DOI 10.1070/IM2015v079n06ABEH002775

Infinite determinantal measures and the ergodic decomposition of infinite Pickrell measures. I. Construction of infinite determinantal measures

A. I. Bufetov

Abstract. This paper is the first in a series of three. We give an explicit description of the ergodic decomposition of infinite Pickrell measures on the space of infinite complex matrices. A key role is played by the construction of σ-finite analogues of determinantal measures on spaces of configurations, including the infinite Bessel process, a scaling limit of the σ-finite analogues of the Jacobi orthogonal polynomial ensembles. Our main result identifies the infinite Bessel process with the decomposing measure of an infinite Pickrell measure.

Keywords: determinantal processes, infinite determinantal measures, ergodic decomposition, infinite harmonic analysis, infinite unitary group, scaling limits, Jacobi polynomials, Harish-Chandra–Itzykson–Zuber orbit integral.

§ 1. Introduction

1.1. Informal outline of main results. The Pickrell family of measures is given by the formula

$$\mu_n^{(s)} = \mathrm{const}_{n,s} \det(1+z^*z)^{-2n-s}\, dz.$$

Here n is a positive integer, s a real number, z an $n\times n$ matrix with complex entries, dz the Lebesgue measure on the space of such matrices, and $\mathrm{const}_{n,s}$ a normalizating constant whose precise value will be obtained below. The measure $\mu_n^{(s)}$ is finite if $s>-1$ and infinite if $s\leqslant -1$. It is clear from the definition that $\mu_n^{(s)}$ is invariant under the actions by the unitary group $U(n)$ of multiplication on the left and on the right.

This work was supported by the project A*MIDEX (no. ANR-11-IDEX-0001-02) of the French Republic Government Programme 'Investing in the Future' carried out by the French National Agency of Scientific Research (ANR). It was also supported by the Programme of Governmental Support of Scientific Research of Young Russian Scholars, Candidates and Doctors of Sciences (grant no. MD-2859.2014.1), the Programme of Fundamental Research of RAS no. I.28P 'Mathematical problems of modern control theory' (project no. 0014-2015-0006 'Ergodic theory and dynamical systems'), the subsidy for governmental support of leading universities of the Russian Federation aimed at raising their competitiveness among world leading scientific and educational centres, distributed to the National Research University 'Higher School of Economics', and the RFBR (grant no. 13-01-12449-ofi_m).

AMS 2010 *Mathematics Subject Classification.* 20C32, 22D40, 28C10, 28D15, 43A05, 60B15, 60G55.

If the constants $\mathrm{const}_{n,s}$ are chosen appropriately, then the Pickrell family of measures has the Kolmogorov property of consistency with respect to natural projections: the push-forward of the Pickrell measure $\mu_{n+1}^{(s)}$ under the natural projection sending each $(n+1)\times(n+1)$ matrix to its upper left $n\times n$ corner is precisely the Pickrell measure $\mu_n^{(s)}$. This consistency property also holds for infinite Pickrell measures provided that n is sufficiently large (see Proposition 1.8 for an exact statement). The consistency property and Kolmogorov's theorem enable one to define the Pickrell family of measures $\mu^{(s)}$, $s\in\mathbb{R}$, on the space of infinite complex matrices. The Pickrell measures are invariant under the actions of the infinite unitary group on the left and right, and the Pickrell family of measures is the natural infinite-dimensional analogue of the canonical unitarily invariant measure on a Grassmann manifold (see the paper of Pickrell [1]).

What is the ergodic decomposition of Pickrell measures with respect to the action of the Cartesian square of the infinite unitary group? The ergodic unitarily invariant probability measures on the space of infinite complex matrices were explicitly described by Pickrell [2], and another approach to their description was given by Vershik and Olshanski [3]. Each ergodic measure is parametrized by an infinite array $x=(x_1,\dots,x_n,\dots)$ of numbers in the half-line satisfying $x_1\geqslant x_2\geqslant\cdots\geqslant 0$ and $x_1+\cdots+x_n+\cdots<+\infty$, and an additional parameter $\widetilde{\gamma}$ that we call the *Gaussian parameter*. Informally speaking, the parameters x_n may be thought of as the 'asymptotic eigenvalues' of an infinite complex matrix and $\widetilde{\gamma}$ as the difference between the 'asymptotic trace' and the sum of the asymptotic eigenvalues (this difference is positive, in particular, for a Gaussian measure).

In 2000 Borodin and Olshanski [4] proved that for finite Pickrell measures the Gaussian parameter vanishes almost surely and the corresponding spectral measure (the measure parametrizing the ergodic decomposition), regarded as a measure on the space of configurations on the half-line $(0,+\infty)$, coincides with the Bessel point process of Tracy and Widom [5]. The correlation functions of this process are expressed as determinants of the Bessel kernel.

Borodin and Olshanski [4] posed the following problem. *Describe the ergodic decomposition of infinite Pickrell measures.* In this paper we give a solution.

The first step is the result in [6] saying that almost all ergodic components of an infinite Pickrell measure are themselves *finite*, only the spectral measure is infinite. Then it is proved, as for finite measures, that the Gaussian parameter vanishes almost surely. The ergodic decomposition is thus given by a σ-finite measure $\mathbb{B}^{(s)}$ on the space of configurations on the half-line $(0,+\infty)$.

How to describe σ-finite measures on the space of configurations? We note that the formalism of correlation functions is completely inapplicable since they are defined only for finite measures.

This paper gives, for a first time, an explicit method for constructing infinite measures on the space of configurations. Since these measures are very closely related to determinantal probability measures, we call them *infinite determinantal measures.*

We give three descriptions of the measure $\mathbb{B}^{(s)}$. The first two may be carried out in much greater generality.

(1) **Inductive limit of determinantal measures.** By definition, $\mathbb{B}^{(s)}$ is supported on those configurations X whose particles accumulate only to zero, not to infinity. Hence $\mathbb{B}^{(s)}$-almost every configuration X admits a maximal particle $x_{\max}(X)$. By choosing an arbitrary $R>0$ and restricting $\mathbb{B}^{(s)}$ to the set $\{X\colon x_{\max}(X)<R\}$, we get a finite measure which becomes determinantal after normalization. The corresponding operator is an orthogonal projection operator whose range can be described explicitly for all $R>0$. Thus the measure $\mathbb{B}^{(s)}$ may be obtained as an inductive limit of finite determinantal measures along an exhausting [1] family of subsets of the space of configurations.

(2) **A determinantal measure multiplied by a multiplicative functional.** More generally, one can reduce $\mathbb{B}^{(s)}$ to a finite determinantal measure by taking its product with an appropriate multiplicative functional. A *multiplicative functional* on the space of configurations may be obtained as the product of the values of a fixed non-negative function over all particles of a configuration:

$$\Psi_g(X)=\prod_{x\in X} g(x).$$

If g is suitably chosen, then the measure

$$\Psi_g\mathbb{B}^{(s)} \tag{1}$$

is finite and becomes determinantal after normalization. The corresponding operator is an orthogonal projection operator with explicitly computable range. Of course, the previous description is the particular case $g=\chi_{(0,R)}$ of this. It is often convenient to take a strictly positive function, for example, $g^{\beta}(x)=\exp(-\beta x)$ for $\beta>0$. The range of the orthogonal projection operator inducing the measure (1) is known explicitly for a large class of functions g, but an explicit formula for its kernel is known only for several concrete functions. These computations will appear in a sequel to this paper.

(3) **A skew product.** As noted above, $\mathbb{B}^{(s)}$-almost every configuration X contains a maximal particle $x_{\max}(X)$. Therefore it is natural for the measure $\mathbb{B}^{(s)}$ to fix the position $x_{\max}(X)$ of the maximal particle and consider the corresponding conditional measure. This is a well-defined determinantal probability measure induced by a projection operator whose range can be found explicitly (using the description of Palm measures of determinantal point processes introduced by Shirai and Takahashi [7]). The σ-finite distribution of the maximal particle can also be found explicitly: the ratios of the measures of intervals are obtained as ratios of suitable Fredholm determinants. The measure $\mathbb{B}^{(s)}$ is thus represented as a skew product whose base is an explicitly known σ-finite measure on the half-line and whose fibres are certain explicitly known determinantal probability measures (see § 1.10 for a detailed presentation).

A key role in the construction of infinite determinantal measures is played by the following result of [8] (see also [9]): the product of a determinantal probability measure and an integrable multiplicative functional is, after normalization, again a determinantal probability measure whose operator can be found explicitly. In

[1] A family of bounded sets B_n is said to be *exhausting* if $B_{n-1}\subset B_n$ and $\bigcup B_n=E$.

particular, if $\mathbb{P}_\Pi$ is a determinantal point process induced by a projection operator Π with range L, then under certain additional assumptions the process $\Psi_g\mathbb{P}_\Pi$ is, after normalization, again the determinantal point process induced by the projection operator onto the subspace $\sqrt{g}\,L$ (a precise statement is given in Proposition B.3).

Informally speaking, if the function g is such that the subspace $\sqrt{g}\,L$ does not lie in L_2, then the measure $\Psi_g\mathbb{P}_\Pi$ ceases to be finite and we obtain precisely an infinite determinantal measure corresponding to a subspace of locally square-integrable functions. This is one of our main constructions (see Theorem 2.11).

The Bessel point process of Tracy and Widom, which governs the ergodic decomposition of finite Pickrell measures, is a scaling limit of Jacobi orthogonal polynomial emsembles. To describe the ergodic decomposition of infinite Pickrell measures, one must consider the scaling limit of infinite analogues of Jacobi orthogonal polynomial emsembles. The resulting infinite determinantal measure is computed in this paper and is called the infinite Bessel point process (see § 1.4 for a precise definition).

Our main result, Theorem 1.11, identifies the ergodic decomposition measure of an infinite Pickrell measure with the infinite Bessel point process.

1.2. Historical remarks. Pickrell measures were introduced by Pickrell [1] in 1987. In 2000, Borodin and Olshanski [4] studied a closely related two-parameter family of measures on the space of infinite Hermitian matrices that are invariant with respect to the natural action of the infinite unitary group by conjugation. Since the existence of such measures (as well as of the original family considered by Pickrell) is proved by a computation going back to the work of Hua Loo-Keng [10], Borodin and Olshanski called their family of measures the *Hua–Pickrell measures*. For various generalizations of the Hua–Pickrell measures see, for example, the papers of Neretin [11] and Bourgade, Nikehbali, Rouault [12]. Pickrell considered only those values of the parameter for which the corresponding measures are finite, while Borodin and Olshanski [4] showed that the infinite Pickrell and Hua–Pickrell measures are also well defined. Borodin and Olshanski [4] also proved that the ergodic decomposition of Hua–Pickrell probability measures is given by a determinantal point process arising as a scaling limit of pseudo-Jacobian orthogonal polynomial ensembles and posed the problem of describing the ergodic decomposition of infinite Hua–Pickrell measures.

The aim of the present paper, which is devoted to Pickrell's original model, is to give an explicit description of the ergodic decomposition of infinite Pickrell measures on spaces of infinite complex matrices.

1.3. Organization of the paper. This paper is the first in a cycle of three giving an explicit description of the ergodic decomposition of infinite Pickrell measures. References to the other parts [13], [14] are organized as follows: Proposition II.2.3, equation (III.9), and so on.

The paper is organized as follows. The introduction begins with a description of the main construction (infinite determinantal measures) in the concrete case of the infinite Bessel point process. We recall the construction of Pickrell measures and the Olshanski–Vershik approach to Pickrell's classification of ergodic unitarily equivalent measures on the space of infinite complex matrices. Then we state the main result of the paper, Theorem 1.11, which identifies the ergodic decomposition measure of an infinite Pickrell measure with the infinite Bessel process (up to the change of variable $y = 4/x$). We conclude the introduction by giving an outline of the proof of Theorem 1.11: the ergodic decomposition measures of Pickrell measures are obtained as scaling limits of their finite-dimensional approximations (the radial parts of finite-dimensional projections of Pickrell measures). First, Lemma 1.14 says that the rescaled radial parts, multiplied by a certain density, converge to the desired ergodic decomposition measure multiplied by the same density. Second, it turns out that the normalized products of the push-forwards of the rescaled radial parts in the space of configurations on the half-line with an appropriately chosen multiplicative functional on the space of configurations converge weakly in the space of measures on the space of configurations. Combining these two facts, we complete the proof of Theorem 1.11.

§ 2 is devoted to a general construction of infinite determinantal measures on the space of configurations $\operatorname{Conf}(E)$ of a complete locally compact metric space E endowed with a σ-finite Borel measure μ.

