

New examples of words for which the binomial complexities and the subword complexity coincide

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I. Definitions and motivations

II. Results: old and new

III. Two key lemmas

IV. Perspectives and open questions

I. Definitions and motivations

1) Basic definitions

- \mathcal{A} = finite set of symbols called **alphabet**.
- $\mathcal{A}^* := \bigcup_{n \in \mathbb{N}} \mathcal{A}^n$ = set of **finite words** written on \mathcal{A} .
- $\mathcal{A}^{\mathbb{N}}$ = set of (right) **infinite words** written on \mathcal{A} .

Def: - $u \in \mathcal{A}^*$ is a **factor** of $w \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$ if u occurs in w as a block of consecutive letters.

- $\mathcal{L}(w)$ = set of factors of w

- $\mathcal{L}_n(w) := \mathcal{L}(w) \cap \mathcal{A}^n$ = set of length n factors of w .

Def: The **subword complexity** of the infinite word $w \in \mathcal{A}^{\mathbb{N}}$ is the function:

$$\begin{aligned} p_w : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto \#\mathcal{L}_n(w) \end{aligned}$$

Def: Let $C \geq 0$. The infinite word $w \in \mathcal{A}^{\mathbb{N}}$ is **C-balanced** if, for every equally long factors $u, v \in \mathcal{L}(w)$ and every letter $a \in \mathcal{A}$,

$$||u|_a - |v|_a| \leq C.$$

2) k -binomial equivalence

- **Scattered factor** = subword with non-necessarily consecutive letters.

- $\binom{u}{x}$ = nb of occurrences of the scattered factor $x \in \mathcal{A}^*$ in $u \in \mathcal{A}^*$.

Ex: - 212 is a scattered factor of 21121, but 122 is not.

$$- \binom{21121}{11} = 3.$$

Def: Let $k \geq 1$. Two finite words $u, v \in \mathcal{A}^*$ are said to be **k -binomially equivalent** if, for every $x \in \mathcal{A}^{\leq k}$,

$$\binom{u}{x} = \binom{v}{x}.$$

Ex: 12211 \sim_2 21121 but 12211 $\not\sim_3$ 21121.

Rk: 1) For $k = 1$, 1-binomial equivalence = abelian equivalence.

2) $(k + 1)$ -binomial equivalence \Rightarrow k -binomial equivalence.

3) u, v with length $\leq k$ are k -binomial equivalent $\iff u = v$.

3) k -binomial complexity

Def: Let $k \geq 1$. The **k -binomial complexity** of the infinite word $w \in \mathcal{A}^{\mathbb{N}}$ is the function:

$$b_w^k : \mathbb{N} \longrightarrow \mathbb{N}$$

$$n \longmapsto \# \mathcal{L}_n(w) / \sim_k$$

Rk: 1) For $k = 1$, $b_w^1 =$ abelian complexity of w .

2) For every $k \geq 1$, $b_w^k \leq b_w^{k+1} \leq p_w$.

3) For every $k \geq 1$ and every $n \leq k$, $b_w^k(n) = p_w(n)$. Consequently, $b_w^k \rightarrow p_w$ as $k \rightarrow \infty$ (pointwise convergence).

Csq: k -binomial complexities form a scale from the abelian complexity to the subword complexity.

4) An important example

Theorem [Rigo, Salimov 15]

The 2-binomial complexity of any binary 1-balanced word coincides with its subword complexity.

Rk: In other words, for $w \in \mathcal{A}^{\mathbb{N}}$ a binary 1-balanced word, and $u, v \in \mathcal{L}(w)$

$$u \sim_2 v \iff u = v.$$

Rk: For binary 1-balanced words, the binomial complexity scale collapses!

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Main objective of this talk

Study of the words for which, as Sturmian words, their k -binomial complexity and subword complexity coincide for some (small) k .

