$



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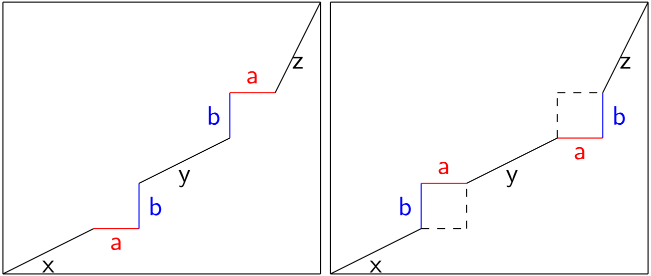

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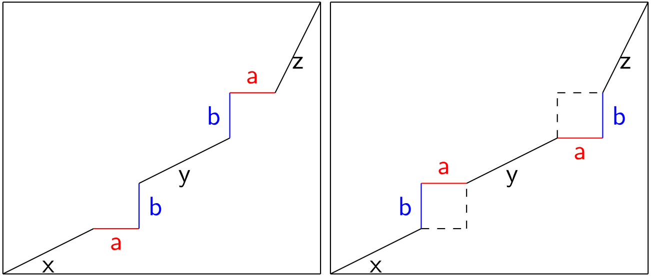

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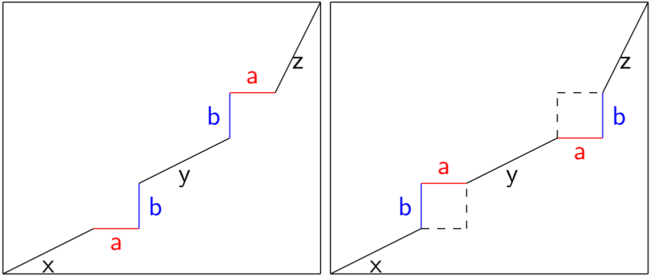

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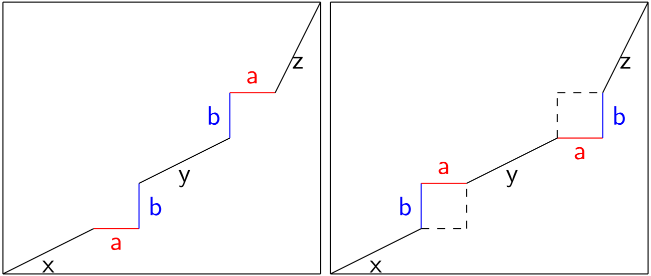

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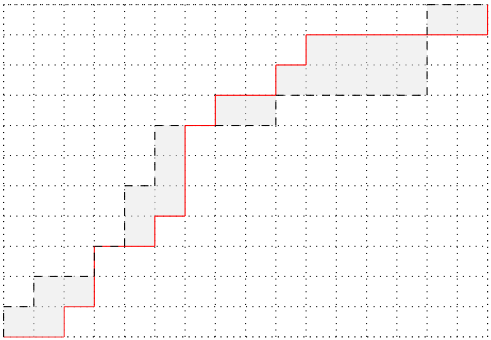
About the proof of the converse

- Assume $u \sim_2 v$ (corresponding above areas have the same value)



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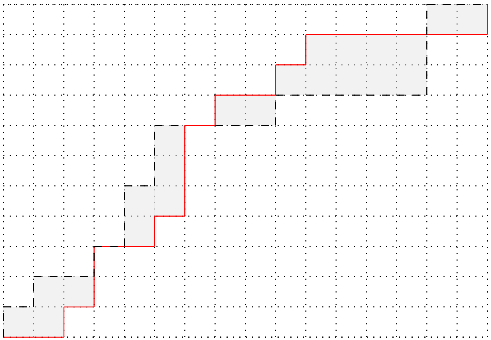

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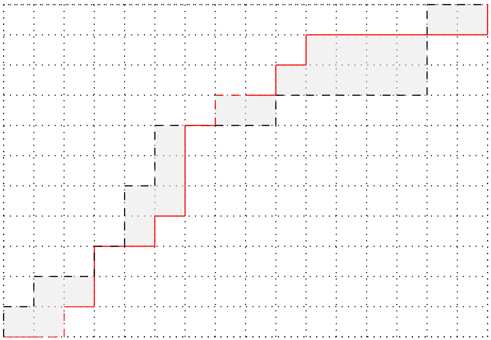