We start with a space H of functions on E that are locally square integrable with respect to μ and with an increasing family of subsets

$$E_0 \subset E_1 \subset \cdots \subset E_n \subset \cdots$$

in E such that for every $n \in \mathbb{N}$ the restriction $\chi_{E_n} H$ is a closed subspace of $L_2(E, \mu)$. If the corresponding projection operator Π_n is locally of trace class, then the projection operator Π_n induces a determinantal measure $\mathbb{P}_n$ on $\operatorname{Conf}(E)$ by the Macchi–Soshnikov theorem. Under certain additional assumptions it follows from the results of [8] (see Corollary B.5 below) that the measures $\mathbb{P}_n$ satisfy the following consistency condition. Let $\operatorname{Conf}(E, E_n)$ be the set of configurations all of whose particles lie in E_n. Then for every $n \in \mathbb{N}$ we have

$$\frac{\mathbb{P}_{n+1}|_{\operatorname{Conf}(E,E_n)}}{\mathbb{P}_{n+1}(\operatorname{Conf}(E, E_n))} = \mathbb{P}_n. \tag{2}$$

The consistency property (2) implies that there is a σ-finite measure $\mathbb{B}$ such that for every $n \in \mathbb{N}$ we have $0 < \mathbb{B}(\operatorname{Conf}(E, E_n)) < +\infty$ and

$$\frac{\mathbb{B}|_{\operatorname{Conf}(E,E_n)}}{\mathbb{B}(\operatorname{Conf}(E, E_n))} = \mathbb{P}_n.$$

The measure $\mathbb{B}$ is called an infinite determinantal measure. An alternative description of infinite determinantal measures uses the formalism of multiplicative functionals. It was proved in [8] (see also [9] and Proposition B.3 below) that the product of a determinantal measure and an integrable multiplicative functional is, after normalization, again a determinantal measure. Taking the product of a determinantal measure and a convergent (but not integrable) multiplicative functional, we obtain an infinite determinantal measure. This reduction of infinite determinantal measures to ordinary ones by taking the product with a multiplicative functional is essential in the proof of Theorem 1.11. We conclude § 2 by proving the existence of the infinite Bessel point process.

The paper has three appendices. In Appendix A we collect the necessary facts about Jacobi orthogonal polynomials, including the recurrence relation between the nth Christoffel–Darboux kernel corresponding to the parameters (α, β) and the $(n-1)$th Christoffel–Darboux kernel corresponding to the parameters $(\alpha+2, \beta)$.

Appendix B is devoted to determinantal point processes on spaces of configurations. We first recall the definition of a space of configurations, its Borel structure, and its topology. Then we define determinantal point processes, recall the Macchi–Soshnikov theorem, and state the rule of transformation of kernels under a change of variable. We also recall the definition of a multiplicative functional on the space of configurations, state the result of [8] (see also [9]) saying that the product of a determinantal point process and a multiplicative functional is again a determinantal point process, and give an explicit representation of the resulting kernel. In particular, we recall the representations from [8], [9] for the kernels of induced processes.

Appendix C is devoted to the construction of Pickrell measures following a computation of Hua Loo–Keng [10] and the observation of Borodin and Olshanski in the infinite case.

1.4. The infinite Bessel point process.

1.4.1. *Outline of the construction.* Take $n \in \mathbb{N}$, $s \in \mathbb{R}$ and endow the cube $(-1,1)^n$ with the measure

$$\prod_{1 \leqslant i < j \leqslant n} (u_i - u_j)^2 \prod_{i=1}^{n} (1-u_i)^s \, du_i. \tag{3}$$

When $s > -1$ the measure (3) is a particular case of the Jacobi orthogonal polynomial ensemble (a determinantal point process induced by the nth Christoffel–Darboux projection operator for Jacobi polynomials). The classical Heine–Mehler asymptotics for Jacobi polynomials yields an asymptotic formula for the Christoffel–Darboux kernels and, therefore, for the corresponding determinantal point processes, whose scaling limit with respect to the scaling

$$u_i = 1 - \frac{y_i}{2n^2}, \qquad i = 1, \dots, n, \tag{4}$$

is the Bessel point process of Tracy and Widom [5]. We recall that the Bessel point process is governed by the operator of projection of $L_2((0,+\infty); \mathrm{Leb})$ onto the subspace of functions whose Hankel transform is supported on $[0,1]$.

For $s \leqslant -1$ the measure (3) is infinite. To describe its scaling limit, we first recall a recurrence relation between the Christoffel–Darboux kernels for Jacobi polynomials and a corollary: recurrence relations between the corresponding orthogonal polynomial ensembles. Namely, the nth Christoffel–Darboux kernel of the Jacobi ensemble with parameter s is a rank-one perturbation of the $(n-1)$th Christoffel–Darboux kernel of the Jacobi ensemble with parameter $s+2$.

This recurrence relation motivates the following construction. Consider the range of the Christoffel–Darboux projection operator. This is a finite-dimensional subspace of polynomials of degree at most $n-1$ multiplied by the weight $(1-u)^{s/2}$. Consider the same subspace for $s \leqslant -1$. Being no longer a subspace of L_2, it is nevertheless a well-defined space of *locally* square-integrable functions. In view of the recurrence relations, our subspace corresponding to a parameter s is a rank-one perturbation of the similar subspace corresponding to the parameter $s+2$, and so on, until we arrive at a value of the parameter (to be denoted by $s+2n_s$) for which the subspace becomes a part of L_2. Thus our initial subspace is a finite-rank perturbation of a closed subspace of L_2, and the rank of this perturbation depends on s but not on n. Taking the scaling limit of this representation, we obtain a subspace of locally square-integrable functions on $(0,+\infty)$, which is again a finite-rank perturbation of the range of the Bessel projection operator which corresponds to the parameter $s+2n_s$.

To every such subspace of locally square-integrable functions we then assign a σ-finite measure on the space of configurations: the *infinite Bessel point process*. The infinite Bessel point process is a scaling limit of the measures (3) under the scaling (4).

1.4.2. *The Jacobi orthogonal polynomial ensemble.* We first consider the case when $s>-1$. Let $P_n^{(s)}$ be the standard Jacobi orthogonal polynomials corresponding to the weight $(1-u)^s$, and let $\widetilde{K}_n^{(s)}(u_1,u_2)$ be the nth Christoffel–Darboux kernel for the Jacobi orthogonal polynomial ensemble (see the formulae (34), (35) in Appendix A). For $s>-1$ we have the following well-known determinantal representation of the measure (3):

$$\mathrm{const}_{n,s}\prod_{1\leqslant i<j\leqslant n}(u_i-u_j)^2\prod_{i=1}^{n}(1-u_i)^s\,du_i=\frac{1}{n!}\det\widetilde{K}_n^{(s)}(u_i,u_j)\prod_{i=1}^{n}du_i, \tag{5}$$

where the normalization constant $\mathrm{const}_{n,s}$ is chosen in such a way that the left-hand side is a probability measure.

1.4.3. *The recurrence relation for Jacobi orthogonal polynomial ensembles.* We write Leb for the ordinary Lebesgue measure on (a subset of) the real axis. Given a finite family of functions $f_1,\dots,f_N$ on the line, we write $\mathrm{span}(f_1,\dots,f_N)$ for the vector space spanned by these functions. The Christoffel–Darboux kernel $\widetilde{K}_n^{(s)}$ is the kernel of the orthogonal projection operator of $L_2([-1,1];\mathrm{Leb})$ onto the subspace

$$\begin{aligned}L_{\mathrm{Jac}}^{(s,n)}&=\mathrm{span}\bigl((1-u)^{s/2},(1-u)^{s/2}u,\dots,(1-u)^{s/2}u^{n-1}\bigr)\\&=\mathrm{span}\bigl((1-u)^{s/2},(1-u)^{s/2+1},\dots,(1-u)^{s/2+n-1}\bigr).\end{aligned}$$

By definition we have a direct-sum decomposition

$$L_{\mathrm{Jac}}^{(s,n)}=\mathbb{C}(1-u)^{s/2}\oplus L_{\mathrm{Jac}}^{(s+2,n-1)}.$$

By Proposition A.1, for every $s>-1$ we have the recurrence relation

$$\widetilde{K}_n^{(s)}(u_1,u_2)=\frac{s+1}{2^{s+1}}P_{n-1}^{(s+1)}(u_1)(1-u_1)^{s/2}P_{n-1}^{(s+1)}(u_2)(1-u_2)^{s/2}+\widetilde{K}_n^{(s+2)}(u_1,u_2)$$

and, therefore, an orthogonal direct sum decomposition

$$L_{\mathrm{Jac}}^{(s,n)}=\mathbb{C}P_{n-1}^{(s+1)}(u)(1-u)^{s/2}\oplus L_{\mathrm{Jac}}^{(s+2,n-1)}.$$

We now pass to the case when $s \leqslant -1$. We define a positive integer n_s by the relation

$$\frac{s}{2} + n_s \in \left(-\frac{1}{2}, \frac{1}{2}\right]$$

and consider the subspace

$$\widetilde{V}^{(s,n)} = \operatorname{span}\bigl((1-u)^{s/2}, (1-u)^{s/2+1}, \dots, P_{n-n_s}^{(s+2n_s-1)}(u)(1-u)^{s/2+n_s-1}\bigr). \qquad (6)$$

By definition we get a direct-sum decomposition

$$L_{\mathrm{Jac}}^{(s,n)} = \widetilde{V}^{(s,n)} \oplus L_{\mathrm{Jac}}^{(s+2n_s, n-n_s)}. \qquad (7)$$

Note that

$$L_{\mathrm{Jac}}^{(s+2n_s, n-n_s)} \subset L_2([-1,1]; \mathrm{Leb})$$

while

$$\widetilde{V}^{(s,n)} \cap L_2([-1,1]; \mathrm{Leb}) = 0.$$

1.4.4. *Scaling limits.* We recall that the scaling limit of the Christoffel–Darboux kernels $\widetilde{K}_n^{(s)}$ of the Jacobi orthogonal polynomial ensemble with respect to the scaling (4) is the Bessel kernel $\widetilde{J}_s$ of Tracy and Widom [5]. (We recall a definition of the Bessel kernel in Appendix A and give a precise statement on the scaling limit in Proposition A.3.)

Clearly, for every β under the scaling (4) we have

$$\lim_{n\to\infty} (2n^2)^{\beta}(1-u_i)^{\beta} = y_i^{\beta}$$

and, for every $\alpha > -1$, the classical Heine–Mehler asymptotics for the Jacobi polynomials yields that

$$\lim_{n\to\infty} 2^{-\frac{\alpha+1}{2}} n^{-1} P_n^{(\alpha)}(u_i)(1-u_i)^{\frac{\alpha-1}{2}} = \frac{J_{\alpha}(\sqrt{y_i})}{\sqrt{y_i}}.$$

It is therefore natural to take the subspace

$$\widetilde{V}^{(s)} = \operatorname{span}\left(y^{s/2}, y^{s/2+1}, \dots, \frac{J_{s+2n_s-1}(\sqrt{y})}{\sqrt{y}}\right)$$

as the scaling limit of the subspaces (6).

Moreover, we already know that the scaling limit of the subspace (7) is the range $\tilde{L}^{(s+2n_s)}$ of the operator $\tilde{J}_{s+2n_s}$.

Thus we arrive at the subspace $\widetilde{H}^{(s)}$,

$$\widetilde{H}^{(s)} = \widetilde{V}^{(s)} \oplus \tilde{L}^{(s+2n_s)}.$$

It is natural to regard $\widetilde{H}^{(s)}$ as the scaling limit of the subspaces $L_{\mathrm{Jac}}^{(s,n)}$ as $n \to \infty$ under the scaling (4).

Note that the subspaces $\widetilde{H}^{(s)}$ consist of functions that are locally square integrable and, moreover, fail to be square integrable only *at zero*: for every $\varepsilon > 0$ the subspace $\chi_{[\varepsilon,+\infty)}\widetilde{H}^{(s)}$ lies in L_2.

1.4.5. *Definition of the infinite Bessel point process.* We now proceed to give a precise description, in this specific case, of one of our main constructions, that of the σ-finite measure $\widetilde{\mathbb{B}}^{(s)}$, the scaling limit of the infinite Jacobi ensembles (3) under the scaling (4). Let $\mathrm{Conf}((0,+\infty))$ be the space of configurations on $(0,+\infty)$. Given any Borel subset $E_0 \subset (0,+\infty)$, we write $\mathrm{Conf}((0,+\infty); E_0)$ for the subspace of configurations all of whose particles lie in E_0. As usual, given a measure $\mathbb{B}$ on X and a measurable subset $Y \subset X$ with $0 < \mathbb{B}(Y) < +\infty$, we write $\mathbb{B}|_Y$ for the restriction of $\mathbb{B}$ to Y.

It will be proved in what follows that for every $\varepsilon > 0$ the subspace $\chi_{(\varepsilon,+\infty)}\widetilde{H}^{(s)}$ is a closed subspace of $L_2((0,+\infty); \mathrm{Leb})$ and that the orthogonal projection operator $\widetilde{\Pi}^{(\varepsilon,s)}$ onto $\chi_{(\varepsilon,+\infty)}\widetilde{H}^{(s)}$ is locally of trace class. By the Macchi–Soshnikov theorem, the operator $\widetilde{\Pi}^{(\varepsilon,s)}$ induces a determinantal measure $\mathbb{P}_{\widetilde{\Pi}^{(\varepsilon,s)}}$ on $\mathrm{Conf}((0,+\infty))$.

