# The Thue-Morse word in base $3/2$

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## I) Automatic words : "numeration system + automata"

$$- \text{base } 2 : \quad n = \sum_{k=0}^{\ell(n)} \underbrace{d_k(n)}_{\in\{0,1\}} 2^k$$

$$x_n := \mathrm{ADD2}(d_0(n) d_1(n) \dots d_{\ell(n)}(n)) := \sum_{k=0}^{\ell(n)} d_k(n) \pmod 2$$

$x \in \{0,1\}^{\mathbb{N}}$ = Thue-Morse word

$$- \text{base } 3/2 : \quad n = \sum_{k=0}^{\ell(n)} \underbrace{d'_k(n)}_{\in\{0,1,2\}} \frac{1}{2}\left(\frac{3}{2}\right)^k$$

Akiyama
Frougny 08
Sacarovitch
(Mahler's Z numbers)

$$t_n := \mathrm{ADD2}(d'_0(n) \dots d'_{\ell(n)}(n)) := \sum_{k=0}^{\ell(n)} d'_k(n) \pmod 2$$

$t \in \{0,1\}^{\mathbb{N}}$ = $\frac{3}{2}$-Thue Morse word.

**Conjecture** (Dekking): $t$ is uniformly recurrent and frequency of symbol "0" equals $1/2$.

... Many other natural questions and conjectures.

## II) Skew-substitutions

"$\frac{3}{2}$-automatic $\Rightarrow$ fixed point of uniform skew-substitution"

[General thm by Rigo-Stip]

Consider an action $A^* \curvearrowright S$ and $\tau : S \times A^* \longrightarrow A^*$ s.t.

$$\tau(s \mid b_0 b_1 \dots b_k) = \tau(s \mid b_0)\, \tau(s.b_0 \mid b_1)\, \tau(s.b_0 b_1 \mid b_2) \dots \tau(s.b_0 b_1 \cdots b_{k-1} \mid b_k)$$

$\rightarrow$ "apply and update state"

- transducer
- skew-product

$S = \{*\}$ → usual substitutions

our case: $S = \mathbb{Z}/2\mathbb{Z}$ with action $i.b = i+1 \pmod 2$, $\forall b \in B$.

"apply $\tau(0|\cdot)$ at even positions,
apply $\tau(1|\cdot)$ at odd positions"

$\tau(0|u)$

**Definition:** $x \in A^{\mathbb{N}}$ is a fixed point if $\tau(0|x) = x$.

**Ex 1.** $\sigma : \mathbb{Z}/2\mathbb{Z} \times \{1,2\}^* \to \{1,2\}^*$

$$\text{even}: \begin{cases} \sigma(0|1) = 1 \\ \sigma(0|2) = 11 \end{cases} \qquad \text{odd} \begin{cases} \sigma(1|1) = 2 \\ \sigma(1|2) = 22 \end{cases}$$

→ Kolakoski word = fixed point of $\sigma$

**Ex 2.** $(p,q)$-Toeplitz word are fixed points (with $S = \mathbb{Z}/p\mathbb{Z}$)

[Cassaigne Karhumäki]

**Ex 3.** there are skew-substitution whose fixed points have:
i) exponential factor complexity, or
ii) undecidable language.

[Endrullis Hendriks]

→ **Ex 4:** $\frac{3}{2}$-Thue-Morse

$A = \{0,1\}$

$$\underset{\text{"duplicate"}}{\text{even}}: \begin{cases} \theta(0|0) = 00 \\ \theta(0|1) = 11 \end{cases} \; 2 \qquad \underset{\text{"invert"}}{\text{odd}}: \begin{cases} \theta(1|0) = 1 \\ \theta(1|1) = 0 \end{cases} \; 1$$

uniform: $|\tau(s,a)| = \ell_s$

$$\mathbf{t} = 0\;0\;1\;1\;1\;0\;1\;1$$
$$\theta(0|\mathbf{t}) = 00\;1\;11\;0\;11\;1\;11\;0\ldots$$

**Thm** [Cassaigne, E, Rigo, Stipulanti]

- $\mathbf{t}$ is the fixed point of $\theta$ ← [Dekking '21]
- factor complexity $= \Theta(n^\alpha)$, $\alpha := \log 3 / \log 3/2 > 2$
- infinite topological rank (so, not morphic)
- $\mathbf{t}$ is uniformly recurrent.

## Main tool

ion

decoration

$$t = 0_{000} \quad 0_{101}\ 1_{011} \quad 1_{110}\,1_{001} \quad 0_{100}\ 1_{010} \quad 1_{111}\ 1_{000}$$

$$t = 0_{00}\ \ 0_{1}\ \ 1_{01}\ \ 1_{11}1_{0}\ \ 0_{10}\ \ 1_{01}\ \ 1_{1}\ \ 1_{00}\ \ 1_{10}1_{0}\ \ 0_{11}\ \ 1_{00}1_{1}$$

$$t = 0_0\, 0_1\, 1_0\, 1_1\, 1_0\, 0_1\, 1_0\, 1_1\, 1_0\, 1_1\, 1_0\, 0_1\, 1_0\, 1_1\, 0_0\, 1_1\, 1_0\, 1_1\, 1_0\, 1_1\, 0_0$$