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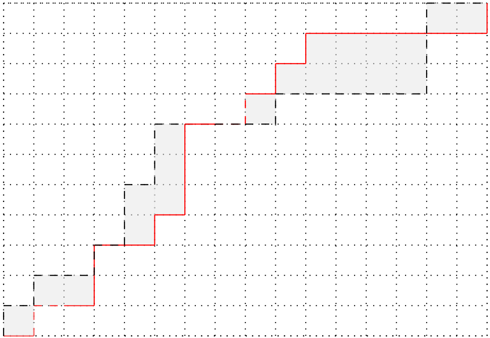

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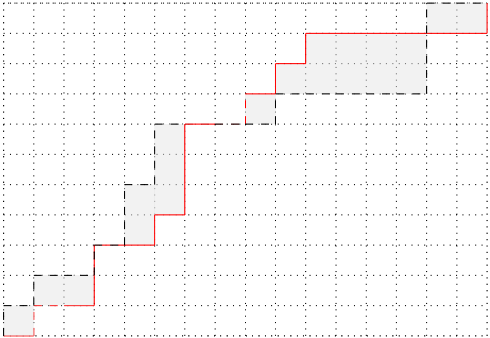

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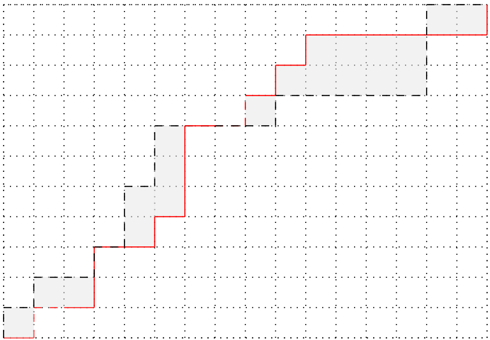

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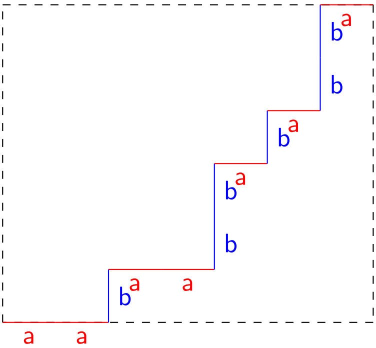
- By induction on the "distance", $u' \equiv^* v$ and so $u \equiv^* v$

Contents

A basic representant of a 2-binomial class

Rewriting $aabaabbababba$ in order to have no factor in $ab\{a, b\}^*ba$



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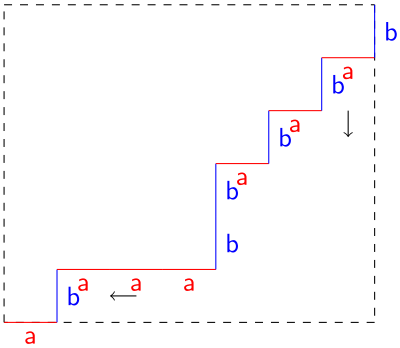

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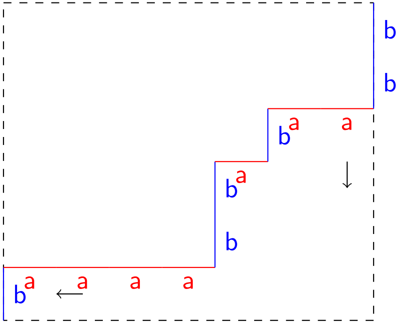

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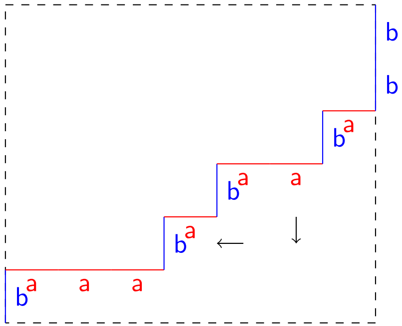

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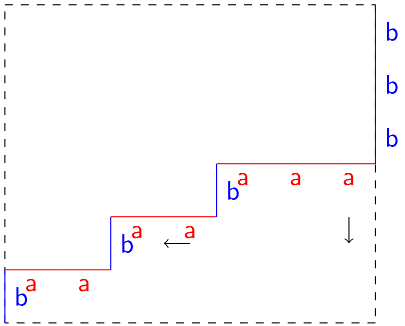