Proposition 1.1. *Let $s \leqslant -1$. Then there is a σ-finite measure $\mathbb{B}^{(s)}$ on* $\mathrm{Conf}((0,+\infty))$ *such that the following conditions hold.*

(1) *The particles of $\mathbb{B}$-almost all configurations do not accumulate at zero.*

(2) *For every $\varepsilon > 0$ we have*

$$0 < \mathbb{B}\bigl(\mathrm{Conf}((0,+\infty); (\varepsilon,+\infty))\bigr) < +\infty,$$

$$\frac{\mathbb{B}|_{\mathrm{Conf}((0,+\infty);(\varepsilon,+\infty))}}{\mathbb{B}(\mathrm{Conf}((0,+\infty); (\varepsilon,+\infty)))} = \mathbb{P}_{\widetilde{\Pi}^{(\varepsilon,s)}}.$$

These conditions uniquely determine the measure $\widetilde{\mathbb{B}}^{(s)}$ up to multiplication by a constant.

Remark. When $s \neq -1, -3, \dots$, we can also write

$$\widetilde{H}^{(s)} = \mathrm{span}(y^{s/2}, \dots, y^{s/2+n_s-1}) \oplus \tilde{L}^{(s+2n_s)}$$

and use the previous construction without any further changes. Note that when $s = -1$ the function $y^{1/2}$ is not square integrable at infinity, whence the need for the definition above. When $s > -1$ we put $\widetilde{\mathbb{B}}^{(s)} = \mathbb{P}_{\tilde{J}_s}$.

Proposition 1.2. *If $s_1 \neq s_2$, then the measures $\widetilde{\mathbb{B}}^{(s_1)}$ and $\widetilde{\mathbb{B}}^{(s_2)}$ are mutually singular.*

The proof of Proposition 1.2 will be deduced from Proposition 1.4, which will in turn be obtained from our main result, Theorem 1.11.

1.5. The modified Bessel point process. In what follows we use the Bessel point process subject to the change of variable $y = 4/x$. To describe it, we consider the half-line $(0,+\infty)$ with the standard Lebesgue measure Leb. Take $s > -1$ and introduce a kernel $J^{(s)}$ by the formula

$$J^{(s)}(x_1,x_2) = \frac{J_s\bigl(\frac{2}{\sqrt{x_1}}\bigr)\frac{1}{\sqrt{x_2}}J_{s+1}\bigl(\frac{2}{\sqrt{x_2}}\bigr) - J_s\bigl(\frac{2}{\sqrt{x_2}}\bigr)\frac{1}{\sqrt{x_1}}J_{s+1}\bigl(\frac{2}{\sqrt{x_1}}\bigr)}{x_1 - x_2},$$

$$x_1 > 0, \qquad x_2 > 0,$$

or, equivalently,

$$J^{(s)}(x,y)=\frac{1}{x_1x_2}\int_0^1 J_s\biggl(2\sqrt{\frac{t}{x_1}}\biggr)J_s\biggl(2\sqrt{\frac{t}{x_2}}\biggr)\,dt.$$

The change of variable $y=4/x$ reduces the kernel $J^{(s)}$ to the kernel $\tilde{J}_s$ of the Bessel point process of Tracy and Widom as considered above (we recall that a change of variables $u_1=\rho(v_1)$, $u_2=\rho(v_2)$ transforms a kernel $K(u_1,u_2)$ to a kernel of the form $K(\rho(v_1),\rho(v_2))\sqrt{\rho'(v_1)\rho'(v_2)}$). Thus the kernel $J^{(s)}$ induces a locally trace-class orthogonal projection operator on $L_2((0,+\infty);\mathrm{Leb})$. Slightly abusing our notation, we denote this operator again by $J^{(s)}$. Let $L^{(s)}$ be the range of $J^{(s)}$. By the Macchi–Soshnikov theorem, the operator $J^{(s)}$ induces a determinantal measure $\mathbb{P}_{J^{(s)}}$ on the space of configurations $\mathrm{Conf}((0,+\infty))$.

1.6. The modified infinite Bessel point process. The involutive homeomorphism $y=4/x$ of the half-line $(0,+\infty)$ induces the corresponding homeomorphism (change of variable) of the space $\mathrm{Conf}((0,+\infty))$. Let $\mathbb{B}^{(s)}$ be the image of $\widetilde{\mathbb{B}}^{(s)}$ under the change of variables. We shall see below that $\mathbb{B}^{(s)}$ is precisely the ergodic decomposition measure for the infinite Pickrell measures.

A more explicit description of $\mathbb{B}^{(s)}$ can be given as follows.

By definition we put

$$L^{(s)}=\biggl\{\frac{\varphi(4/x)}{x},\ \varphi\in\tilde{L}^{(s)}\biggr\}.$$

(The behaviour of determinantal measures under a change of variables is described in § B.5.)

We similarly write $V^{(s)},H^{(s)}\subset L_{2,\mathrm{loc}}((0,+\infty);\mathrm{Leb})$ for the images of the subspaces $\widetilde{V}^{(s)}$, $\widetilde{H}^{(s)}$ under the change of variable $y=4/x$:

$$V^{(s)}=\biggl\{\frac{\varphi(4/x)}{x},\ \varphi\in\widetilde{V}^{(s)}\biggr\},\qquad H^{(s)}=\biggl\{\frac{\varphi(4/x)}{x},\ \varphi\in\widetilde{H}^{(s)}\biggr\}.$$

By definition we have

$$V^{(s)}=\operatorname{span}\biggl(x^{-s/2-1},\dots,\frac{J_{s+2n_s-1}(2/\sqrt{x})}{\sqrt{x}}\biggr),\qquad H^{(s)}=V^{(s)}\oplus L^{(s+2n_s)}.$$

We shall see that for all $R>0$, $\chi_{(0,R)}H^{(s)}$ is a closed subspace of $L_2((0,+\infty);\mathrm{Leb})$. Let $\Pi^{(s,R)}$ be the corresponding orthogonal projection. By definition, $\Pi^{(s,R)}$ is locally of trace class and, by the Macchi–Soshnikov theorem, $\Pi^{(s,R)}$ induces a determinantal measure $\mathbb{P}_{\Pi^{(s,R)}}$ on $\mathrm{Conf}((0,+\infty))$.

The measure $\mathbb{B}^{(s)}$ is characterized by the following conditions.

(1) The set of particles of $\mathbb{B}^{(s)}$-almost every configuration is bounded.

(2) For every $R>0$ we have

$$0<\mathbb{B}\bigl(\mathrm{Conf}((0,+\infty);(0,R))\bigr)<+\infty,$$

$$\frac{\mathbb{B}|_{\mathrm{Conf}((0,+\infty);(0,R))}}{\mathbb{B}(\mathrm{Conf}((0,+\infty);(0,R)))}=\mathbb{P}_{\Pi^{(s,R)}}.$$

These conditions uniquely determine $\mathbb{B}^{(s)}$ up to a constant.

Remark. When $s \neq -1, -3, \dots$, we can also write

$$H^{(s)} = \operatorname{span}(x^{-s/2-1}, \dots, x^{-s/2-n_s+1}) \oplus L^{(s+2n_s)}.$$

Let $\mathscr{I}_{1,\mathrm{loc}}((0,+\infty);\mathrm{Leb})$ be the space of locally trace-class operators on the space $L_2((0,+\infty);\mathrm{Leb})$ (see § B.3 for a detailed definition). The following proposition describes the asymptotic behaviour of the operators $\Pi^{(s,R)}$ as $R \to \infty$.

Proposition 1.3. *Let $s \leqslant -1$. Then the following assertions hold.*

(1) *As $R \to \infty$, we have $\Pi^{(s,R)} \to J^{(s+2n_s)}$ in $\mathscr{I}_{1,\mathrm{loc}}((0,+\infty);\mathrm{Leb})$.*

(2) *The corresponding measures also converge as $R \to \infty$:*

$$\mathbb{P}_{\Pi^{(s,R)}} \longrightarrow \mathbb{P}_{J^{(s+2n_s)}}$$

weakly in the space of probability measures on $\mathrm{Conf}((0,+\infty))$.

As above, for $s > -1$ we put $\mathbb{B}^{(s)} = \mathbb{P}_{J^{(s)}}$. Proposition 1.2 is equivalent to the following assertion.

Proposition 1.4. *If $s_1 \neq s_2$, then the measures $\mathbb{B}^{(s_1)}$ and $\mathbb{B}^{(s_2)}$ are mutually singular.*

We will obtain Proposition 1.4 in the last section of the paper as a corollary of our main result, Theorem 1.11.

We now represent the measure $\mathbb{B}^{(s)}$ as the product of a determinantal probability measure and a multiplicative functional. Here we restrict ourselves to a specific example of such representation, but we shall see in what follows that the construction holds in much greater generality. We introduce a function S on the space of configurations $\mathrm{Conf}((0,+\infty))$ putting

$$S(X) = \sum_{x \in X} x.$$

Of course, the function S may take the value ∞, but the following proposition shows that the set of such configurations has $\mathbb{B}^{(s)}$-measure zero.

Proposition 1.5. *For every $s \in \mathbb{R}$ we have $S(X) < +\infty$ almost surely with respect to the measure $\mathbb{B}^{(s)}$ and, for any $\beta > 0$,*

$$\exp\bigl(-\beta S(X)\bigr) \in L_1\bigl(\mathrm{Conf}((0,+\infty)); \mathbb{B}^{(s)}\bigr).$$

Furthermore, we shall now see that the measure

$$\frac{\exp(-\beta S(X))\mathbb{B}^{(s)}}{\int_{\mathrm{Conf}((0,+\infty))} \exp(-\beta S(X))\, d\mathbb{B}^{(s)}}$$

is determinantal.

Proposition 1.6. *For all $s \in \mathbb{R}$ and $\beta > 0$, the subspace*

$$\exp\left(-\frac{\beta x}{2}\right) H^{(s)} \tag{8}$$

is a closed subspace of $L_2\bigl((0,+\infty);\mathrm{Leb}\bigr)$ and the operator of orthogonal projection onto (8) is locally of trace class.

Let $\Pi^{(s,\beta)}$ be the operator of orthogonal projection onto the subspace (8).

By Proposition 1.6 and the Macchi–Soshnikov theorem, the operator $\Pi^{(s,\beta)}$ induces a determinantal probability measure on the space $\mathrm{Conf}((0,+\infty))$.

Proposition 1.7. *For all $s \in \mathbb{R}$ and $\beta > 0$ we have*

$$\frac{\exp(-\beta S(X))\mathbb{B}^{(s)}}{\int_{\mathrm{Conf}((0,+\infty))} \exp(-\beta S(X))\, d\mathbb{B}^{(s)}} = \mathbb{P}_{\Pi^{(s,\beta)}}. \tag{9}$$

1.7. Unitarily invariant measures on spaces of infinite matrices.

1.7.1. *Pickrell measures.* Let $\mathrm{Mat}(n,\mathbb{C})$ be the space of complex $n \times n$ matrices:

$$\mathrm{Mat}(n,\mathbb{C}) = \bigl\{z = (z_{ij}),\ i = 1,\dots,n,\ j = 1,\dots,n\bigr\}.$$

Let $\mathrm{Leb} = dz$ be the Lebesgue measure on $\mathrm{Mat}(n,\mathbb{C})$. For $n_1 < n$ let

$$\pi^n_{n_1}\colon \mathrm{Mat}(n,\mathbb{C}) \to \mathrm{Mat}(n_1,\mathbb{C})$$

be the natural projection sending each matrix $z = (z_{ij})$, $i,j = 1,\dots,n$, to its upper left corner, that is, the matrix $\pi^n_{n_1}(z) = (z_{ij})$, $i,j = 1,\dots,n_1$.

Following Pickrell [2], we take $s \in \mathbb{R}$ and introduce a measure $\widetilde{\mu}^{(s)}_n$ on $\mathrm{Mat}(n,\mathbb{C})$ by the formula

$$\widetilde{\mu}^{(s)}_n = \det(1 + z^* z)^{-2n-s} dz.$$

The measure $\widetilde{\mu}^{(s)}_n$ is finite if and only if $s > -1$.

The measures $\widetilde{\mu}^{(s)}_n$ have the following property of consistency with respect to the projections $\pi^n_{n_1}$.

