On modulo-recurrence and window complexity in infinite words

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Definitions, notations

Let $\mathcal{A}$ be an alphabet. An infinite word u in $\mathcal{A}^\omega$, is written $u = u_0 u_1 u_2 u_3 \cdots$ where $u_i \in_A$, for all $i \geq 0$.
Let $u = u_0 u_1 u_2 u_3 \cdots$ be an infinite word.$\mathcal{F}_n(u)$ is the set of factors of u of length n:

$$\mathcal{F}_n(u) = \{u_k u_{k+1} \cdots u_{k+n-1} : k \geq 0\} \quad .$$

The set of all the factors of u is denoted $\mathcal{F}(u)$.
A factor v of length n of a word $u = u_0 u_1 u_2 \cdots$ appears in u at position k if $v = u_k u_{k+1} \cdots u_{k+n-1}$.

Definitions, notations

Definition

An infinite word u is said to be

- *recurrent if any factor of u appears infinitely many times in u,*
- *uniformly recurrent if for all $n \in \mathbb{N}$, there exists N such that any factor of u of length N contains any factor of u of length n.*

Definitions, notations

Definition

The complexity of an infinite word u *is the map from* $\mathbb{N}$ *to* $\mathbb{N}^*$ *defined by* $P_u(n) = \#\mathcal{F}_n(u)$, *where* $\#\mathcal{F}_n(u)$ *is the number of elements in* $\mathcal{F}_n(u)$.

Definition

A Sturmian word u *is an infinite word such that* $P_u(n) = n + 1$ *for all integer* $n \geq 0$.

Window complexity

A factor v appears in u at position $\equiv i \mod k$ (i modulo k) if there exists l_i such that $v = u_{l_i k+i} u_{l_i k+i+1} \cdots u_{l_i k+i+|v|-1}$.

Definition

A factor v of length $n \geq 2$ in an infinite word u is a n-window factor of u if v appears in u at a position multiple of n, $i.e \equiv 0 \mod n$.

Window complexity

The set of n-window factors of u, is denoted $\mathcal{F}_n^w(u)$:

$$\mathcal{F}_n^w(u) = \left\{ u_{kn} u_{kn+1} \cdots u_{(k+1)n-1} : k \geq 0 \right\} \quad .$$

Definition

Let $u = u_0 u_1 u_2 \cdots$ be an infinite word. The window complexity function of u is the map $P_u^w : \mathbb{N} \longrightarrow \mathbb{N}^*$ defined by

$$P_u^w(n) = \#\mathcal{F}_n^w(u) \quad .$$

Window complexity

For a fixed infinite word u, we have the following comparision:

$$P_u^w \leq P_u \tag{1}$$

Proposition (Kaboré, Tapsoba, 2007)

Let u be a Sturmian words. Then, we have $P_u^w = P_u$.

Modulo-recurrent words

The modulo-recurrent words are introduced in 2007.

Definition

An infinite word $u = u_0 u_1 u_2 \cdots$ *is said to be modulo-recurrent if, for any* $k \geq 1$, *every factor* w *of* u *appears in* u *at every position modulo* k, *i.e.,*

$$\forall i \in \{0, 1, \ldots, k-1\}, \exists l_i \in \mathbb{N} : w = u_{kl_i+i} u_{kl_i+i+1} \cdots u_{kl_i+i+|w|-1} \, .$$

Modulo-recurrent words

Examples:

- Unary word: $u = aaaaaaa\ldots$
- Sturmian words
- Champernowne word

Remark

The modulo-recurrent words are recurrent words.

Modulo-recurrent words

A characterization of modulo-recurrent words using window complexity:

Theorem (Cassaigne, Kaboré, Tapsoba, 2010)

Let u be a recurrent infinite word. Then, the following assertions are equivalent:

1. *The word u is modulo-recurrent.*
2. $\forall n \geq 0,\ P_u^w(n) = P_u(n)$.

Uniformly modulo-recurrent words

Definition

An infinite word $u = u_0 u_1 u_2 \cdots$ is said to be uniformly modulo-recurrent if, for any $k \geq 1$, $n \geq 0$ there exists N such that every factor w of u of length n appears in any factor v of u of length N at every position modulo k.

We have the following:

Proposition

Let u be an infinite word. Then, the following statements are equivalent

1. *u is uniformly recurrent and modulo-recurrent.*
2. *u is uniformly modulo-recurrent.*

Uniformly modulo-recurrent words

Proof.

- Clearly $(2) \Longrightarrow (1)$.
- Show that $(1) \Longrightarrow (2)$. Let $k \geq 1$ and $n \geq 0$. Let $w \in \mathcal{F}_n(u)$.