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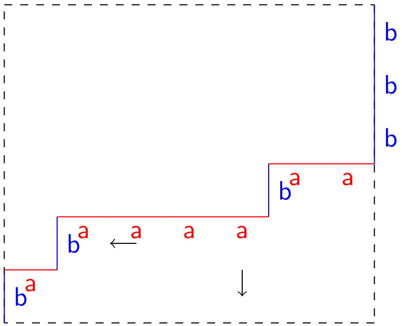

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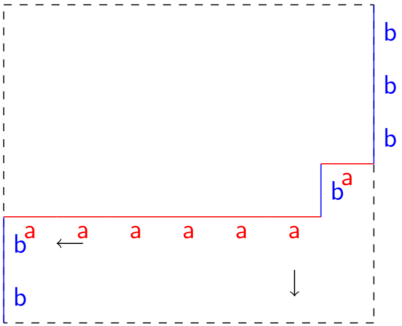
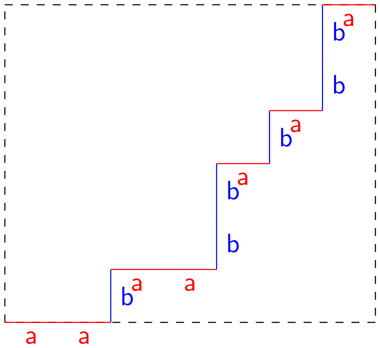


For abritrary w, we get a word in $b^*a^*ba^*b^*$ or in $b^*a^*b^*$. Let denote it $\mathrm{final}(w)$.

Another basic representant of a 2-binomial class

Rewriting $aabaabbababba$ in order to have no factor in $ba\{a, b\}^*ab$



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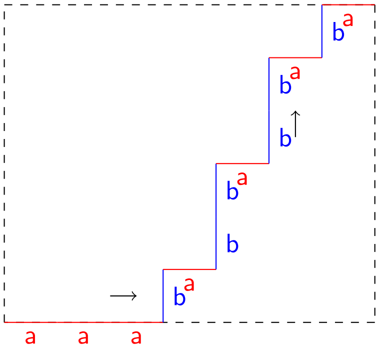

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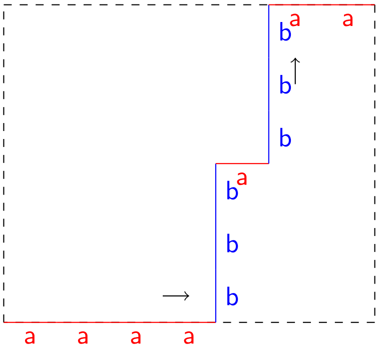


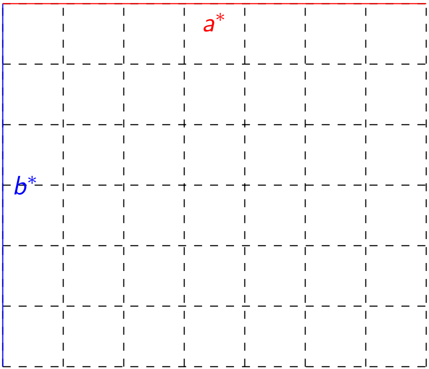
For abritrary w, we get a word in $a^*b^*ab^*a^*$ or in $a^*b^*a^*$. Let denote it $\mathrm{init}(w)$.

Direct construction of representants

If $u \sim_2 v$ then $\mathrm{init}(u) = \mathrm{init}(v)$ and $\mathrm{final}(u) = \mathrm{final}(v)$.

Proof. A reconstruction result

Case $\binom{w}{ab} = 0$, $w \in b^* a^*$, $w = \mathrm{init}(w) = \mathrm{final}(w)$