Proposition 1.8. *Let $s \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that $n + s > 0$. Then for every matrix $\tilde z \in \mathrm{Mat}(n,\mathbb{C})$ we have*

$$\int_{(\pi^{n+1}_n)^{-1}(\tilde z)} \det(1+z^* z)^{-2n-2-s}\, dz = \frac{\pi^{2n+1}(\Gamma(n+1+s))^2}{\Gamma(2n+2+s)\Gamma(2n+1+s)} \det(1+\tilde z^* \tilde z)^{-2n-s}.$$

We now consider the space $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ of infinite complex matrices whose rows and columns are indexed by positive integers:

$$\mathrm{Mat}(\mathbb{N},\mathbb{C}) = \bigl\{z = (z_{ij}),\ i,j \in \mathbb{N},\ z_{ij} \in \mathbb{C}\bigr\}.$$

Let $\pi^\infty_n\colon \mathrm{Mat}(\mathbb{N},\mathbb{C}) \to \mathrm{Mat}(n,\mathbb{C})$ be the natural projection sending each infinite matrix $z \in \mathrm{Mat}(\mathbb{N},\mathbb{C})$ to its upper left $n \times n$ corner, that is, the matrix (z_{ij}), $i,j = 1,\dots,n$.

For $s > -1$ it follows from Proposition 1.8 and Kolmogorov's existence theorem [15] that there is a unique probability measure $\mu^{(s)}$ on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ such that for every $n \in \mathbb{N}$ we have

$$(\pi^\infty_n)_* \mu^{(s)} = \pi^{-n^2} \prod_{l=1}^{n} \frac{\Gamma(2l+s)\Gamma(2l-1+s)}{(\Gamma(l+s))^2} \widetilde{\mu}^{(s)}_n.$$

If $s \leqslant -1$, then Proposition 1.8 along with Kolmogorov's existence theorem [15] enables us to conclude that for every $\lambda > 0$ there is a unique infinite measure $\mu^{(s,\lambda)}$ on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ with the following properties.

(1) For every $n \in \mathbb{N}$ satisfying $n + s > 0$ and every compact set $Y \subset \mathrm{Mat}(n, \mathbb{C})$ we have $\mu^{(s,\lambda)}(Y) < +\infty$. Hence the push-forwards $(\pi_n^\infty)_* \mu^{(s,\lambda)}$ are well defined.

(2) For every $n \in \mathbb{N}$ satisfying $n + s > 0$ we have

$$(\pi_n^\infty)_* \mu^{(s,\lambda)} = \lambda \biggl(\prod_{l=n_0}^{n} \pi^{-2n} \frac{\Gamma(2l+s)\Gamma(2l-1+s)}{(\Gamma(l+s))^2} \biggr) \widetilde{\mu}^{(s)}.$$

The measures $\mu^{(s,\lambda)}$ are called *infinite Pickrell measures*. Slightly abusing our notation, we shall omit the superscript λ and write $\mu^{(s)}$ for a measure defined up to a multiplicative constant. A detailed definition of infinite Pickrell measures is given by Borodin and Olshanski [4], p. 116.

Proposition 1.9. *For all $s_1, s_2 \in \mathbb{R}$ with $s_1 \neq s_2$ the Pickrell measures $\mu^{(s_1)}$ and $\mu^{(s_2)}$ are mutually singular.*

Proposition 1.9 may be obtained from Kakutani's theorem in the spirit of [4] (see also [11]).

Let $U(\infty)$ be the infinite unitary group. An infinite matrix $u = (u_{ij})_{i,j \in \mathbb{N}}$ belongs to $U(\infty)$ if there is a positive integer n_0 such that the matrix (u_{ij}), $i, j \in [1, n_0]$, is unitary, $u_{ii} = 1$ for $i > n_0$, and $u_{ij} = 0$ whenever $i \neq j$, $\max(i, j) > n_0$. The group $U(\infty) \times U(\infty)$ acts on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ by multiplication on both sides: $T_{(u_1,u_2)} z = u_1 z u_2^{-1}$. The Pickrell measures $\mu^{(s)}$ are by definition invariant under this action. The role of Pickrell (and related) measures in the representation theory of $U(\infty)$ is reflected in [3], [16], [17].

It follows from Theorem 1 and Corollary 1 in [18] that the measures $\mu^{(s)}$ admit an ergodic decomposition. Theorem 1 in [6] says that for every $s \in \mathbb{R}$ almost all ergodic components of $\mu^{(s)}$ are finite. We now state this result in more detail. Recall that a $(U(\infty) \times U(\infty))$-invariant measure on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is said to be *ergodic* if every $(U(\infty) \times U(\infty))$-invariant Borel subset of $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ either has measure zero or has a complement of measure zero. Equivalently, ergodic probability measures are extreme points of the convex set of all $(U(\infty) \times U(\infty))$-invariant probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$. We denote the set of all ergodic $(U(\infty) \times U(\infty))$-invariant probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ by $\mathfrak{M}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, \mathbb{C}))$. The set $\mathfrak{M}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ is a Borel subset of the set of all probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ (see, for example, [18]). Theorem 1 in [6] says that for every $s \in \mathbb{R}$ there is a unique σ-finite Borel measure $\overline{\mu}^{(s)}$ on $\mathfrak{M}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ such that

$$\mu^{(s)} = \int_{\mathfrak{M}_{\mathrm{erg}}(\mathrm{Mat}(\mathbb{N}, \mathbb{C}))} \eta \, d\overline{\mu}^{(s)}(\eta).$$

Our main result is an explicit description of the measure $\overline{\mu}^{(s)}$ and its identification, after a change of variable, with the infinite Bessel point process considered above.

1.8. Classification of ergodic measures. We recall a classification of ergodic $(U(\infty)\times U(\infty))$-invariant probability measures on $\operatorname{Mat}(\mathbb{N},\mathbb{C})$. This classification was obtained by Pickrell [2], [19]. Vershik [20] and Vershik and Olshanski [3] suggested another approach to it in the case of unitarily invariant measures on the space of infinite Hermitian matrices, and Rabaoui [21], [22] adapted the Vershik–Olshanski approach to the original problem of Pickrell. We also follow the approach of Vershik and Olshanski.

We take a matrix $z\in\operatorname{Mat}(\mathbb{N},\mathbb{C})$, put $z^{(n)}=\pi_n^\infty z$, and let

$$\lambda_1^{(n)}\geqslant\cdots\geqslant\lambda_n^{(n)}\geqslant 0$$

be the eigenvalues of the matrix

$$(z^{(n)})^*z^{(n)}$$

counted with multiplicities and arranged in non-increasing order. To stress the dependence on z, we write $\lambda_i^{(n)}=\lambda_i^{(n)}(z)$.

Theorem. (1) *Let η be an ergodic $(U(\infty)\times U(\infty))$-invariant Borel probability measure on* $\operatorname{Mat}(\mathbb{N},\mathbb{C})$. *Then there are non-negative real numbers*

$$\gamma\geqslant 0,\quad x_1\geqslant x_2\geqslant\cdots\geqslant x_n\geqslant\cdots\geqslant 0,$$

satisfying the condition $\gamma\geqslant\sum_{i=1}^\infty x_i$ and such that for η-almost all matrices $z\in$ $\operatorname{Mat}(\mathbb{N},\mathbb{C})$ *and every $i\in\mathbb{N}$ we have*

$$x_i=\lim_{n\to\infty}\frac{\lambda_i^{(n)}(z)}{n^2},\qquad \gamma=\lim_{n\to\infty}\frac{\operatorname{tr}(z^{(n)})^*z^{(n)}}{n^2}. \tag{10}$$

(2) *Conversely, suppose we are given any non-negative real numbers $\gamma\geqslant 0$, $x_1\geqslant x_2\geqslant\cdots\geqslant x_n\geqslant\cdots\geqslant 0$ such that $\gamma\geqslant\sum_{i=1}^\infty x_i$. Then there is a unique ergodic $(U(\infty)\times U(\infty))$-invariant Borel probability measure η on* $\operatorname{Mat}(\mathbb{N},\mathbb{C})$ *such that the relations* (10) *hold for η-almost all matrices $z\in\operatorname{Mat}(\mathbb{N},\mathbb{C})$.*

We define the *Pickrell set* $\Omega_P\subset\mathbb{R}_+\times\mathbb{R}_+^{\mathbb{N}}$ by the formula

$$\Omega_P=\biggl\{\omega=(\gamma,x)\colon x=(x_n),\ n\in\mathbb{N},\ x_n\geqslant x_{n+1}\geqslant 0,\ \gamma\geqslant\sum_{i=1}^\infty x_i\biggr\}.$$

By definition, Ω_P is a closed subset of the product $\mathbb{R}_+\times\mathbb{R}_+^{\mathbb{N}}$ endowed with the Tychonoff topology. For $\omega\in\Omega_P$ let η_ω be the corresponding ergodic probability measure.

The Fourier transform of η_ω may be described explicitly as follows. For every $\lambda\in\mathbb{R}$ we have

$$\int_{\operatorname{Mat}(\mathbb{N},\mathbb{C})}\exp(i\lambda\operatorname{Re}z_{11})\,d\eta_\omega(z)=\frac{\exp\bigl(-4\bigl(\gamma-\sum_{k=1}^\infty x_k\bigr)\lambda^2\bigr)}{\prod_{k=1}^\infty(1+4x_k\lambda^2)}. \tag{11}$$

We denote the right-hand side of (11) by $F_\omega(\lambda)$. Then for all $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ we have

$$\int_{\operatorname{Mat}(\mathbb{N},\mathbb{C})}\exp\bigl(i(\lambda_1\operatorname{Re}z_{11}+\cdots+\lambda_m\operatorname{Re}z_{mm})\bigr)\,d\eta_\omega(z)=F_\omega(\lambda_1)\cdot\ldots\cdot F_\omega(\lambda_m).$$

Thus the Fourier transform is completely determined and, therefore, the measure η_ω is completely described.

An explicit construction of the measures η_ω is as follows. Letting all entries of z be independent identically distributed complex Gaussian random variables with expectation 0 and variance $\widetilde{\gamma}$, we get a Gaussian measure with parameter $\widetilde{\gamma}$. Clearly, this measure is unitarily invariant and, by Kolmogorov's zero-one law, ergodic. It corresponds to the parameter $\omega = (\widetilde{\gamma}, 0, \dots, 0, \dots)$: all x-coordinates are equal to zero (indeed, the eigenvalues of a Gaussian matrix grow at the rate of $\sqrt{n}$ rather than n). We now consider two infinite sequences $(v_1, \dots, v_n, \dots)$ and $(w_1, \dots, w_n, \dots)$ of independent identically distributed complex Gaussian random variables with variance $\sqrt{x}$ and put $z_{ij} = v_i w_j$. This yields a measure which is clearly unitarily invariant and, by Kolmogorov's zero-one law, ergodic. This measure corresponds to a parameter $\omega \in \Omega_P$ such that $\gamma(\omega) = x$, $x_1(\omega) = x$, and all other parameters are equal to zero. Following Vershik and Olshanski [3], we call such measures *Wishart measures* with parameter x. In the general case we put $\widetilde{\gamma} = \gamma - \sum_{k=1}^{\infty} x_k$. The measure η_ω is then an infinite convolution of the Wishart measures with parameters $x_1, \dots, x_n, \dots$ and a Gaussian measure with parameter $\widetilde{\gamma}$. This convolution is well defined since the series $x_1 + \dots + x_n + \cdots$ converges.

The quantity $\widetilde{\gamma} = \gamma - \sum_{k=1}^{\infty} x_k$ will therefore be called the *Gaussian parameter* of the measure η_ω. We shall prove that the Gaussian parameter vanishes for almost all ergodic components of Pickrell measures.

By Proposition 3 in [18], the set of ergodic $(U(\infty) \times U(\infty))$-invariant measures is a Borel subset in the space of all Borel measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ endowed with the natural Borel structure (see, for example, [23]). Furthermore, writing η_ω for the ergodic Borel probability measure corresponding to a point $\omega \in \Omega_P$, $\omega = (\gamma, x)$, we see that the correspondence $\omega \to \eta_\omega$ is a Borel isomorphism between the Pickrell set Ω_P and the set of ergodic $(U(\infty) \times U(\infty))$-invariant probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$.

The ergodic decomposition theorem (Theorem 1 and Corollary 1 in [18]) says that each Pickrell measure $\mu^{(s)}$, $s \in \mathbb{R}$, induces a unique decomposition measure $\overline{\mu}^{(s)}$ on Ω_P such that

$$\mu^{(s)} = \int_{\Omega_P} \eta_\omega \, d\overline{\mu}^{(s)}(\omega). \tag{12}$$

The integral is understood in the ordinary weak sense (see [18]).

When $s > -1$ the measure $\overline{\mu}^{(s)}$ is a probability measure on Ω_P. When $s \leqslant -1$, it is infinite.

We put

$$\Omega_P^0 = \biggl\{ (\gamma, \{x_n\}) \in \Omega_P \colon x_n > 0 \text{ for all } n,\ \gamma = \sum_{n=1}^{\infty} x_n \biggr\}.$$

Of course, the subset Ω_P^0 is *not closed* in Ω_P. We introduce the map

$$\mathrm{conf} \colon \Omega_P \to \mathrm{Conf}((0, +\infty))$$

sending each point $\omega \in \Omega_P$, $\omega = (\gamma, \{x_n\})$, to the configuration

$$\mathrm{conf}(\omega) = (x_1, \dots, x_n, \dots) \in \mathrm{Conf}((0, +\infty)).$$

The map $\omega \to \mathrm{conf}(\omega)$ is bijective when restricted to Ω_P^0.