II. Results: old and new

1) Words whose abelian complexity coincides with their subword complexity

Proposition [Rigo, Stipulanti, Whiteland 24]

The abelian/ 1 -binomial complexity of an infinite word w coincides with its subword complexity if and only if there exist d **distinct** letters a_1, \dots, a_d and $(d - 1)$ positive integers k_1, \dots, k_{d-1} such that

$$w = a_1^{k_1} a_2^{k_2} \dots a_{d-1}^{k_{d-1}} a_d^\omega,$$

where a_d^ω denotes the constant infinite word $a_d a_d a_d a_d \dots$

Proof. \Leftarrow Let $u \in \mathcal{L}(w)$. Remark that its abelian class uniquely determine its position in w .

\Rightarrow Let $u = au'a \in \mathcal{L}(w)$ with $a \in \mathcal{A}$. Then $au' \sim_1 u'a$, so $au' = u'a$, i.e., u' starts and ends by a . After finitely many iterations: $u = a^{|u|}$.

2) Words whose 2-binomial complexity coincides with their subword complexity (I)

Rk: If $b_w^1 = p_w$, then $b_w^2 = p_w$.

Theorem [Rigo, Salimov 15]

The 2-binomial complexity of any binary 1-balanced word coincides with its subword complexity.

Theorem [Lejeune, Rigo, Rosenfeld 20]

The 2-binomial complexity of the Tribonacci word $w_{\text{tribo}} = 12131211121312\dots$ (fixed point of $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$) coincides with its subword complexity.

Rk: Their proof is computer-assisted and based on the method of *templates*.

Rk: Their proof could be adapted to obtain an algorithm deciding whether the k -binomial complexity of a (purely) morphic word coincides with its subword complexity (might require some restrictions on the morphisms).

Conjecture [Lejeune, Rigo, Rosenfeld 20]

The 2-binomial complexity of any Arnoux-Rauzy word coincides with its subword complexity.

2) Words whose 2-binomial complexity coincides with their subword complexity (II)

Theorem [V. 25]

Let $d \geq 2$.

- i. If w is a 1-balanced d -ary word, then its 2-binomial complexity is equal to its subword complexity.
- ii. If w has subword complexity $n \in \mathbb{N}_{>0} \mapsto n + (d - 1)$, then its 2-binomial complexity is equal to its subword complexity.
- iii. If w is a hypercubic billiard word in dimension d , then its 2-binomial complexity is equal to its subword complexity.
- iv. If w is obtained as the coloring of a Sturmian word with another Sturmian word, then its 2-binomial complexity is equal to its subword complexity.

Rk: As Arnoux-Rauzy words, all these words can be thought of as generalizations of Sturmian words.

Rk: Colorings of Sturmian words with another Sturmian word have recently been introduced by Dvořáková, Masáková and Pelantová (2024).

3) Words whose k -binomial complexity coincides with their subword complexity

Rk: If $b_w^2 = p_w$, then $b_w^k = p_w$ for every $k \geq 3$.

Theorem [Rigo, Stipulanti, Whiteland 24]

For every $k \geq 3$, there exists a binary infinite word w whose k -binomial complexity, but not its $(k-1)$ -binomial complexity, coincides with its subword complexity.

More precisely, if $\sigma : 1 \mapsto 12, 2 \mapsto 21$ is the Thue-Morse substitution, then for every $k \geq 2$ and every **Sturmian word** $w \in \{1, 2\}^{\mathbb{N}}$, the k -binomial complexity of $\sigma^{k-2}(w)$ coincides with its subword complexity, but not its $(k-1)$ -binomial complexity.

III. The two key lemmas (and applications)

1) The first key lemma

Notation: For $\mathcal{B} \subseteq \mathcal{A}$, $\pi_{\mathcal{B}} : \mathcal{A} \rightarrow \mathcal{A}$ denotes the substitution defined by

$$\pi_{\mathcal{B}}(a) = \begin{cases} a & \text{if } a \in \mathcal{B}, \\ \epsilon \text{ (the empty word)} & \text{otherwise.} \end{cases}$$

Ex: $\pi_{2,3}(121312) = 232$.

Key Lemma 1

Let $w \in \mathcal{A}^{\mathbb{N}}$ and $k \geq 1$. If, for every pair of distinct letters $i, j \in \mathcal{A}$, $b_{\pi_{i,j}(w)}^k = p_{\pi_{i,j}(w)}$, then $b_w^k = p_w$.