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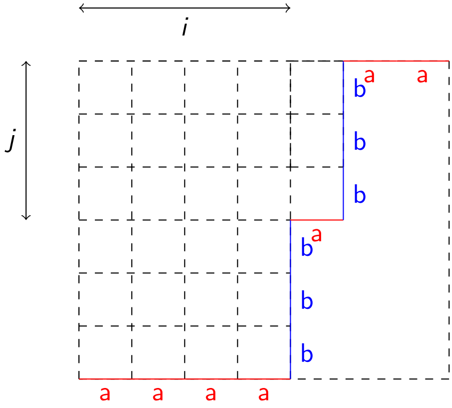

$$\mathrm{init}(w) = a^i b^{|w|_b - j} a b^j a^{|w|_a - 1 - j}$$

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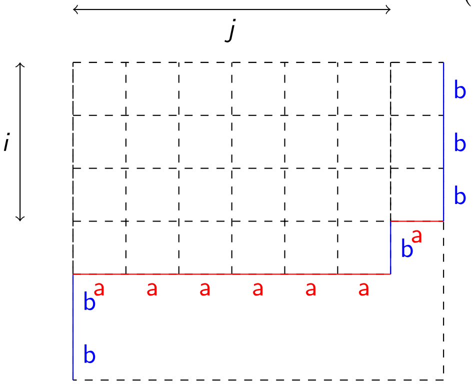


$$\mathrm{final}(w) = b^{|w|_b - i - 1} a^j b a^{|w|_a - j} b^i$$

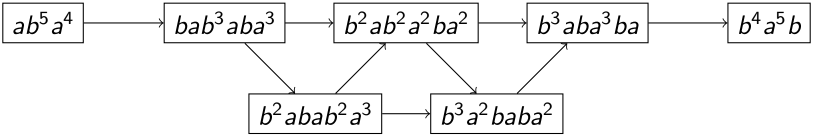
Structure of the graph $G_2(w) = ([w]_{\sim_2}, \rightarrow)$

Set $[w]_{\sim_2} = \{u \sim_2 w\}$.

Lemma

Given any word w, the graph $G_2(w)$ is a connected directed acyclic graph with :

- $\mathrm{init}(w)$ = *unique root of* $G_2(w)$
- $\mathrm{final}(w)$ = *unique sink of* $G_2(w)$.
- *for any* w, $\mathrm{final}(w) \leq_{lex} w \leq_{lex} \mathrm{init}(w)$



No loop since $xabybaz \rightarrow xbayabz$

Bibliographical remarks

Remark 1.
Idea of direct computations of representants can be found in [Teh, Kwa 2015]

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Theorem ([Rigo Salimov 2015])

$$\#(\{a, b\}^n / \sim_2) = \frac{n^3 + 5n + 6}{6}.$$

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Remark 3. [Lejeune Rigo Rosenfeld 2020]

- $LL(\sim_2, \{a, b\}) := \{w \in \{a, b\}^* \mid \forall u \in [w]_{\sim_2} : w \leq u\}$
- Proof that $LL(\sim_2, \{a, b\})$ is regular proving $LL(\sim_2, \{a, b\}) = \mathrm{init}(\{a, b\}^*)$ that is the set of words without any factor on the form $xbayabz$.

Classes of 2-binomial that are singletons

Theorem

Are equivalent:

1. $\#[w]_{\sim_2} = 1$ *(word w is unique in its 2-binomial equivalence)*
2. $w \notin \Sigma^* ab \Sigma^* ba \Sigma^* \cup \Sigma^* ba \Sigma^* ab \Sigma^*$ *with* $\Sigma = \{a, b\}$
3. $w \in b^*a^* + b^*ab^* + b^*aba^* + a^*b^* + a^*ba^* + a^*bab^*$

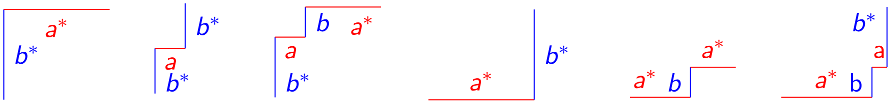

Figure: Shapes of words w with $\#[w]_{\sim_2} = 1$

Proof

- $1 \Leftrightarrow 2 \Leftrightarrow 3$: Mateescu, Salomaa 2004

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4. $\mathrm{init}(w) = \mathrm{final}(w)$

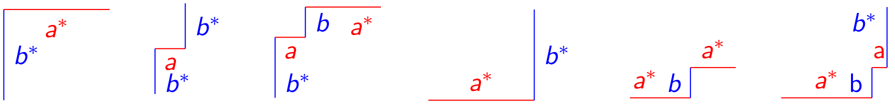

Figure: Shapes of words w with $\#[w]_{\sim_2} = 1$

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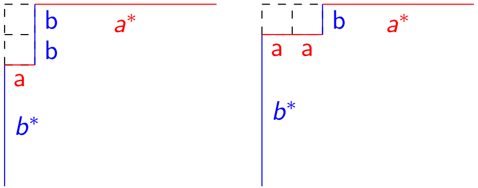
- $\Leftrightarrow 4$: follows from $\mathrm{init}(w) \sim_2 \mathrm{final}(w)$