Remark. The map conf is defined by counting multiplicities of the 'asymptotic eigenvalues' x_n. Moreover, if $x_{n_0} = 0$ for some n_0, then x_{n_0} and all subsequent terms are discarded and the resulting configuration is finite. Nevertheless we shall see that $\overline{\mu}^{(s)}$-almost all configurations are infinite and, $\overline{\mu}^{(s)}$-almost surely, all multiplicities are equal to one. We shall also prove that $\Omega_P \setminus \Omega_P^0$ has $\overline{\mu}^{(s)}$-measure zero for all s.

1.9. Statement of the main result. We first state an analogue of the Borodin–Olshanski ergodic decomposition theorem [4] for finite Pickrell measures.

Proposition 1.10. *Suppose that $s > -1$. Then $\overline{\mu}^{(s)}(\Omega_P^0) = 1$ and the map $\omega \to \mathrm{conf}(\omega)$, which is bijective $\overline{\mu}^{(s)}$-almost everywhere, identifies the measure $\overline{\mu}^{(s)}$ with the determinantal measure $\mathbb{P}_{J^{(s)}}$.*

Our main result is the following explicit description of the ergodic decomposition for infinite Pickrell measures.

Theorem 1.11. *Suppose that $s \in \mathbb{R}$ and let $\overline{\mu}^{(s)}$ be the decomposition measure (defined in (12)) of the Pickrell measure $\mu^{(s)}$. Then the following assertions hold.*

(1) $\overline{\mu}^{(s)}(\Omega_P \setminus \Omega_P^0) = 0$.

(2) *The map $\omega \to \mathrm{conf}(\omega)$, which is bijective $\overline{\mu}^{(s)}$-almost everywhere, identifies the measure $\overline{\mu}^{(s)}$ with the infinite determinantal measure $\mathbb{B}^{(s)}$.*

1.10. A skew-product representation of the measure $\mathbb{B}^{(s)}$. Note that $\mathbb{B}^{(s)}$-almost all configurations X may accumulate only at zero and, therefore, admit a maximal particle, which we denote by $x_{\max}(X)$. We are interested in the distribution of values of $x_{\max}(X)$ with respect to $\mathbb{B}^{(s)}$. By definition, for every $R > 0$, $\mathbb{B}^{(s)}$ takes finite values on the sets $\{X : x_{\max}(X) < R\}$. Furthermore, again by definition, the following relation holds for $R > 0$ and $R_1, R_2 \leqslant R$:

$$\frac{\mathbb{B}^{(s)}(\{X : x_{\max}(X) < R_1\})}{\mathbb{B}^{(s)}(\{X : x_{\max}(X) < R_2\})} = \frac{\det(1 - \chi_{(R_1,+\infty)}\Pi^{(s,R)}\chi_{(R_1,+\infty)})}{\det(1 - \chi_{(R_2,+\infty)}\Pi^{(s,R)}\chi_{(R_2,+\infty)})}.$$

The push-forward of the measure $\mathbb{B}^{(s)}$ is a well-defined σ-finite Borel measure on $(0, +\infty)$. We denote it by $\xi_{\max}\mathbb{B}^{(s)}$. Of course, the measure $\xi_{\max}\mathbb{B}^{(s)}$ is defined up to multiplication by a positive constant.

Question. What is the asymptotic behaviour of $\xi_{\max}\mathbb{B}^{(s)}(0, R)$ as $R \to \infty$ and as $R \to 0$?

We again denote the kernel of the operator $\Pi^{(s,R)}$ by $\Pi^{(s,R)}$. Consider the function $\varphi_R(x) = \Pi^{(s,R)}(x, R)$. By definition,

$$\varphi_R(x) \in \chi_{(0,R)} H^{(s)}.$$

Let $\overline{H}^{(s,R)}$ be the orthogonal complement of the one-dimensional subspace spanned by $\varphi_R(x)$ in $\chi_{(0,R)} H^{(s)}$. In other words, $\overline{H}^{(s,R)}$ is the subspace of all functions in $\chi_{(0,R)}H^{(s)}$ that vanish at R. Let $\overline{\Pi}^{(s,R)}$ be the operator of orthogonal projection onto $\overline{H}^{(s,R)}$.

Proposition 1.12. *We have*

$$\mathbb{B}^{(s)} = \int_0^\infty \mathbb{P}_{\overline{\Pi}^{(s,R)}}\, d\xi_{\max}\mathbb{B}^{(s)}(R).$$

Proof. This follows immediately from the definition of $\mathbb{B}^{(s)}$ and the characterization of Palm measures for determinantal point processes (see the paper [24] by Shirai and Takahashi). □

1.11. The general scheme of ergodic decomposition.

1.11.1. *Approximation.* Let $\mathfrak{F}$ be the family of σ-finite $(U(\infty)\times U(\infty))$-invariant measures μ on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ for which there is a positive integer n_0 (depending on μ) such that for all $R>0$ we have

$$\mu\Bigl(\Bigl\{z\colon \max_{1\leqslant i,j\leqslant n_0}|z_{ij}|<R\Bigr\}\Bigr)<+\infty.$$

By definition, all Pickrell measures belong to the class $\mathfrak{F}$.

We recall a result of [6]: every ergodic measure of class $\mathfrak{F}$ is finite and, therefore, the ergodic components of any measure in $\mathfrak{F}$ are finite almost surely. (The existence of an ergodic decomposition for every measure $\mu\in\mathfrak{F}$ follows from the ergodic decomposition theorem for actions of inductively compact groups, that is, inductive limits of compact groups, established in [18].) The classification of finite ergodic measures now yields that for every measure $\mu\in\mathfrak{F}$ there is a unique σ-finite Borel measure $\overline{\mu}$ on the Pickrell set Ω_P such that

$$\mu=\int_{\Omega_P}\eta_\omega\, d\overline{\mu}(\omega). \tag{13}$$

Our next aim is to construct, following Borodin and Olshanski [4], a sequence of finite-dimensional approximations of $\overline{\mu}$.

With every matrix $z\in\mathrm{Mat}(\mathbb{N},\mathbb{C})$ and number $n\in\mathbb{N}$ we associate the sequence

$$(\lambda_1^{(n)},\lambda_2^{(n)},\dots,\lambda_n^{(n)})$$

of all eigenvalues of the matrix $(z^{(n)})^*z^{(n)}$ arranged in non-increasing order. Here

$$z^{(n)}=(z_{ij})_{i,j=1,\dots,n}.$$

For $n\in\mathbb{N}$ we define the map

$$\mathfrak{r}^{(n)}\colon \mathrm{Mat}(\mathbb{N},\mathbb{C})\to\Omega_P$$

by the formula

$$\mathfrak{r}^{(n)}(z)=\biggl(\frac{1}{n^2}\operatorname{tr}(z^{(n)})^*z^{(n)},\frac{\lambda_1^{(n)}}{n^2},\frac{\lambda_2^{(n)}}{n^2},\dots,\frac{\lambda_n^{(n)}}{n^2},0,0,\dots\biggr).$$

It is clear from the definition that for all $n \in \mathbb{N}$ and $z \in \mathrm{Mat}(\mathbb{N},\mathbb{C})$ we have

$$\mathfrak{r}^{(n)}(z) \in \Omega_P^0.$$

For every measure $\mu \in \mathfrak{F}$ and all sufficiently large $n \in \mathbb{N}$ the push-forwards $(\mathfrak{r}^{(n)})_*\mu$ are well defined since the unitary group is compact. We now prove that for any $\mu \in \mathfrak{F}$ the measures $(\mathfrak{r}^{(n)})_*\mu$ approximate the ergodic decomposition measure $\overline{\mu}$.

We start with the direct description of the map sending each measure $\mu \in \mathfrak{F}$ to its ergodic decomposition measure $\overline{\mu}$.

Following Borodin and Olshanski [4], we consider the set $\mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C})$ of all *regular* matrices z, that is, matrices with the following properties.

(1) For every k the limit $\lim_{n\to\infty} \frac{1}{n^2}\lambda_n^{(k)} =: x_k(z)$ exists.

(2) The limit $\lim_{n\to\infty} \frac{1}{n^2}\operatorname{tr}(z^{(n)})^* z^{(n)} =: \gamma(z)$ exists.

Since the set of regular matrices has full measure with respect to any finite ergodic $(U(\infty)\times U(\infty))$-invariant measure, the existence of the ergodic decomposition (13) implies that

$$\mu\bigl(\mathrm{Mat}(\mathbb{N},\mathbb{C}) \setminus \mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C})\bigr) = 0.$$

We define a map

$$\mathfrak{r}^{(\infty)}\colon \mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C}) \to \Omega_P$$

by the formula

$$\mathfrak{r}^{(\infty)}(z) = \bigl(\gamma(z), x_1(z), x_2(z), \dots, x_k(z), \dots\bigr).$$

The ergodic decomposition theorem [18] and the classification of ergodic unitarily invariant measures (in the form of Vershik and Olshanski) yield the important equality

$$(\mathfrak{r}^{(\infty)})_*\mu = \overline{\mu}. \tag{14}$$

Remark. This equality has a simple analogue in the context of De Finetti's theorem. To obtain the ergodic decomposition of an exchangeable measure on the space of binary sequences, it suffices to consider the push-forward of the initial measure under the almost surely defined map sending each sequence to the frequency of zeros in it.

Given a complete separable metric space Z, we write $\mathfrak{M}_{\mathrm{fin}}(Z)$ for the space of all finite Borel measures on Z endowed with the weak topology. We recall (see [23]) that $\mathfrak{M}_{\mathrm{fin}}(Z)$ is itself a complete separable metric space: the weak topology is induced, for example, by the Lévy–Prokhorov metric.

We now prove that the measures $(\mathfrak{r}^{(n)})_*\mu$ approximate the measure $(\mathfrak{r}^{(\infty)})_*\mu = \overline{\mu}$ as $n \to \infty$. For finite measures μ the following result was obtained by Borodin and Olshanski [4].

Proposition 1.13. *Let μ be a finite unitarily invariant measure on* $\mathrm{Mat}(\mathbb{N},\mathbb{C})$. *Then, as $n \to \infty$, we have*

$$(\mathfrak{r}^{(n)})_*\mu \to (\mathfrak{r}^{(\infty)})_*\mu$$

weakly in $\mathfrak{M}_{\mathrm{fin}}(\Omega_P)$.

Proof. Let $f\colon \Omega_P \to \mathbb{R}$ be continuous and bounded. By definition, for every infinite matrix $z \in \mathrm{Mat}_{\mathrm{reg}}(\mathbb{N}, \mathbb{C})$ we have $\mathfrak{r}^{(n)}(z) \to \mathfrak{r}^{(\infty)}(z)$ as $n \to \infty$ and, therefore,

$$\lim_{n\to\infty} f(\mathfrak{r}^{(n)}(z)) = f(\mathfrak{r}^{(\infty)}(z)).$$

Hence, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} f(\mathfrak{r}^{(n)}(z))\, d\mu(z) = \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} f(\mathfrak{r}^{(\infty)}(z))\, d\mu(z).$$

Changing variables, we arrive at the convergence

$$\lim_{n\to\infty} \int_{\Omega_P} f(\omega)\, d(\mathfrak{r}^{(n)})_*\mu = \int_{\Omega_P} f(\omega)\, d(\mathfrak{r}^{(\infty)})_*\mu.$$

This establishes the desired weak convergence. □

For σ-finite measures $\mu \in \mathfrak{F}$, the result of Borodin and Olshanski can be modified as follows.

Lemma 1.14. *Let $\mu \in \mathfrak{F}$. Then there is a positive bounded continuous function f on the Pickrell set Ω_P such that the following conditions hold.*

(1) *$f \in L_1(\Omega_P, (\mathfrak{r}^{(\infty)})_*\mu)$ and $f \in L_1(\Omega_P, (\mathfrak{r}^{(n)})_*\mu)$ for all sufficiently large $n \in \mathbb{N}$.*

(2) *As $n \to \infty$, we have*

$$f(\mathfrak{r}^{(n)})_*\mu \to f(\mathfrak{r}^{(\infty)})_*\mu$$

weakly in $\mathfrak{M}_{\mathrm{fin}}(\Omega_P)$.

A proof of Lemma 1.14 will be given in the third part of this paper.

Remark. The argument above shows that the explicit characterization of the ergodic decomposition of Pickrell measures in Theorem 1.11 relies on the abstract result (Theorem 1 in [18]) that *a priori* guarantees the existence of the ergodic decomposition. Theorem 1.11 does not give an alternative proof of the existence of this ergodic decomposition.