Corollary

Let $w \in \mathcal{A}^{\mathbb{N}}$. If, for every pair of distinct letters $i, j \in \mathcal{A}$, $\pi_{i,j}(w)$ is a 1-balanced word, then $b_w^2 = p_w$.

2) Proof of Key Lemma 1

Let $u, v \in \mathcal{L}(w)$ and assume that $u \sim_k v$. The goal is to show that $u = v$.

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- **Step 1.** For every pair of distinct letters $i, j \in \mathcal{A}$ and every finite word $x \in \{i, j\}^{\leq k} \subseteq \mathcal{A}^{\leq k}$,

$$\binom{\pi_{i,j}(u)}{x} = \binom{u}{x} = \binom{v}{x} = \binom{\pi_{i,j}(v)}{x}.$$

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- **Step 2.** A word is uniquely determined by all its binary projections, so $u = v$.

Ex: If $\pi_{1,2}(u) = 11212211$,

$$\pi_{1,3}(u) = 11133311,$$

$$\pi_{2,3}(u) = 232332,$$

then $u = ??$

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3) Application 1: d -ary 1-balanced words

Lemma

If $w \in \mathcal{A}^{\mathbb{N}}$ is C -balanced, then for every pair of distinct letters $i, j \in \mathcal{A}$, $\pi_{i,j}(w)$ is also C -balanced.

Csq: If w is 1-balanced, then $b_w^2 = p_w$.

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- Then there exist $u, v \in \mathcal{L}(w)$ and $a \in \{i, j\}$ such that $|\pi_{i,j}(u)| = |\pi_{i,j}(v)|$ and $|\pi_{i,j}(u)|_a - |\pi_{i,j}(v)|_a \geq C + 1$.

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- Since $\pi_{i,j}(u), \pi_{i,j}(v)$ are binary words, $|\pi_{i,j}(v)|_b - |\pi_{i,j}(u)|_b = |\pi_{i,j}(u)|_a - |\pi_{i,j}(v)|_a \geq C + 1$ (where $b \in \{i, j\} \setminus \{a\}$).

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- If $v = ps$ with $|p| = |u|$, then

$$|u|_a - |p|_a \geq |u|_a - |v|_a = |\pi_{i,j}(u)|_a - |\pi_{i,j}(v)|_a \geq C + 1.$$

A contradiction.

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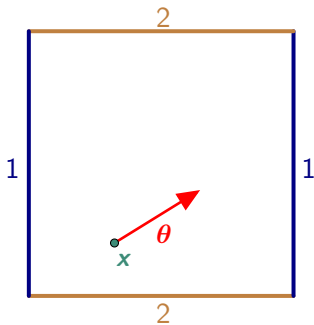
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- If $u = ps$ with $|p| = |v|$, then

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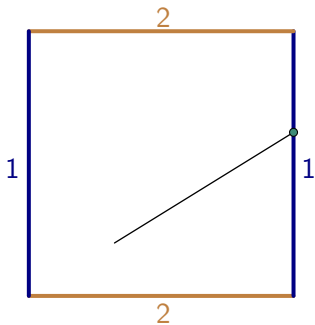
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4) Application 2: hypercubic billiard words



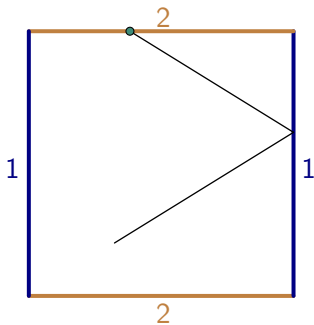
$$w(x, \theta) =$$

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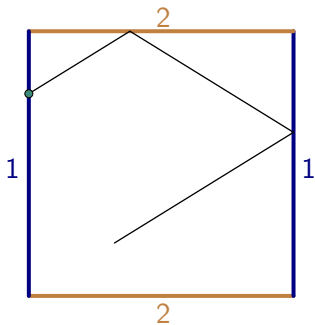
$$w(x, \theta) = 1$$