Contents

Enumerating $[w]_{\sim_2}$ w.r.t. $\binom{w}{ab}$

$\binom{w}{ab} = 2$



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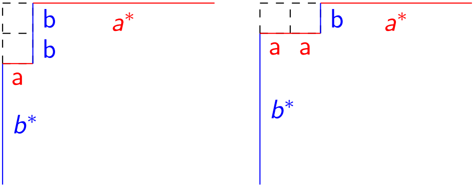

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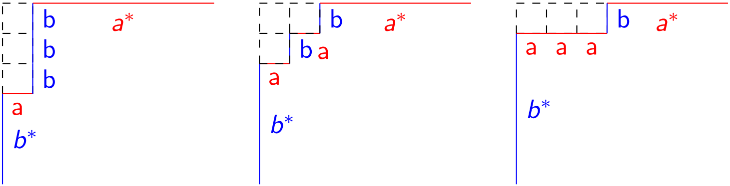

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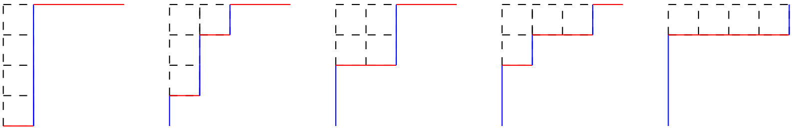

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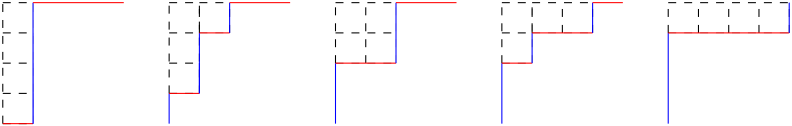
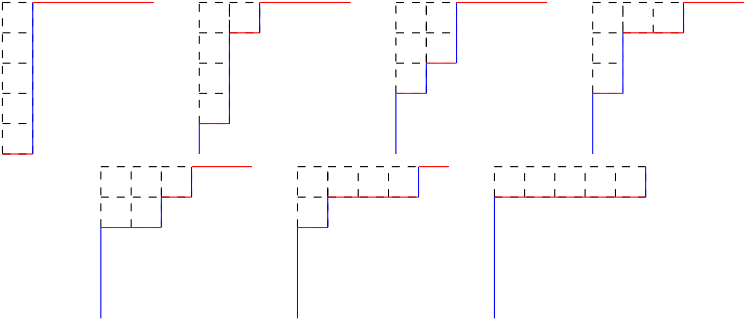


$$\binom{w}{ab} = 5$$



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- *Bijection between elements of* $[w]_{\sim_2}$ *and*
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- Remark: the graph $([w]_{\sim_2}, \to)]$ is not isomorphic to the correspondant lattice of partitions.

The lattice of integer partitions

- Let λ be a partition of the integer n: $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ with
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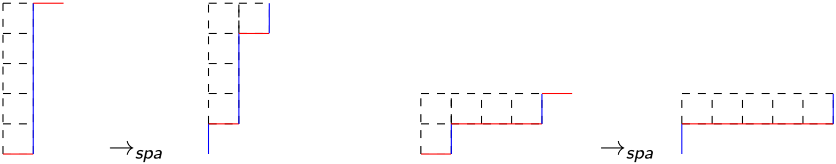
  - and one of the two cases holds:
    1. $k = j + 1$;
    2. $\mu_j = \mu_k$ (or, equivalently, $\lambda_j = \lambda_k + 2$).

New rewriting rules



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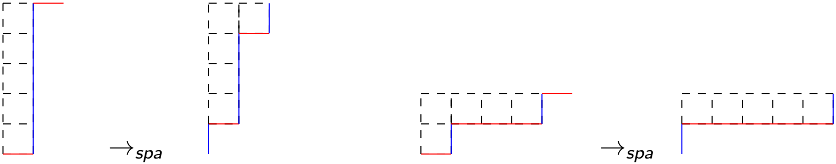

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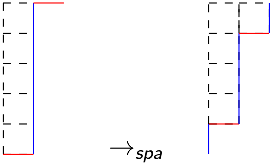