1.11.2. *Convergence of probability measures on the Pickrell set.* We recall the definition of a natural *'forgetful' map*

$$\mathrm{conf}\colon \Omega_P \to \mathrm{Conf}((0, +\infty)).$$

This map sends each point $\omega = (\gamma, x)$, $x = (x_1, \dots, x_n, \dots)$, to the configuration $\mathrm{conf}(\omega) = (x_1, \dots, x_n, \dots)$.

For $\omega \in \Omega_P$, $\omega = (\gamma, x)$, $x = (x_1, \dots, x_n, \dots)$, $x_n = x_n(\omega)$, we put

$$S(\omega) = \sum_{n=1}^{\infty} x_n(\omega).$$

In other words, we put $S(\omega) = S(\mathrm{conf}(\omega))$ and, slightly abusing the notation, denote both maps by the same letter. Take $\beta > 0$ and for every $n \in \mathbb{N}$ consider the measures

$$\exp(-\beta S(\omega))\mathfrak{r}^{(n)}(\mu^{(s)}).$$

Proposition 1.15. *For all* $s \in \mathbb{R}$ *and* $\beta > 0$ *we have*

$$\exp(-\beta S(\omega)) \in L_1(\Omega_P, \mathfrak{r}^{(n)}(\mu^{(s)})).$$

Introduce probability measures

$$\nu^{(s,n,\beta)} = \frac{\exp(-\beta S(\omega))\mathfrak{r}^{(n)}(\mu^{(s)})}{\int_{\Omega_P} \exp(-\beta S(\omega))\, d\mathfrak{r}^{(n)}(\mu^{(s)})}.$$

We now reconsider the probability measure $\mathbb{P}_{\Pi^{(s,\beta)}}$ on the space $\operatorname{Conf}((0,+\infty))$ (see (9)) and define a measure $\nu^{(s,\beta)}$ on Ω_P by the following requirements:

(1) $\nu^{(s,\beta)}(\Omega_P \setminus \Omega_P^0) = 0$;

(2) $\operatorname{conf}_* \nu^{(s,\beta)} = \mathbb{P}_{\Pi^{(s,\beta)}}$.

The following proposition plays a key role in the proof of Theorem 1.11.

Proposition 1.16. *For all* $\beta > 0$ *and* $s \in \mathbb{R}$ *we have*

$$\nu^{(s,n,\beta)} \to \nu^{(s,\beta)}$$

weakly on $\mathfrak{M}_{\mathrm{fin}}(\Omega_P)$ *as* $n \to \infty$.

Proposition 1.16 will be proved in §§ III.3, III.4. Combining Proposition 1.16 with Lemma 1.14, we shall complete the proof of our main result, Theorem 1.11.

To prove the weak convergence of the measures $\nu^{(s,n,\beta)}$, we first study scaling limits of the radial parts of finite-dimensional projections of infinite Pickrell measures.

1.12. The radial part of a Pickrell measure. Following Pickrell, we associate with every matrix $z \in \operatorname{Mat}(n,\mathbb{C})$ the tuple $(\lambda_1(z),\dots,\lambda_n(z))$ of eigenvalues of z^*z, arranged in non-decreasing order. We introduce the map

$$\mathfrak{rad}_n\colon \operatorname{Mat}(n,\mathbb{C}) \to \mathbb{R}_+^n$$

by the formula

$$\mathfrak{rad}_n\colon z \to \bigl(\lambda_1(z),\dots,\lambda_n(z)\bigr). \tag{15}$$

The map (15) extends naturally to $\operatorname{Mat}(\mathbb{N},\mathbb{C})$, and we denote this extension again by $\mathfrak{rad}_n$. Thus the map $\mathfrak{rad}_n$ sends each infinite matrix to the tuple of squared eigenvalues of its upper left corner of size $n \times n$.

We now define the *radial part* of the Pickrell measure $\mu_n^{(s)}$ as the push-forward of $\mu_n^{(s)}$ under the map $\mathfrak{rad}_n$. Since the finite-dimensional unitary groups are compact and, by definition, $\mu_n^{(s)}$ is finite on compact sets for every s and all sufficiently large n, the push-forward is well defined for all sufficiently large n, even when the measure $\mu^{(s)}$ is infinite.

Slightly abusing the notation, we write dz for the Lebesgue measure on $\operatorname{Mat}(n,\mathbb{C})$, and $d\lambda$ for the Lebesgue measure on $\mathbb{R}_+^n$.

For the push-forward of the Lebesgue measure $\mathrm{Leb}^{(n)} = dz$ under the map $\mathfrak{rad}_n$ we now have

$$(\mathfrak{rad}_n)_*(dz) = \operatorname{const}(n) \prod_{i<j} (\lambda_i - \lambda_j)^2 \, d\lambda$$

(see, for example, [25], [26]), where const(n) is a positive constant depending only on n.

Then the radial part of $\mu_n^{(s)}$ takes the form

$$(\mathfrak{rad}_n)_*\mu_n^{(s)} = \mathrm{const}(n,s)\prod_{i<j}(\lambda_i-\lambda_j)^2\frac{1}{(1+\lambda_i)^{2n+s}}\,d\lambda,$$

where const(n,s) is a positive constant depending only on n and s.

Following Pickrell, we introduce new variables $u_1,\dots,u_n$ by the formula

$$u_i=\frac{x_i-1}{x_i+1}. \tag{16}$$

Proposition 1.17. *In the coordinates* (16), *the radial part* $(\mathfrak{rad}_n)_*\mu_n^{(s)}$ *of the measure* $\mu_n^{(s)}$ *is given on the cube* $[-1,1]^n$ *by the formula*

$$(\mathfrak{rad}_n)_*\mu_n^{(s)} = \mathrm{const}(n,s)\prod_{i<j}(u_i-u_j)^2\prod_{i=1}^{n}(1-u_i)^s\,du_i \tag{17}$$

(*the constant* const(n,s) *may change from one formula to another*).

If $s>-1$, then the constant const(n,s) may be chosen in such a way that the right-hand side is a probability measure. If $s\leqslant -1$, then there is no canonical normalization and the left-hand side is defined up to an arbitrary positive constant.

When $s>-1$ Proposition 1.17 yields a determinantal representation for the radial part of the Pickrell measure. Namely, the radial part is identified with the Jacobi orthogonal polynomial ensemble in the coordinates (16). Passing to a scaling limit, we obtain the Bessel point process up to the change of variable $y=4/x$.

We shall similarly prove that when $s\leqslant -1$, the scaling limit of the measures (17) is equal to the modified infinite Bessel point process introduced above. Furthermore, if we multiply the measures (17) by the density $\exp(-\beta S(X)/n^2)$, then the resulting measures are finite and determinantal, and their weak limit after an appropriate choice of scaling is equal to the determinantal measure $\mathbb{P}_{\Pi^{(s,\beta)}}$ as given in (9). This weak convergence is a key step in the proof of Proposition 1.16.

Thus the study of the case $s\leqslant -1$ requires a new object: infinite determinantal measures on the space of configurations. In the next section we proceed to the general construction and description of properties of infinite determinantal measures.

Grigori Olshanski posed the problem to me, and I am greatly indebted to him. I an deeply grateful to Alexei M. Borodin, Yanqi Qiu, Klaus Schmidt, and Maria V. Shcherbina for useful discussions.

§ 2. Construction and properties of infinite determinantal measures

2.1. Preliminary remarks on σ-finite measures. Let Y be a Borel space. We consider a representation

$$Y=\bigcup_{n=1}^{\infty}Y_n$$

of Y as a countable union of an increasing sequence of subsets Y_n, $Y_n \subset Y_{n+1}$. As above, given a measure μ on Y and a subset $Y' \subset Y$, we write $\mu|_{Y'}$ for the restriction of μ to Y'. Suppose that for every n we are given a probability measure $\mathbb{P}_n$ on Y_n. The following proposition is obvious.

Proposition 2.1. *A σ-finite measure $\mathbb{B}$ on Y with*

$$\frac{\mathbb{B}|_{Y_n}}{\mathbb{B}(Y_n)} = \mathbb{P}_n \tag{18}$$

exists if and only if for all N and n with $N > n$ we have

$$\frac{\mathbb{P}_N|_{Y_n}}{\mathbb{P}_N(Y_n)} = \mathbb{P}_n.$$

The condition (18) uniquely determines the measure $\mathbb{B}$ up to multiplication by a constant.

Corollary 2.2. *If $\mathbb{B}_1$, $\mathbb{B}_2$ are σ-finite measures on Y such that for all $n \in \mathbb{N}$ we have*

$$0 < \mathbb{B}_1(Y_n) < +\infty, \qquad 0 < \mathbb{B}_2(Y_n) < +\infty$$

and

$$\frac{\mathbb{B}_1|_{Y_n}}{\mathbb{B}_1(Y_n)} = \frac{\mathbb{B}_2|_{Y_n}}{\mathbb{B}_2(Y_n)},$$

then there is a positive constant $C > 0$ such that $\mathbb{B}_1 = C\mathbb{B}_2$.

2.2. The unique extension property.

2.2.1. *Extension from a subset.* Let E be a standard Borel space, μ a σ-finite measure on E, L a closed subspace of $L_2(E, \mu)$, Π the operator of orthogonal projection onto L, and $E_0 \subset E$ a Borel subset. We say that the subspace L has the *property of unique extension* from E_0 if every function $\varphi \in L$ satisfying $\chi_{E_0}\varphi = 0$ must be identically equal to zero and the subspace $\chi_{E_0}L$ is closed. In general, the condition that every function $\varphi \in L$ satisfying $\chi_{E_0}\varphi = 0$ is always the zero function, does not imply that the restricted subspace $\chi_{E_0}L$ is closed. Nevertheless we have the following obvious corollary of the open mapping theorem.

Proposition 2.3. *Let L be a closed subspace such that every function $\varphi \in L$ with $\chi_{E_0}\varphi = 0$ is the zero function. In this case the subspace $\chi_{E_0}L$ is closed if and only if there is an $\varepsilon > 0$ such that for all $\varphi \in L$ we have*

$$\|\chi_{E \setminus E_0}\varphi\| \leqslant (1 - \varepsilon)\|\varphi\|. \tag{19}$$

If this condition holds, then the natural restriction map $\varphi \to \chi_{E_0}\varphi$ is an isomorphism of Hilbert spaces. If the operator $\chi_{E \setminus E_0}\Pi$ is compact, then the condition (19) holds.

Remark. In particular, the condition (19) holds *a fortiori* if the operator $\chi_{E \setminus E_0}\Pi$ is Hilbert–Schmidt or, equivalently, if the operator $\chi_{E \setminus E_0}\Pi\chi_{E \setminus E_0}$ belongs to the trace class.

We immediately obtain some corollaries of Proposition 2.3.

Corollary 2.4. *Let g be a bounded non-negative Borel function on E such that*

$$\inf_{x\in E_0} g(x) > 0. \tag{20}$$

If the condition (19) holds, then the subspace $\sqrt{g}\,L$ is closed in $L_2(E,\mu)$.

Remark. The apparently superfluous square root is put here to keep the notation consistent with other results in this paper.

Corollary 2.5. *Under the hypotheses of Proposition 2.3, if (19) holds and the Borel function $g\colon E \to [0,1]$ satisfies (20), then the operator Π^g of orthogonal projection onto $\sqrt{g}\,L$ is given by the formula*

$$\Pi^g = \sqrt{g}\,\Pi(1+(g-1)\Pi)^{-1}\sqrt{g} = \sqrt{g}\,\Pi(1+(g-1)\Pi)^{-1}\Pi\sqrt{g}. \tag{21}$$

In particular, the operator Π^{E_0} of orthogonal projection onto the subspace $\chi_{E_0}L$ takes the form

$$\Pi^{E_0} = \chi_{E_0}\Pi(1-\chi_{E\setminus E_0}\Pi)^{-1}\chi_{E_0} = \chi_{E_0}\Pi(1-\chi_{E\setminus E_0}\Pi)^{-1}\Pi\chi_{E_0}. \tag{22}$$

Corollary 2.6. *Under the hypotheses of Proposition 2.3 suppose that the condition (19) holds. Then for every subset $Y \subset E_0$, whenever the operator $\chi_Y\Pi^{E_0}\chi_Y$ belongs to the trace class, so does the operator $\chi_Y\Pi\chi_Y$ and we have*

$$\operatorname{tr}\chi_Y\Pi^{E_0}\chi_Y \geqslant \operatorname{tr}\chi_Y\Pi\chi_Y.$$

Indeed, it is clear from (22) that if the operator $\chi_Y\Pi^{E_0}$ is Hilbert–Schmidt, then so is $\chi_Y\Pi$. The inequality between traces is also immediate from (22).