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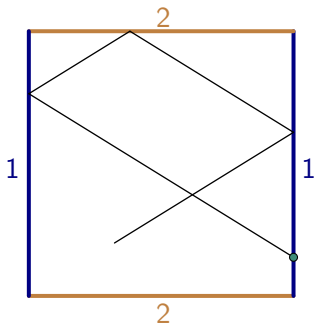
$$w(x, \theta) = 12$$

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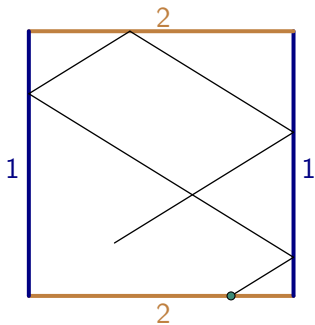
$$w(x, \theta) = 121$$

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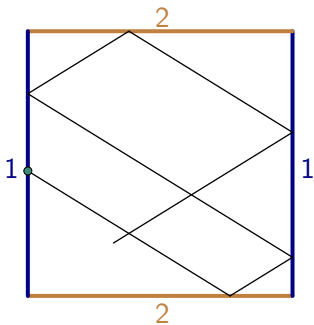
$$w(x, \theta) = 1211$$

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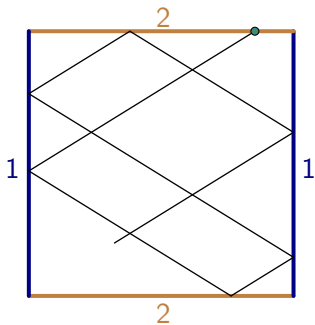
$$w(x, \theta) = 12112$$

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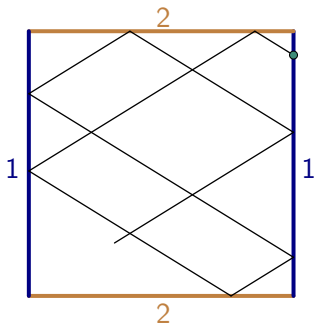
$$w(x, \theta) = 121121$$

4) Application 2: hypercubic billiard words



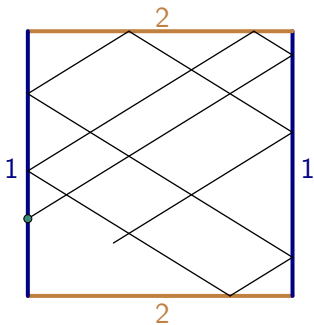
$$w(x, \theta) = 1211212$$

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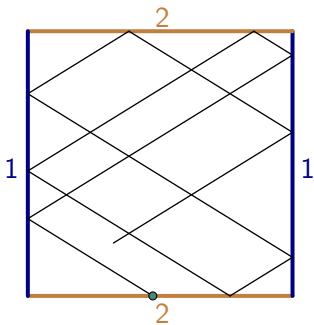
$$w(x, \theta) = 12112121$$

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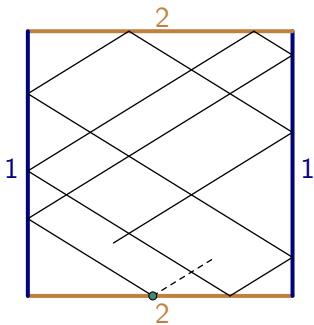
$$w(x, \theta) = 121121211$$

4) Application 2: hypercubic billiard words



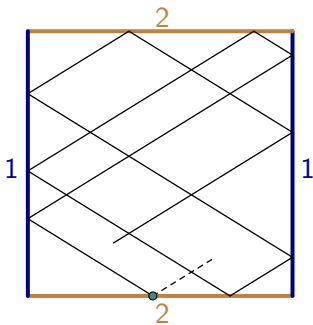
$$w(x, \theta) = 1211212112$$

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$$w(x, \theta) = 1211212112\dots$$

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Fact 1: Square billiard words are 1-balanced.

Fact 2: If $w \in \{1, \dots, d\}^{\mathbb{N}}$ is a hypercubic billiard word, then for every pair of distinct letters $i, j \in \{1, \dots, d\}$, $\pi_{i,j}(w)$ is a square billiard word.