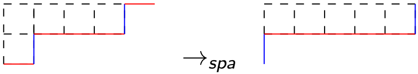

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- $|w|_a$ and $|w|_b$ can be exchanged
- *spa* = special palindromes
- Atanasiu et al. 2002: two binary words are 2-binomial equivalent if and only if they are palindromic amiable (they can be obtained by successive replacing of any palindrome by another one with same numbers of *a* and *b*)

Contents

Fair words: definition

w is fair: $\binom{w}{ab} = \binom{w}{ba}$ for each pair of letters (a, b)

- Why fair? [Černý 2009]
  - "*Imagine members of $k \geq 1$ rivaling groups $a_1, a_2, \ldots, a_k$ want to pass a narrow door. In which order they should pass?*"
  - "*A solution to this problem is fair, if, for any two distinct groups a_i, a_j, the number of member pairs, where a member of a_i precedes a member of a_j, is the same as of those where the order is reversed.*"

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$$D(x, y) = \left\{ w \in \{a, b\}^* \mid \binom{w}{x} = \binom{w}{y} \right\}$$

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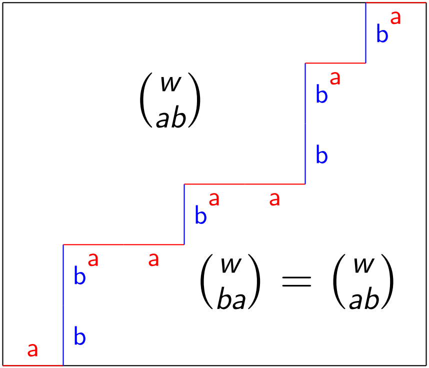
  - Fair words = elements of $D(ab, ba)$

Binary fair words



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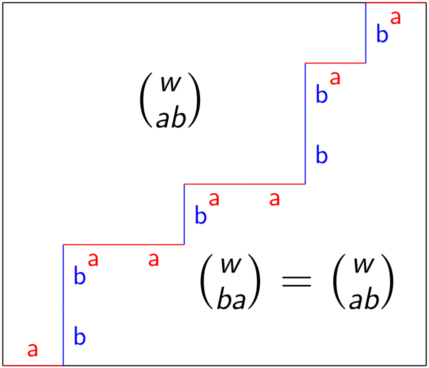

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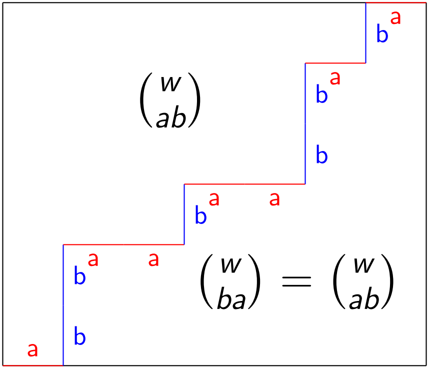
Binary fair words



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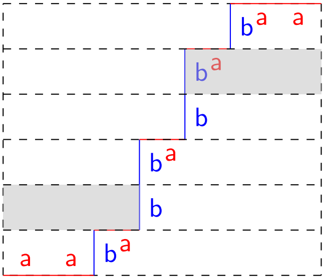
- Hence no fair word w exists with $|w|_a$ and $|w|_b$ both odd

Fair words: palindromes [Černý 2009]

- Palindromes are fair



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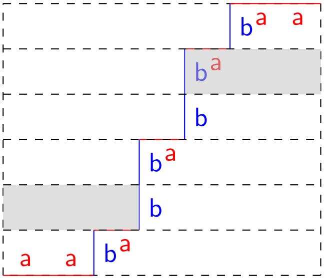
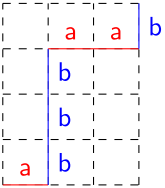


- Smallest fair word unpalindromic:

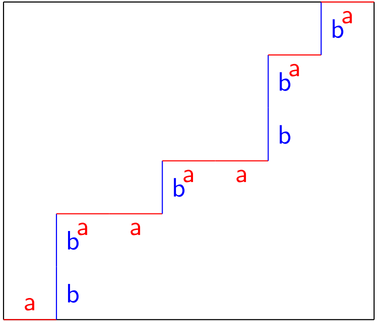
abbbaab



Remark. *abbbaab* $\sim_2$ *bababab* which is a palindrome

Balanced fair words

- A fair word w is a word whose representation cuts in two parts of same areas the rectangle of width $|w|_a$ and heigth $|w|_b$.



