Inverse scattering by interfaces

Mourad Sini *

Abstract

The scattering of time-harmonic acoustic or electromagnetic waves by an obstacle $D \subset \mathbb{R}^n$, $n = 2, 3$, with smooth boundary, reads as follows. Let u_i be an incident wave, i.e. a solution of $\Delta u^i(x) + \kappa^2 u^i(x) = 0,\ x \in \mathbb{R}^n$. We denote by u^s the scattered wave corresponding to the incident wave u^i. The total wave $u := u^i + u^s$ solves one of the two following problems:

1. *The impenetrable obstacle scattering problem* with impedance boundary condition

$$\Delta u(x) + \kappa^2 u(x) = 0,\ x \in \mathbb{R}^n \setminus \bar{D}, \tag{1}$$

$$\left[\frac{\partial}{\partial \nu} + i\lambda(x)\right] u(x) = 0,\ x \in \partial D, \tag{2}$$

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left[\frac{\partial u^s(x)}{\partial r} - i\kappa u^s(x)\right] = 0,\ r = |x|, \tag{3}$$

where κ is called the wavenumber, $\lambda(x) \in L^\infty(\partial D)$ is referred to as the impedance of the boundary of the obstacle and ν is the outward unit normal vector of the boundary ∂D.

2. *The penetrable obstacle scattering*

$$\nabla \cdot \gamma(x)\nabla u(x) + \kappa^2 n(x)u(x) = 0,\ x \in \mathbb{R}^n, \tag{4}$$

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left[\frac{\partial u^s(x)}{\partial r} - i\kappa u^s(x)\right] = 0,\ r = |x|, \tag{5}$$

where γ is the permittivity and n is the permeability, modeling for instance the propagation of electromagnetic waves. They can also model the acoustic propagation where $\gamma(x)$ is constant and n is the index of refraction. We consider the obstacle D placed in a homogeneous medium. In this case, these coefficients can be represented in forms: $\gamma(x) := (1 + \gamma_D(x)\chi_D(x))$, and $n = 1 + n_D(x)\chi_D(x)$, where $\gamma_D \in L^\infty(D)$, $n_D \in L^\infty(D)$, and χ_D is the characteristic function of D. We assume that $\gamma_D \geq C > 0$.

In the system (1)–(3) and (4)–(5), the equations (3) and (5) are called the Sommerfeld radiation conditions, which guarantee the unique solvability of problems (1)–(2) and (4).

The asymptotic behavior of the scattered field u^s at infinity can be represented by

$$u^s(x) = \frac{e^{i\kappa r}}{r^{\frac{n-1}{2}}} u^\infty(\hat{x}) + O(r^{-\frac{n+1}{2}}),\ r \to \infty, \tag{6}$$

where $\hat{x} := x/r$ and u^∞ is an analytic function on the unit sphere $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ referred to as the *far field pattern* of the scattered field u^s. In the case of an incident plane wave, i.e., $u^i(x) = e^{i\kappa x\cdot d}$, with $d \in \mathbb{S}^{n-1}$ being the direction of incidence, we denote the far field pattern by $u^\infty(\hat{x}, d)$ to indicate its dependence on the incident direction d.

The problems (1)–(3) and (4)–(5), which are referred to as forward scattering problems, are well-posed (see, e.g. [3]). We consider the following problem:

Problem. *Reconstruct the shape of the obstacle D from the measured far field pattern $u^\infty(\hat{x}, d)$ for all incident directions $d \in \mathbb{S}^{n-1}$ and propagation directions $\hat{x} \in \mathbb{S}^{n-1}$.*

*RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria.

One way of solving the problem is by constructing special solutions for the models mentioned above and then extract the information about the interface from their behaviors. There are two types of such solutions introduced in the literature. We cite first the Green's type solutions proposed by Isakov [5] in the mid 80's. Based on this type of solutions, several reconstruction methods have been proposed in the mid 90s, known as the sampling methods (Linear sampling, Factorization, MUSIC, etc.), see [1] [2], [6] and the probing methods, see [4], [7] for details. The second type of solutions are the complex geometrical optics solutions (CGO solutions in short), firstly introduced by Sylvester and Uhlmann in the mid 80's, see [8].

After explaining how these special solutions have been used to build up the mentioned methods, we discuss briefly some relations between the ones using Green's type solutions. As a next step, we address the question of the accuracy of the reconstruction methods. To our understanding the parameters involved in the accuracy are listed as follows:

- Use of regularization (due to the ill posed nature of the problem),
- The used frequency κ,
- The geometry of the surface of the obstacle and the material surrounding it,
- The type of the used waves (this is mainly for models governed by a multivelocity systems such as elasticity).

These points will be discussed, briefly also!, regarding the two families of methods mentioned above. We will see that there is a clear distinction between Green's type solutions and CGO solutions regarding the stability issue, for instance. The second half of the talk will be devoted to the mathematical analysis of the methods using CGO solutions and, if time allows it, we will give some results related to the elasticity models where the existence of two types of waves (Pressure and Shear waves) makes the analysis reacher. Finally, we list a couple of open questions.

References

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