→ $(\mathbb{Z}/2\mathbb{Z})^\infty$ = set of parity sequences

→ we get an action $T : A^* \curvearrowright (\mathbb{Z}/2\mathbb{Z})^\infty$

$$(z_0, z_1, \dots).b = (z_0 + 1,\ z_1 + |\tau(z_0|b)|,\ z_2 + |\tau(z_1|\tau(z_0|b))|, \dots)$$

**Thm** [Cassaigne, E, Rigo, Stipulanti] $T$ is conjugate to an odometer

i.e.

$$\begin{tikzcd} (\mathbb{Z}/2\mathbb{Z})^\infty \arrow[r, "T"] & (\mathbb{Z}/2\mathbb{Z})^\infty \\ \mathbb{Z}_2 \arrow[u] \arrow[r, "+1"'] & \mathbb{Z}_2 \arrow[u] \end{tikzcd}$$

($\mathbb{Z}_2$ = 2-adic integers)

- For Kolakoski: $\mathbb{Z}_2$ becomes a profinite completion of free group

**Cor.** $t$ is uniformly recurrent (+ much more)

## III) Frequencies

- general method for computing letter frequencies $\xrightarrow{\text{sliding block codes}}$ factor frequencies (shift-invariant measures)

**Thm** [C.E.R.S.] $\forall n \in \mathbb{N}, a \in A, i \in \mathbb{Z}$:

$$\mu_n(a,i) := \lim_{N\to\infty} \frac{2^n}{N} \#\{0 \le k < N : x_k = a,\ k \equiv i \pmod{2^n}\} = 1/2,$$

"equidistibution across residue classes and letters"

Cor. frequency of "0" in $t$ = 1/2 (Dekking's conjecture).

Proof.

Classical desubstitution argument gives

$$\boxed{\mu_n(a,3i) = \sum_{\substack{b\in A\\ 0\le \ell<3}} c_\ell \cdot \mu_{n+1}(b, 2i-\ell)}$$

PF-equation

$[c_0 = -c_1 = c_2 = 1/3]$

→ informed guess: $(\mu_n(a,i) = 1/2)_{n,a,i}$ is the unique solution

- $\vec{\mu}_n \in \mathbb{R}^{A\times(\mathbb{Z}/2\mathbb{Z})^n}$ → unbounded dimension
  → Perron Frobenius theory is difficult to apply

Strategy: i) lift to $\mathbb{Z}_2$
ii) pass to Fourier domain
iii) prove uniform contraction for associated operator.

Let $\bar{\mu}_n : \mathbb{Z}_2 \to \mathbb{R}^A$ [Continuous]
$z \mapsto (\mu_n(a, z_n))_{a\in A}$

$[\hat{\mathbb{Z}}_2 \cong \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}]$

general theory → $$\bar{\mu}_n(z) = \sum_{\chi\in\hat{\mathbb{Z}}_2} \underbrace{\hat{\mu}_n(\chi)}_{\in\, \mathbb{C}^A} \underbrace{\chi(z)}_{\chi:\, \mathbb{Z}_2\to\mathbb{C}}$$

→ $$\hat{\mu}_n(3\chi) = \sum_{\substack{\zeta\in\hat{\mathbb{Z}}_2\\ 2\zeta=\chi}} \underbrace{\Phi(\zeta)}_{A\times A \text{ matrix}} \cdot \hat{\mu}_{n+1}(\zeta)$$ ← sum has 2 terms.

→ $$\boxed{\hat{\mu}_n(3^k\chi) = \sum_{\substack{\zeta\in\hat{\mathbb{Z}}_2\\ 2^k\zeta=\chi}} \Phi_k(\zeta)\cdot\hat{\mu}_{n+k}(\zeta),}$$ $\forall k\in\mathbb{N}$

← sum has $2^k$ elements.

Fourier PF-equation

trick: $\mu_n(a,i) = 1/2$ unique solution
iff

$S_n(z)=0$ unique solution for

$$\boxed{\hat{S}_n(3^k r) = \sum_{\substack{s\in\hat{\mathbb{Z}}_2 \\ 2^k s = r}} \Psi_k(s)\, \hat{S}_{n+k}(s)}$$

$[S_n(z) = \bar{\mu}_n(0,z) - \bar{\mu}_n(1,z)]$

$\downarrow$ implies

$$\|\hat{S}_n\|_{L^2}^2 \leq \|\Psi_k\|_\infty \cdot \|\hat{S}_{n+k}\|_{L^2}^2 , \quad \forall n,k\in\mathbb{N}.$$

(contraction inequality)

We showed $\|\Psi_2\|_\infty < 1$,

so contraction inequality forces $\hat{S}_n \equiv 0, \forall n$

Conclusion: $\mu_n(a,i) = 1/2, \ \forall n\in\mathbb{N}, i\in\mathbb{Z}, a\in A.$

# IV) Perspectives

i) our results show dynamical disjointness:

$\frac{3}{2}$-Thue-Morse subshift is top. disjoint from $(\mathbb{Z}_2, +1)$

→ as measure-theoretical systems is tougher.

ii) As long as the induced action $T: B^* \curvearrowright S^\infty$ is abelian,
the situation seems to be reasonably tractable.
$[\, s.uv = s.vu, \ \forall u,v \in B^*, \ \forall s \in S^\infty \,]$

iii) Many questions:

- natural classes of fixed points of skew-substitutions?
- study of other properties?
  - → dimension group
  - → infinite special words
  - → automorphism group