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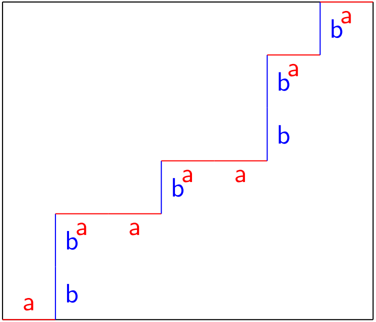

- Balanced words represent segments

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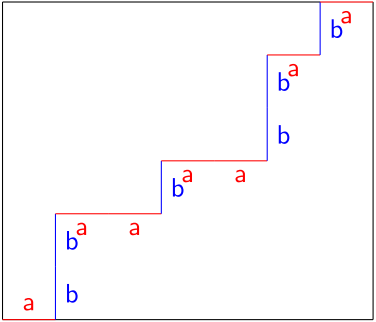

- Balanced words represent segments
- Question: does there exists balanced fair words ?

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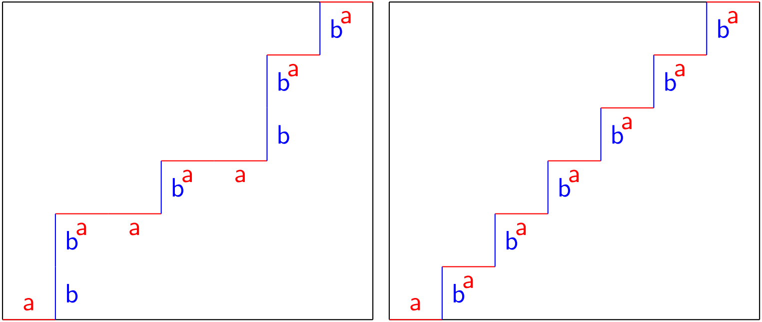

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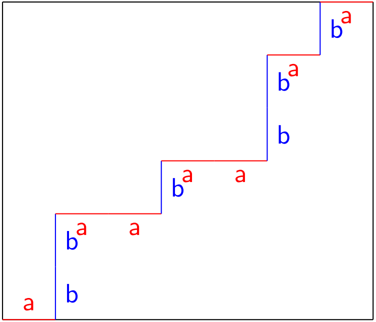

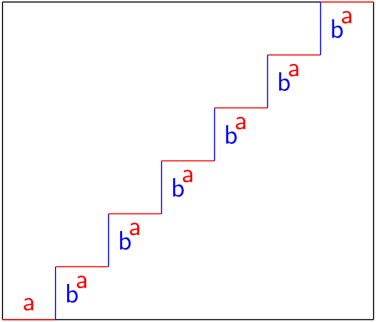

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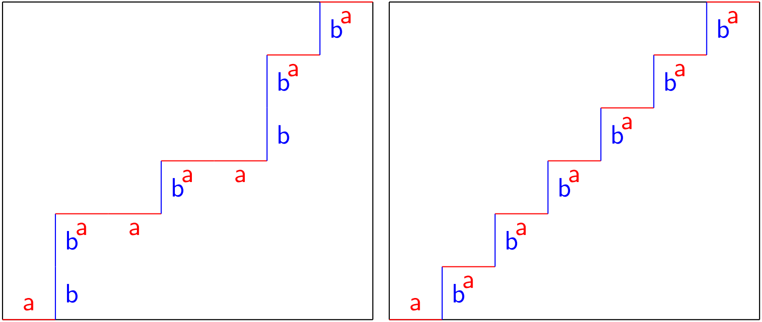

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  - Palindromic balanced representants may not be unique: $abba \sim_2 baab$

An answer to Problem 23 in [Černý 2009]

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- Similar proof for $\mathrm{init}(w)$

Contents

Number of binary fair words [Prodinger 1979]

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For each word $w = a_1 \dots a_n$ *(*$a_i \in \{a, b\}$*), with* $\sigma(a) = +1$*,* $\sigma(b) = -1$*:*

$$\binom{w}{ab} - \binom{w}{ba} = \frac{1}{2} \sum_{k=1}^{n} \sigma(a_k)(n + 1 - 2k)$$

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Theorem (Prodinger 1979)

Number of fair words of length n $\sim 2^{2\lfloor (n-1)/2 \rfloor + 1} \left(\frac{3}{\pi}\right)^{1/2} \left\lfloor \frac{n}{2} \right\rfloor^{-3/2}$

In the On-Line Encyclopedia of Integer Sequences (OEIS)

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The answer is yes!

$$\sum_{i=1}^{n}(\alpha + \beta i - w_i)^2 \text{ is minimal when } \beta = 0 \Leftrightarrow w_1 \cdots w_n \text{ is fair.}$$

(And then $\alpha = |w|_1/n$)

Some elements of the proof of this minimization result

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- The converse acts similarly.