2.2.2. *Examples: the Bessel kernel and the modified Bessel kernel.*

Proposition 2.7. (1) *For every $\varepsilon > 0$ the operator $\tilde{J}_s$ has the property of unique extension from $(\varepsilon,+\infty)$.*

(2) *For every $R > 0$ the operator $J^{(s)}$ has the property of unique extension from $(0,R)$.*

Proof. Part (1) follows directly from the uncertainty principle for the Hankel transform: a function and its Hankel transform cannot both be supported on a set of finite measure [27], [28] (note that the uncertainty principle in [27] is stated only for $s > -1/2$, but the more general uncertainty principle in [28] is directly applicable when $s \in [-1,1/2]$; see also [29]) and the following bound, which clearly holds by the definitions for every $R > 0$:

$$\int_0^R \tilde{J}_s(y,y)\,dy < +\infty.$$

The second part follows from the first by the change of variable $y = 4/x$. □

2.3. Inductively determinantal measures. Let E be a locally compact metric space and $\mathrm{Conf}(E)$ the space of configurations on E endowed with the natural Borel structure (see, for example, [30], [31] and § B.1 below).

Given a Borel subset $E' \subset E$, we write $\mathrm{Conf}(E, E')$ for the subspace of those configurations all of whose particles lie in E'. Given a measure $\mathbb{B}$ on X and a measurable subset $Y \subset X$ with $0 < \mathbb{B}(Y) < +\infty$, we write $\mathbb{B}|_Y$ for the restriction of $\mathbb{B}$ to Y.

Let μ be a σ-finite Borel measure on E.

Take a Borel subset $E_0 \subset E$ and assume that for every bounded Borel subset $B \subset E \setminus E_0$ we are given a closed subspace $L^{E_0 \cup B} \subset L_2(E, \mu)$ such that the corresponding projection operator $\Pi^{E_0 \cup B}$ belongs to $\mathscr{I}_{1,\mathrm{loc}}(E, \mu)$. We also make the following assumption.

Assumption 1. (1) $\|\chi_B \Pi^{E_0 \cup B}\| < 1,\ \chi_B \Pi^{E_0 \cup B} \chi_B \in \mathscr{I}_1(E, \mu)$.

(2) *For any subsets* $B^{(1)} \subset B^{(2)} \subset E \setminus E_0$ *we have*

$$\chi_{E_0 \cup B^{(1)}} L^{E_0 \cup B^{(2)}} = L^{E_0 \cup B^{(1)}}.$$

Proposition 2.8. *Under these assumptions there is a σ-finite measure $\mathbb{B}$ on* $\mathrm{Conf}(E)$ *with the following properties.*

(1) *For $\mathbb{B}$-almost all configurations, only finitely many particles lie in $E \setminus E_0$.*

(2) *For every bounded Borel subset $B \subset E \setminus E_0$ we have* $0 < \mathbb{B}(\mathrm{Conf}(E;\, E_0 \cup B)) < +\infty$ *and*

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E; E_0 \cup B)}}{\mathbb{B}(\mathrm{Conf}(E; E_0 \cup B))} = \mathbb{P}_{\Pi^{E_0 \cup B}}.$$

We call such a measure $\mathbb{B}$ an *inductively determinantal measure.*

Proposition 2.8 follows immediately from Proposition 2.1 combined with Proposition B.3 and Corollary B.5. Note that the conditions (1) and (2) determine the measure uniquely up to multiplication by a constant.

We now give a sufficient condition for an inductively determinantal measure to be an actual finite determinantal measure.

Proposition 2.9. *Consider a family of projections $\Pi^{E_0 \cup B}$ satisfying Assumption 1, and let $\mathbb{B}$ be the corresponding inductively determinantal measure. If there are $R > 0$ and $\varepsilon > 0$ such that for all bounded Borel subsets $B \subset E \setminus E_0$ we have*

(1) $\|\chi_B \Pi^{E_0 \cup B}\| < 1 - \varepsilon$,

(2) $\operatorname{tr} \chi_B \Pi^{E_0 \cup B} \chi_B < R$,

then there is an operator $\Pi \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$ of projection onto a closed subspace $L \subset L_2(E, \mu)$ with the following properties:

(1) $L^{E_0 \cup B} = \chi_{E_0 \cup B} L$ *for all* B;

(2) $\chi_{E \setminus E_0} \Pi \chi_{E \setminus E_0} \in \mathscr{I}_1(E, \mu)$;

(3) *the measures $\mathbb{B}$ and $\mathbb{P}_\Pi$ coincide up to multiplication by a constant.*

Proof. By our assumptions, for every bounded Borel subset $B \subset E \setminus E_0$ we have a closed subspace $L^{E_0 \cup B}$ (the range of the operator $\Pi^{E_0 \cup B}$) which has the property of unique extension from E_0. The uniform bounds for the norms of the operators $\chi_B \Pi^{E_0 \cup B}$ imply the existence of a closed subspace L such that $L^{E_0 \cup B} = \chi_{E_0 \cup B} L$.

Since by assumption the projection operator $\Pi^{E_0\cup B}$ belongs to $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$, we obtain for every bounded subset $Y\subset E$ that

$$\chi_Y\Pi^{E_0\cup Y}\chi_Y\in\mathscr{I}_1(E,\mu).$$

Using Corollary 2.6 for the subset $E_0\cup Y$, we now obtain that

$$\chi_Y\Pi\chi_Y\in\mathscr{I}_1(E,\mu).$$

Thus the operator Π of orthogonal projection onto L is locally of trace class and, therefore, induces a unique determinantal probability measure $\mathbb{P}_\Pi$ on $\mathrm{Conf}(E)$. Using Corollary 2.6 again, we obtain that

$$\operatorname{tr}\chi_{E\setminus E_0}\Pi\chi_{E\setminus E_0}\leqslant R. \qquad \square$$

We now give sufficient conditions for the measure $\mathbb{B}$ to be infinite.

Proposition 2.10. *If at least one of the following assumptions holds, then the measure $\mathbb{B}$ is infinite.*

(1) *For every $\varepsilon>0$ there is a bounded Borel subset $B\subset E\setminus E_0$ such that*

$$\|\chi_B\Pi^{E_0\cup B}\|>1-\varepsilon.$$

(2) *For every $R>0$ there is a bounded Borel subset $B\subset E\setminus E_0$ such that*

$$\operatorname{tr}\chi_B\Pi^{E_0\cup B}\chi_B>R.$$

Proof. We recall that

$$\frac{\mathbb{B}(\mathrm{Conf}(E;E_0))}{\mathbb{B}(\mathrm{Conf}(E;E_0\cup B))}=\mathbb{P}_{\Pi^{E_0\cup B}}(\mathrm{Conf}(E;E_0))=\det(1-\chi_B\Pi^{E_0\cup B}\chi_B).$$

If the first assumption holds, then it follows immediately that the top eigenvalue of the self-adjoint trace-class operator $\chi_B\Pi^{E_0\cup B}\chi_B$ is greater than $1-\varepsilon$, whence

$$\det(1-\chi_B\Pi^{E_0\cup B}\chi_B)\leqslant\varepsilon.$$

If the second assumption holds, then

$$\det(1-\chi_B\Pi^{E_0\cup B}\chi_B)\leqslant\exp(-\operatorname{tr}\chi_B\Pi^{E_0\cup B}\chi_B)\leqslant\exp(-R).$$

In both cases the ratio

$$\frac{\mathbb{B}(\mathrm{Conf}(E;E_0))}{\mathbb{B}(\mathrm{Conf}(E;\,E_0\cup B))}$$

can be made arbitrarily small by an appropriate choice of the subset B. It follows that the measure $\mathbb{B}$ is infinite. $\square$

2.4. General construction of infinite determinantal measures. By the Macchi–Soshnikov theorem, under some additional assumptions, one can construct a determinantal measure from an operator of orthogonal projection or, equivalently, from a closed subspace of $L_2(E,\mu)$. In a similar way, an infinite determinantal measure may be assigned to every subspace H of *locally* square-integrable functions.

We recall that $L_{2,\mathrm{loc}}(E,\mu)$ is the space of all measurable functions $f\colon E\to\mathbb{C}$ such that for every bounded set $B\subset E$ we have

$$\int_B |f|^2\,d\mu<+\infty. \tag{23}$$

By choosing an exhausting family B_n of bounded sets (for example, balls with fixed centre whose radii tend to infinity) and using (23) for $B=B_n$, we endow the space $L_{2,\mathrm{loc}}(E,\mu)$ with a countable family of seminorms that makes it into a complete separable metric space. Of course, the resulting topology is independent of the choice of the exhausting family of sets.

Let $H\subset L_{2,\mathrm{loc}}(E,\mu)$ be a vector subspace. When $E'\subset E$ is a Borel set such that $\chi_{E'}H$ is a closed subspace of $L_2(E,\mu)$, we denote by $\Pi^{E'}$ the operator of orthogonal projection onto the subspace $\chi_{E'}H\subset L_2(E,\mu)$. We now fix a Borel subset $E_0\subset E$. Informally speaking, E_0 is the set where the particles may accumulate. We impose the following conditions on E_0 and H.

Assumption 2. (1) *For every bounded Borel set $B\subset E$, the space $\chi_{E_0\cup B}H$ is a closed subspace of $L_2(E,\mu)$.*

(2) *For every bounded Borel set $B\subset E\setminus E_0$ we have*

$$\Pi^{E_0\cup B}\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu),\qquad \chi_B\Pi^{E_0\cup B}\chi_B\in\mathscr{I}_1(E,\mu).$$

(3) *If $\varphi\in H$ satisfies $\chi_{E_0}\varphi=0$, then $\varphi=0$.*

If a subspace H and the subset E_0 are such that every function $\varphi\in H$ with $\chi_{E_0}\varphi=0$ must be the zero function, then we say that H has the *property of unique extension* from E_0.

Theorem 2.11. *Let E be a locally compact metric space and μ a σ-finite Borel measure on E. If a subspace $H\subset L_{2,\mathrm{loc}}(E,\mu)$ and a Borel subset $E_0\subset E$ satisfy Assumption 2, then there is a σ-finite Borel measure $\mathbb{B}$ on $\mathrm{Conf}(E)$ with the following properties.*

(1) *$\mathbb{B}$-almost every configuration has at most finitely many particles outside E_0.*

(2) *For every (possibly empty) bounded Borel set $B\subset E\setminus E_0$ we have $0<\mathbb{B}(\mathrm{Conf}(E;E_0\cup B))<+\infty$ and*

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E;E_0\cup B)}}{\mathbb{B}(\mathrm{Conf}(E;E_0\cup B))}=\mathbb{P}_{\Pi^{E_0\cup B}}.$$

The conditions (1) and (2) determine the measure $\mathbb{B}$ uniquely up to multiplication by a positive constant.

We write $\mathbf{B}(H,E_0)$ for the one-dimensional cone of non-zero infinite determinantal measures induced by H and E_0 and, slightly abusing the notation, write $\mathbb{B}=\mathbb{B}(H,E_0)$ for a representative of this cone.

Remark. If B is a bounded set, then, by definition,

$$\mathbf{B}(H, E_0) = \mathbf{B}(H, E_0 \cup B).$$

Remark. If $E' \subset E$ is a Borel subset such that $\chi_{E_0 \cup E'}$ is a closed subspace of $L_2(E, \mu)$ and the operator $\Pi^{E_0 \cup E'}$ of orthogonal projection onto $\chi_{E_0 \cup E'} H$ satisfies

$$\Pi^{E_0 \cup E'} \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu), \qquad \chi_{E'} \Pi^{E_0 \cup E'} \chi_{E'} \in \mathscr{I}_1(E, \mu),$$

then, exhausting E' by bounded sets, we easily obtain from Theorem 2.11 that $0 < \mathbb{B}(\mathrm{Conf}(E; E_0 \cup E')) < +\infty$ and

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E; E_0 \cup E')}}{\mathbb{B}(\mathrm{Conf}(E; E_0 \cup E'))} = \mathbb{P}_{\Pi^{E_0 \cup E'}}.$$

2.5. Change of variables for infinite determinantal measures. Any homeomorphism $F\colon E \to E$ induces a homeomorphism of $\mathrm{Conf}(E)$, which will again be denoted by F. For every configuration $X \in \mathrm{Conf}(E)$, the particles of $F(X)$ are of the form $F(x)$ for all $x \in X$.

We now assume that the measures $F_*\mu$ and μ are equivalent, and let $\mathbb{B} = \mathbb{B}(H, E_0)$ be an infinite determinantal measure. We introduce the subspace

$$F^*H = \biggl\{ \varphi(F(x)) \sqrt{\frac{dF_*\mu}{d\mu}},\ \varphi \in H \biggr\}.$$

The following proposition is easily obtained from the definitions.