Csq: If w is a hypercubic billiard word, then $b_w^2 = p_w$.

5) Application 3: words with subword complexity $n \mapsto n + (d - 1)$

Lemma [Ferenczi, Mauduit, 97]

Let \mathcal{A} be a d -ary alphabet with $d \geq 3$, and let $w \in \mathcal{A}^{\mathbb{N}}$ be a word with subword complexity $n \in \mathbb{N}_{>0} \mapsto n + (d - 1)$.

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(i) If w is recurrent, then there exist:

- a Sturmian word $w_0 \in \{1, 2\}^{\mathbb{N}}$,
- a partition $\mathcal{A} = \mathcal{B} \sqcup \mathcal{C} \sqcup \mathcal{D}$ where $\mathcal{B} = \{b_1, \dots, b_{N_{\mathcal{B}}}\}$, $\mathcal{C} = \{c_1, \dots, c_{N_{\mathcal{C}}}\}$ and $\mathcal{D} = \{d_1, \dots, d_{N_{\mathcal{D}}}\}$, with both $\mathcal{B} \neq \emptyset$ and $\mathcal{C} \sqcup \mathcal{D} \neq \emptyset$,
- and an integer $k \in \mathbb{N}$,

such that $w = S^k(\sigma(w_0))$, where S denotes the shift and $\sigma : \{1, 2\} \rightarrow \mathcal{A}^*$ is the substitution

$$1 \mapsto b_1 \dots b_{N_{\mathcal{B}}} c_1 \dots c_{N_{\mathcal{C}}}, \quad 2 \mapsto b_1 \dots b_{N_{\mathcal{B}}} d_1 \dots d_{N_{\mathcal{D}}}.$$

5) Application 3: words with subword complexity $n \mapsto n + (d - 1)$

Lemma [Ferenczi, Mauduit, 97]

Let \mathcal{A} be a d -ary alphabet with $d \geq 3$, and let $w \in \mathcal{A}^{\mathbb{N}}$ be a word with subword complexity $n \in \mathbb{N}_{>0} \mapsto n + (d - 1)$.

(i) If w is recurrent, then there exist:

- a Sturmian word $w_0 \in \{1, 2\}^{\mathbb{N}}$,
- a partition $\mathcal{A} = \mathcal{B} \sqcup \mathcal{C} \sqcup \mathcal{D}$ where $\mathcal{B} = \{b_1, \dots, b_{N_{\mathcal{B}}}\}$, $\mathcal{C} = \{c_1, \dots, c_{N_{\mathcal{C}}}\}$ and $\mathcal{D} = \{d_1, \dots, d_{N_{\mathcal{D}}}\}$, with both $\mathcal{B} \neq \emptyset$ and $\mathcal{C} \sqcup \mathcal{D} \neq \emptyset$,
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(ii) If w is not recurrent, then there exist $1 \leq d' < d$, a d' -letter sub-alphabet $\mathcal{B} \subset \mathcal{A}$, and a recurrent word $w_0 \in \mathcal{B}^{\mathbb{N}}$ with subword complexity $n \in \mathbb{N}_{>0} \mapsto n + (d' - 1)$ such that

$$w = a_1 \dots a_{d-d'} w_0,$$

where the letters a_i are the $d - d'$ distinct elements of $\mathcal{A} \setminus \mathcal{B}$.

5) Application 3: words with subword complexity $n \mapsto n + (d - 1)$

Lemma

If $p_w(n) = n + (d - 1)$ for every $n \in \mathbb{N}_{>0}$, then for every pair of distinct letters $i, j \in \{1, \dots, d\}$, $\pi_{i,j}(w)$ is 1-balanced.

Proof (in the recurrent case). $w = S^k(\sigma(w_0))$ so $\pi_{i,j}(w) = S^{k'}(\pi_{i,j} \circ \sigma(w_0))$.