Since u is modulo-recurrent, for all $i \in \{0, \ldots, k-1\}$ there exists $m_{w,\,i} \in \mathbb{N}$ such that $w = u_{km_{w,\,i}+i} \cdots u_{km_{w,\,i}+i+n-1}$. So, take

$$m = \underbrace{\max}_{w \in \mathcal{F}_n(u),\, i \in \{0,\ldots,k-1\}} (km_{w,\,i} + i) \text{ and } z = u_0 u_1 \ldots u_{m+n-1}.$$

Then, for any $i \in \{0, \ldots, k-1\}$ and for any $w \in \mathcal{F}_n(u)$, z contains at least one occurrence of w at a position $\equiv i \mod k$. □

Uniformly modulo-recurrent words

Proof.

Furthermore, since u is uniformly recurrent, there exists $N \in \mathbb{N}$ such that, any factor v of length N contains also the factor z:

$$\forall v \in \mathcal{F}_N(u) : z \in \mathcal{F}(v).$$

Hence, for any $i \in \{0, \ldots, k-1\}$ and for any $w \in \mathcal{F}_n(u)$, v contains one occurrence of w at position $\equiv i \mod k$. So, u is uniformely modulo-recurrent. ☐

Uniformly modulo-recurrent words

Remark

1. *Modulo-recurrence does not imply uniform modulo-recurrence. For example, Champernowne word is modulo-recurrent but it is not uniformly recurrent.*
2. *Uniform recurrence does not imply uniform modulo-recurrence. For example, Thue-Morse word is uniformly recurrent but it is not modulo-recurrent.*

Strongly modulo-recurrent words

Definition

An infinite binary word u *is said to be strongly modulo-recurrent if for any morphism* σ *on* $\mathcal{A}^*$ *verifying the following property*

$$\forall a, b \in \mathcal{A} : a \neq b \Rightarrow gcd(|\sigma(a)|, |\sigma(b)|) = 1$$

then, the word $\sigma(u)$ *is modulo-recurrent.*

Theorem

Every Sturmian word is strongly modulo-recurrent.

Strongly modulo-recurrent words on binary alphabet

Proof.

Let u be a Sturmian word on $\mathcal{A} = \{a, b\}$ and τ a morphism on $\mathcal{A}^*$ such that $|\tau(a)|$ and $|\tau(b)|$ are coprime.
By Rauzy's pairs construction we can find a Sturmian morphism σ such that for all $x \in \mathcal{A}$, $|\sigma(x)| = |\tau(x)|$. So, $\sigma(u)$ is Sturmian. Thus, $\sigma(u)$ is modulo-recurrent. Let w be a factor of u. Then, for all $k > 1$ and $i \geq 0$, $\sigma(w)$ appears in $\sigma(u)$ at some position n with $n \equiv i \mod k$ since $\sigma(u)$ is modulo-recurrent. So, we can write $u = pws$ with p verifying $|\sigma(p)| = n$. Since, $\tau(u) = \tau(p)\tau(w)\tau(s)$ with $|\tau(p)| = n$, then $\tau(w)$ appears in $\tau(u)$ at the position n. $\square$

Strongly modulo-recurrent words

Theorem

Champernowne words are strongly modulo-recurrent.

Proof.

Let u be a Champernowne binary word and τ a morphism on $\mathcal{A}^*$ such that $gcd\left(|\tau(a)|, |\tau(b)|\right) = 1$. Let w be a factor of u, m and i some integers. Write $|\tau(a)| = x$, $|\tau(b)| = y$ and consider the set

$$E = \left\{|\tau(w)| + nx + py : n, \, p \in \mathbb{N}\right\}.$$

Since $gcd(x, y) = 1$, there exist $u, \, v \in \mathbb{Z}$ such that $ux + vy = 1$. So, the set E is such that $E = \mathbb{N} - H$ where H is a finite subset of $\mathbb{N}$. Thus, there exists n_0 such that for all $l \geq n_0$ one has $l \in E$. Let $j = |\tau(w)| + nx + py \in E$ such that $j \equiv 1 \mod m$. Since the word $w' = (wa^n b^p)^m$ occurs in u then $\tau(w')$ appears in $\tau(u)$ at some position m_0. It follows that $\tau(w)$ appears in $\tau(u)$ at the positions m_0, $m_0 + j$, $m_0 + 2j$, …, $m_0 + (m-1)j$. Among these positions, one is $\equiv i \mod m$ because $j \equiv 1 \mod m$. □

Strongly modulo-recurrent words

Remark

If a word u is strongly modulo-recurrent then u is modulo-recurrent.

Indeed, for any strongly modulo-recurrent word u, $Id_{\mathcal{A}}(u)$ is modulo-recurrent, where Id is the identity morphism.

About window complexity of the Thue-Morse word

The Thue-Morse word, $\mathbf{t}$, is one of the classical infinite words which has been the most studied. It is generated by the morphism $\theta : a \mapsto ab,\ b \mapsto ba$.