A characterization of fair words

Result

A word w over $\{a, b\}$ is fair ($\binom{w}{ab} = \binom{w}{ba}$)
if and only if
it has the same sum of positions of b's than its reverse

Proof.

Consequence of:

- $\binom{\tilde{w}}{ab} = \binom{w}{ba}$
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  - $$S_b(w) = \frac{|w|_b(|w|_b + 1)}{2} + \binom{w}{ab}.$$
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Remark. This proves a conjecture of Wiseman in OEIS A222955 (and A359402)

Complementary remark 1

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A word over $\{a, b\}$ is fair if and only if the average of position of b is $\frac{|w|+1}{2}$

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Hence $S_b(w)/|w|_b = \frac{(|w|+1)}{2}$

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Conversely if $S_b(w)/|w|_b = \frac{(|w|+1)}{2}$ then $\binom{w}{ab} = \frac{|w|_a|w|_b}{2} = \binom{w}{ba}$. $\square$

Complementary remark 2

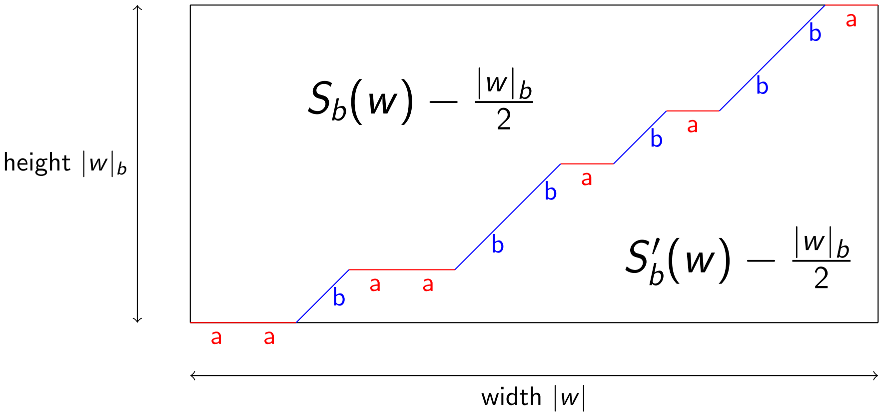

Figure: Geometrical interpretations of $S_b(w)$ and $S'_b(w)$

$S_b(w) = \sum_{pb \text{ prefix of } w} |pb|$ and $S'_b(w) = \sum_{bs \text{ suffix of } w} |bs|$

Contents

Rewriting using fair words

Let recall:

- $u \sim_2 v$ if and only if u and v are palindromic amiable [Atanasiu et al. 2002]
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Thus a natural definition.

fair amiable words

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Fact

$u \sim_2 v \Leftrightarrow u \equiv_{pa} v \Leftrightarrow u \equiv_{fair} v$

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  Hence the number of different fair words in $\mu^2(w)$ is in $\Omega(\#Fact(w))$

Number of different fair factors in a word

- [Droubay, Justin, Pirillo 2001:
  A word w has at most $|w| + 1$ different palindromic factors.

- What about the number of fair words ? Is the difference between the number of fair words and $|w|$ bounded?

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- One can also prove that the maximal difference between the number of fair factors and of palindromes in a word w can be in $\Theta(|w|^2)$.

Fair length

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- No longer true when considering the fair lengths instead of the palindromic lengths
  - Fair lengths of w = minimal k such that $w = \phi_1\phi_2\cdots\phi_k$ with ϕ_i fair
  - The fair lengths of factors of the Thue-More word are bounded by 4.

Contents

Some questions to conclude

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- The original Černý's question:

  Count the number of fair words over an arbitrary alphabet

  Are the following remarks useful?
  - A word over an alphabet is completely and uniquely defined by its projections on binary subalphabets
  - A word is fair if and only if its projections on binary alphabets are fair