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Case 1: $i, j \in \mathcal{B}$ or \mathcal{C} or \mathcal{D} . Then

$$\begin{aligned}\pi_{i,j} \circ \sigma(1) &= ij \text{ (if } i, j \in \mathcal{B} \text{ or } \mathcal{C}) && \text{or } \epsilon \text{ (if } i, j \in \mathcal{D}), \\ \pi_{i,j} \circ \sigma(2) &= ij \text{ (if } i, j \in \mathcal{B} \text{ or } \mathcal{D}) && \text{or } \epsilon \text{ (if } i, j \in \mathcal{C}).\end{aligned}$$

Thus $\pi_{i,j}(w)$ is the periodic word of period ij or ji .

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Case 2: $i \in \mathcal{C}$ and $j \in \mathcal{D}$. Then

$$\pi_{i,j} \circ \sigma(1) = i \quad \text{and} \quad \pi_{i,j} \circ \sigma(2) = j.$$

Thus $\pi_{i,j}(w)$ is Sturmian.

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Case 3: $i \in \mathcal{B}$ and $j \in \mathcal{C}$ or \mathcal{D} . Then

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Thus $\pi_{i,j} \circ \sigma$ is a Sturmian morphism, so $\pi_{i,j}(w)$ is Sturmian.

6) Application 4: the Tribonacci word?

$w_{\text{tribo}} = 121312112131212131211121312131211121212121212111213121 \dots$

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Fact 1: None of the three binary projections of the Tribonacci word is 1-balanced.

Ex: $u = 11211211211211$, $v = 21211211211212 \in \mathcal{L}_{14}(\pi_{1,2}(w_{\text{tribo}}))$
and $|u|_1 - |v|_1 = 2$,
 $u = 1111$, $v = 3113 \in \mathcal{L}_4(\pi_{1,3}(w_{\text{tribo}}))$ and $|u|_1 - |v|_1 = 2$,
 $u = 22322322322322$, $v = 32322322322323 \in \mathcal{L}_{14}(\pi_{2,3}(w_{\text{tribo}}))$
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Fact 2: None of the three binary projections of the Tribonacci word has its 2-binomial complexity equal to its subword complexity.

Ex: $u = 2112112112112112$, $v = 1212112112112121 \in \mathcal{L}_{16}(\pi_{1,2}(w_{\text{tribo}}))$
 and $u \sim_2 v$,
 $u = 311113$, $v = 131131 \in \mathcal{L}_6(\pi_{1,3}(w_{\text{tribo}}))$ and $u \sim_2 v$,
 $u = 3223223223223223$, $v = 2323223223223232 \in \mathcal{L}_{16}(\pi_{2,3}(w_{\text{tribo}}))$
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7) Coloring of a word

Coloring of a word. Let $w = 010010100100101001010\dots \in \{0, 1\}^{\mathbb{N}}$ be the Fibonacci word. Then

$$w' = 010010100100101001010\dots$$

is a new word, written on the ternary alphabet $\{0, 1, 1\}$, and build from w by coloring the letter 1 in **blue** and **brown**.

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Formally. Let \mathcal{A} and \mathcal{B} be **disjoint** alphabets. Let $w_0 \in \mathcal{A}^{\mathbb{N}}$, $w_1 \in \mathcal{B}^{\mathbb{N}}$ and $a \in \mathcal{A}$. The **coloring of the letter a in w_0 by w_1** is the word $\text{color}(w_0, a, w_1) \in (\mathcal{A} \sqcup \mathcal{B} \setminus \{a\})^{\mathbb{N}}$ defined as follows: for every $n \in \mathbb{N}$,

$$\text{color}(w_0, a, w_1)[n] := \begin{cases} w_0[n] & \text{if } w_0[n] \neq a, \\ w_1[k-1] & \text{if } w_0[n] = a \text{ and } |w_0[0:n]|_a = k. \end{cases}$$

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8) The second key lemma

Key Lemma 2

Let \mathcal{A} and \mathcal{B} be disjoint alphabets. Let $w_0 \in \mathcal{A}^{\mathbb{N}}$ and $w_1 \in \mathcal{B}^{\mathbb{N}}$. If there exists $k \geq 1$ such that $b_{w_0}^k = p_{w_0}$ and $b_{w_1}^k = p_{w_1}$, then for every letter $a \in \mathcal{A}$, the k -binomial complexity of $\text{color}(w_0, a, w_1)$ coincides with its subword complexity.