Since the Thue-Morse word is not modulo-recurrent then $P^w_{\mathbf{t}} \neq P_{\mathbf{t}}$. So, it seems interesting to look at finely the known inequality $P^w_{\mathbf{t}}(n) \leq P_{\mathbf{t}}(n)$ for all n.

About window complexity of the Thue-Morse word

One has:

Theorem

The window complexity of $\mathbf{t}$ *satisfies:*

$$\forall n \in \mathbb{N},\ P^w(n) = P\left(\frac{n}{2^{\upsilon_2(n)}}\right)$$

where $\upsilon_2(n)$ *denotes diadic valuation of* n*. Moreover, we have* $P^w(n) = P(n)$ *if* n *is odd.*

The proof of this Theorem uses the following wellknown property of $\mathbf{t}$:

Lemma

For all $j,\ n \in \mathbb{N}$ *we have* $\mathbf{t}_{2^j+n} = \overline{\mathbf{t}_n}$.

About window complexity of the Thue-Morse word

From the previous theorem we deduce that for the Thue-Morse word, the set $\Pi = \{n \in \mathbb{N} / P^w(n) = P(n)\}$ is infinite.
By this result and the inequality "$P^w \leq P$" it becomes interesting to ask the following question: Does there exist some words for which the window complexity satisfies:

$$\forall n \geq 2,\ \ P^w(n) < P(n)?$$

An example of recurrent word with bounded window-complexity

Here we provide a recurrent and non eventually periodic word with bounded window-complexity to give an answer to the above question.
Consider the sequence of positive integers (k_n) defined by $k_0 = 1$, and $k_{n+1} = (2k_i)!$ for all $n \geq 0$.
The first few terms of (k_n) are:

n	0	1	2	3	4	$\cdots$
k_n	1	2	24	$48!$	$(2 \times 48!)!$	$\cdots$

The sequence (k_n) is strictly increasing.

An example of recurrent word with bounded window-complexity

Lemma

$\forall i \geq 1 : \ k_i > \sum_{j=0}^{i-1} k_j$

Lemma

Let i *and* n *be two positive integers such that* $2k_{i-1} < n \leq 2k_i$*. Then* n *divides* k_{i+1}*.*

Proof.

Since $k_{n+1} = (2k_i)!$ and $n \leq 2k_i$ then n divides k_{n+1}. ☐

An example of recurrent word with bounded window-complexity

Now, define a sequence of words x_n related to the sequence (k_i) as follows: $x_0 = 1$ and $x_{i+1} = x_i x_i 0^{k_{i+1}-2k_i}$.

$$
\begin{aligned}
x_0 &= 1 \\
x_1 &= 11 \\
x_2 &= 11110^{20} \\
x_3 &= 11110^{20}11110^{48!-28} \\
x_4 &= 11110^{20}11110^{48!-28}11110^{20}11110^{[2(48!)]!-48!-28} \\
&\vdots
\end{aligned}
$$

The sequence (x_i) tends to an infinite word $u = u_0 u_1 u_2 \cdots$:

$$u = \lim x_n = 11110^{20}11110^{48!-28} \ldots$$

An example of recurrent word with bounded window-complexity

The word $u = u_0 u_1 u_2 \cdots$ can be defined equivently by $u_0 = 1$ and $u_n = 1$ when there exists $I \subset \mathbb{N}$ such that $n = \sum_{i \in I} k_i$ and $u_n = 0$ otherwise.

We have

$$\forall i \geq 0 : u \in \{x_i,\, 0^{k_i}\}^{\omega}.$$

An example of recurrent word with bounded window-complexity

Proposition

1. *The word u is recurrent*
2. *The word u is not eventually periodic.*

Indeed, it is sufficient to observe that x_i is a prefix of u and appears at least twice in u for all $i \geq 0$. The second statement is obtained by observing that u contains factors of the form $0^{k_i - 2k_{i-1}}$ where $(k_i - 2k_{i-1})$ is an unbounded sequence.

An example of recurrent word with bounded window-complexity

The following table gives the first few values of $\mathcal{F}_n^w(u)$ and $P_u^w(n)$.

n	$\mathcal{F}_n^w(u)$	$P_u^w(n)$
1	$\{0, 1\}$	2
2	$\{00, 11\}$	2
3	$\{000, 100, 111\}$	3
4	$\{0000, 1111\}$	2
5	$\{11110, 00000, 00001, 11100\}$	4
6	$\{111100, 000000\}$	2
7	$\{1111000, 0000000, 0001111\}$	4
$\vdots$	$\ldots$	$\ldots$

An example of recurrent word with bounded window-complexity

Proposition

The window complexity of u *satisfies*

$$\forall n \in \mathbb{N},\ P^w(n) \leq 4.$$

Corollaire

The window complexity of u *satisfies:*

$$\forall n \geq 2,\ P^w(u) < P(n).$$

Thank you for your attention!