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C-ex: The Tribonacci word.

9) Preliminary remarks

Let $w := \text{color}(w_0, a, w_1) \in (\mathcal{A} \sqcup \mathcal{B} \setminus \{a\})^{\mathbb{N}}$ with $w_0 \in \mathcal{A}^{\mathbb{N}}$ and $w_1 \in \mathcal{B}^{\mathbb{N}}$.

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Rk3: $w = \text{color}(\sigma(w), a, \pi_{\mathcal{B}}(w))$ and for every $u \in \mathcal{L}(w)$,
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10) Proof of the second key lemma

Let $w = \text{color}(w_0, a, w_1) \in (\mathcal{A} \sqcup \mathcal{B} \setminus \{a\})^{\mathbb{N}}$ with $w_0 \in \mathcal{A}^{\mathbb{N}}$ and $w_1 \in \mathcal{B}^{\mathbb{N}}$ be such that $b_{w_0}^k = p_{w_0}$ and $b_{w_1}^k = p_{w_1}$.

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- **Step 2.** The following identity holds: for every $x \in \mathcal{A}^*$,

$$\binom{\sigma(u)}{x} = \sum_{\substack{y \in (\mathcal{A} \sqcup \mathcal{B} \setminus \{a\})^{|x|} \\ \sigma(y) = x}} \binom{u}{y}.$$

10) Proof of the second key lemma

Let $w = \text{color}(w_0, a, w_1) \in (\mathcal{A} \sqcup \mathcal{B} \setminus \{a\})^{\mathbb{N}}$ with $w_0 \in \mathcal{A}^{\mathbb{N}}$ and $w_1 \in \mathcal{B}^{\mathbb{N}}$ be such that $b_{w_0}^k = p_{w_0}$ and $b_{w_1}^k = p_{w_1}$.

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Consequently $\sigma(u) \sim_k \sigma(v)$, and then $\sigma(u) = \sigma(v)$.

- **Conclusion.**

$$u = \text{color}(\sigma(u), a, \pi_{\mathcal{B}}(u)) = \text{color}(\sigma(v), a, \pi_{\mathcal{B}}(v)) = v$$

IV. Perspectives and open questions

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The 2-binomial complexity of any binary 1-balanced word coincides with its subword complexity.

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Open Question 1

What about the converse under some additional assumptions?

2) The general case

Important remark

All known *interesting* words whose 2-binomial complexity coincides with their subword complexity are strongly related to Sturmian words. In particular, they can all be considered as genuine generalizations of Sturmian words on d -ary alphabets.

Open Question 2

Is there a natural class of words that can be seen as a *genuine generalization of Sturmian words*, for which the 2-binomial complexity of at least one (resp. all) of its elements differs from its subword complexity?

Open Question 3

In the opposite direction, is there a class of recurrent words, *genuinely unrelated to Sturmian words*, such that the 2-binomial complexity of at least one (resp. all) of its elements coincides with its subword complexity?

3) An example: Cassaigne-Selmer words

Let $c_1, c_2 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}^*$ be the substitutions defined by

$$c_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 13 \\ 3 \mapsto 2 \end{cases} \quad \text{and} \quad c_2 : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 13 \\ 3 \mapsto 3 \end{cases}$$

Def: $w \in \{1, 2, 3\}^{\mathbb{N}}$ is a Cassaigne-Selmer word (or a \mathcal{C} -adic word) if there exists a directive sequence $(s_n)_n \subset \{c_1, c_2\}^{\mathbb{N}}$ such that:

- both substitutions c_1 and c_2 occur infinitely often (+ another technical condition)
- $s_1 \circ s_2 \circ \dots \circ s_n(1) \xrightarrow{n \rightarrow \infty} w$.

Rk: The substitutions c_1 and c_2 are build from a *multidimensional continued fraction algorithm* introduced by Cassaigne in 2015.

Numerical experiments

Up to length $n = 99$, each factor of a Cassaigne-Selmer word generated by a periodic directive sequence with period at most 5 is alone in its 2-binomial class.

Rk: As Arnoux-Rauzy words, Cassaigne-Selmer words are dendric.

Thank you!