Proposition 2.12. *The push-forward of the infinite determinantal measure*

$$\mathbb{B} = \mathbb{B}(H, E_0)$$

is given by

$$F_*\mathbb{B} = \mathbb{B}(F^*H, F(E_0)).$$

2.6. Example: infinite orthogonal polynomial ensembles. Let ρ be a non-negative function on $\mathbb{R}$, not identically equal to zero. Take $N \in \mathbb{N}$ and endow the set $\mathbb{R}^N$ with the measure

$$\prod_{1 \leqslant i,j \leqslant N} (x_i - x_j)^2 \prod_{i=1}^{N} \rho(x_i)\, dx_i. \tag{24}$$

If for all $k = 0, \dots, 2N-2$ we have

$$\int_{-\infty}^{+\infty} x^k \rho(x)\, dx < +\infty,$$

then the measure (24) is finite and, after normalization, yields a determinantal point process on $\mathrm{Conf}(\mathbb{R})$.

Given a finite family of functions $f_1,\dots,f_N$ on the real line, we write $\operatorname{span}(f_1,\dots,f_N)$ for the vector space spanned by these functions. For a general function ρ we introduce a subspace $H(\rho)\subset L_{2,\mathrm{loc}}(\mathbb{R};\mathrm{Leb})$ by the formula

$$H(\rho)=\operatorname{span}\bigl(\sqrt{\rho(x)},x\sqrt{\rho(x)},\dots,x^{N-1}\sqrt{\rho(x)}\,\bigr).$$

The following obvious proposition shows that (24) is an infinite determinantal measure.

Proposition 2.13. *Let ρ be a non-negative continuous function on $\mathbb{R}$, and let $(a,b)\subset\mathbb{R}$ be a non-empty interval such that the restriction of ρ to (a,b) is positive and bounded. Then the measure (24) is an infinite determinantal measure of the form $\mathbb{B}(H(\rho),(a,b))$.*

2.7. Infinite determinantal measures and multiplicative functionals. Our next aim is to show that under some additional assumptions every infinite determinantal measure can be represented as the product of a finite determinantal measure and a multiplicative functional.

Proposition 2.14. *Let $\mathbb{B}=\mathbb{B}(H,E_0)$ be the infinite determinantal measure induced by a subspace $H\subset L_{2,\mathrm{loc}}(E,\mu)$ and a Borel set E_0, let $g\colon E\to(0,1]$ be a positive Borel function such that $\sqrt{g}\,H$ is a closed subspace of $L_2(E,\mu)$, and let Π^g be the corresponding projection operator. Assume further that*

(1) $\sqrt{1-g}\,\Pi^{E_0}\sqrt{1-g}\,\in\mathscr{I}_1(E,\mu)$;

(2) $\chi_{E\setminus E_0}\Pi^g\chi_{E\setminus E_0}\mathscr{I}_1(E,\mu)$;

(3) $\Pi^g\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$.

Then the multiplicative functional Ψ_g is $\mathbb{B}$-almost surely positive and $\mathbb{B}$-integrable, and we have

$$\frac{\Psi_g\mathbb{B}}{\int_{\mathrm{Conf}(E)}\Psi_g\,d\mathbb{B}}=\mathbb{P}_{\Pi^g}.$$

The proof uses several auxiliary propositions.

First, we have the following simple corollary of the unique extension property.

Proposition 2.15. *Suppose that $H\subset L_{2,\mathrm{loc}}(E,\mu)$ has the property of unique extension from E_0, and let $\psi\in L_{2,\mathrm{loc}}(E,\mu)$ be such that $\chi_{E_0\cup B}\psi\in\chi_{E_0\cup B}H$ for every bounded Borel set $B\subset E\setminus E_0$. Then $\psi\in H$.*

Proof. Indeed, for every B there is a function $\psi_B\in L_{2,\mathrm{loc}}(E,\mu)$ such that $\chi_{E_0\cup B}\psi_B=\chi_{E_0\cup B}\psi$. Take bounded Borel sets B_1 and B_2 and note that $\chi_{E_0}\psi_{B_1}=\chi_{E_0}\psi_{B_2}=\chi_{E_0}\psi$, whence the unique extension property yields that $\psi_{B_1}=\psi_{B_2}$. Therefore all the functions ψ_B are equal to each other and to ψ, which thus belongs to H. □

Our next proposition gives a sufficient condition for a subspace of locally square-integrable functions to be closed in L_2.

Proposition 2.16. *Let $L\subset L_{2,\mathrm{loc}}(E,\mu)$ be a subspace with the following properties.*

(1) *For every bounded Borel set $B\subset E\setminus E_0$ the subspace $\chi_{E_0\cup B}L$ is closed in $L_2(E,\mu)$.*

(2) *The natural restriction map* $\chi_{E_0\cup B}L \to \chi_{E_0}L$ *is an isomorphism of Hilbert spaces, and the norm of its inverse is bounded above by a positive constant independent of* B.

Then L *is a closed subspace of* $L_2(E,\mu)$ *and the natural restriction map* $L \to \chi_{E_0}L$ *is an isomorphism of Hilbert spaces.*

Proof. If L contains a function with non-integrable square, then the inverse of the restriction isomorphism $\chi_{E_0\cup B}L \to \chi_{E_0}L$ will have an arbitrarily large norm for an appropriate choice of B. Hence it follows from the unique extension property and Proposition 2.15 that L is closed. □

We now proceed to prove Proposition 2.14.

We first check that the following inclusion holds for every bounded Borel set $B \subset E \setminus E_0$:

$$\sqrt{1-g}\,\Pi^{E_0\cup B}\sqrt{1-g}\,\in \mathscr{I}_1(E,\mu).$$

Indeed, by the definition of an infinite determinantal measure we have

$$\chi_B\Pi^{E_0\cup B} \in \mathscr{I}_2(E,\mu),$$

whence, *a fortiori*,

$$\sqrt{1-g}\,\chi_B\Pi^{E_0\cup B} \in \mathscr{I}_2(E,\mu).$$

We now recall that

$$\Pi^{E_0} = \chi_{E_0}\Pi^{E_0\cup B}(1-\chi_B\Pi^{E_0\cup B})^{-1}\Pi^{E_0\cup B}\chi_{E_0}.$$

Then the relation

$$\sqrt{1-g}\,\Pi^{E_0}\sqrt{1-g}\,\in \mathscr{I}_1(E,\mu)$$

implies that

$$\sqrt{1-g}\,\chi_{E_0}\Pi^{E_0\cup B}\chi_{E_0}\sqrt{1-g}\,\in \mathscr{I}_1(E,\mu)$$

or, equivalently,

$$\sqrt{1-g}\,\chi_{E_0}\Pi^{E_0\cup B} \in \mathscr{I}_2(E,\mu).$$

We conclude that

$$\sqrt{1-g}\,\Pi^{E_0\cup B} \in \mathscr{I}_2(E,\mu)$$

or, in other words,

$$\sqrt{1-g}\,\Pi^{E_0\cup B}\sqrt{1-g}\,\in \mathscr{I}_1(E,\mu),$$

as required.

We now check that the subspace $\sqrt{g}\,H\chi_{E_0\cup B}$ is closed in $L_2(E,\mu)$. This follows directly from the closure of $\sqrt{g}\,H$, the property of unique extension from E_0 (the subspace $\sqrt{g}\,H$ possesses it since H does) and the assumption $\chi_{E\setminus E_0}\Pi^g\chi_{E\setminus E_0} \in \mathscr{I}_1(E,\mu)$.

Let $\Pi^{g\chi_{E_0\cup B}}$ be the operator of orthogonal projection onto $\sqrt{g}\,H\chi_{E_0\cup B}$.

It follows from the above that for every bounded Borel set $B \subset E \setminus E_0$, the multiplicative functional Ψ_g is $\mathbb{P}_{\Pi^{E_0\cup B}}$-almost surely positive and, furthermore,

$$\frac{\Psi_g\mathbb{P}_{\Pi^{E_0\cup B}}}{\int \Psi_g\,d\mathbb{P}_{\Pi^{E_0\cup B}}} = \mathbb{P}_{\Pi^{g\chi_{E_0\cup B}}}.$$

Therefore for every bounded Borel set $B \subset E \setminus E_0$ we have

$$\frac{\Psi_{g\chi_{E_0\cup B}}\mathbb{B}}{\int \Psi_{g\chi_{E_0\cup B}}\,d\mathbb{B}} = \mathbb{P}_{\Pi^{g\chi_{E_0\cup B}}}. \tag{25}$$

It remains to note that the assertion of Proposition 2.14 follows immediately from (25). □

2.8. Infinite determinantal measures obtained as finite-rank perturbations of probability determinantal measures.

2.8.1. *Construction of finite-rank perturbations.* We now consider infinite determinantal measures induced by those subspaces H that are obtained by adding a finite-dimensional subspace V to a closed subspace $L \subset L_2(E,\mu)$.

Let $Q \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be the operator of orthogonal projection onto the closed subspace $L \subset L_2(E,\mu)$, and let V a finite-dimensional subspace of $L_{2,\mathrm{loc}}(E,\mu)$ with $V \cap L_2(E,\mu) = 0$. We put $H = L + V$. Let $E_0 \subset E$ be a Borel subset. We impose the following restrictions on L, V and E_0.

Assumption 3. (1) $\chi_{E\setminus E_0} Q \chi_{E\setminus E_0} \in \mathscr{I}_1(E,\mu)$;

(2) $\chi_{E_0} V \subset L_2(E,\mu)$;

(3) *if* $\varphi \in V$ *satisfies* $\chi_{E_0}\varphi \in \chi_{E_0} L$, *then* $\varphi = 0$;

(4) *if* $\varphi \in L$ *satisfies* $\chi_{E_0}\varphi = 0$, *then* $\varphi = 0$.

Proposition 2.17. *If* L, V *and* E_0 *satisfy Assumption* 3, *then the subspace* $H = L + V$ *and the set* E_0 *satisfy Assumption* 2.

In particular, for every bounded Borel set B the subspace $\chi_{E_0\cup B} L$ is closed. This can be seen by taking $E' = E_0 \cup B$ in the following obvious proposition.

Proposition 2.18. *Let* $Q \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ *be the operator of orthogonal projection onto a closed subspace* $L \subset L_2(E,\mu)$. *Let* $E' \subset E$ *be a Borel subset such that* $\chi_{E'} Q \chi_{E'} \in \mathscr{I}_1(E,\mu)$ *and, for every function* $\varphi \in L$, *the equality* $\chi_{E'}\varphi = 0$ *implies that* $\varphi = 0$. *Then the subspace* $\chi_{E'} L$ *is closed in* $L_2(E,\mu)$.

Thus the subspace H and the Borel subset E_0 determine an infinite determinantal measure $\mathbb{B} = \mathbb{B}(H, E_0)$. The measure $\mathbb{B}(H, E_0)$ is infinite by Proposition 2.10.

2.8.2. *Multiplicative functionals of finite-rank perturbations.* Proposition 2.14 has the following immediate corollary.

Corollary 2.19. *Suppose that* L, V *and* E_0 *induce an infinite determinantal measure* $\mathbb{B}$. *Let* $g\colon E \to (0,1]$ *be a positive measurable function with the following properties*:

(1) $\sqrt{g}\,V \subset L_2(E,\mu)$;

(2) $\sqrt{1-g}\,\Pi\sqrt{1-g} \in \mathscr{I}_1(E,\mu)$.

Then the multiplicative functional Ψ_g *is* $\mathbb{B}$*-almost surely positive and* $\mathbb{B}$*-integrable and we have*

$$\frac{\Psi_g\mathbb{B}}{\int \Psi_g\,d\mathbb{B}} = \mathbb{P}_{\Pi^g},$$

where Π^g *is the operator of orthogonal projection onto the closed subspace* $\sqrt{g}\,L + \sqrt{g}\,V$.

2.9. Example: the infinite Bessel point process. We are now ready to prove Proposition 1.1 on the existence of the infinite Bessel point process $\widetilde{\mathbb{B}}^{(s)}$, $s \leqslant -1$. To do this, we need the following property of the ordinary Bessel point process $\tilde{J}_s$, $s > -1$. As above, we denote the range of the projection operator $\tilde{J}_s$ by $\widetilde{L}_s$.

Lemma 2.20. *Take an arbitrary $s > -1$. Then the following assertions hold.*

(1) *For every $R > 0$ the subspace $\chi_{(R,+\infty)}\widetilde{L}_s$ is closed in $L_2((0,+\infty);\mathrm{Leb})$ and the corresponding projection operator $\tilde{J}_{s,R}$ is locally of trace class.*

(2) *For every $R > 0$ we have*

$$\mathbb{P}_{\tilde{J}_s}\bigl(\mathrm{Conf}((0,+\infty);(R,+\infty))\bigr) > 0,$$

$$\frac{\mathbb{P}_{\tilde{J}_s}|_{\mathrm{Conf}((0,+\infty);(R,+\infty))}}{\mathbb{P}_{\tilde{J}_s}\bigl(\mathrm{Conf}((0,+\infty);(R,+\infty))\bigr)} = \mathbb{P}_{\tilde{J}_{s,R}}.$$

Proof. Clearly, for every $R > 0$ we have

$$\int_0^R \tilde{J}_s(x,x)\,dx < +\infty$$

or, equivalently,

$$\chi_{(0,R)}\tilde{J}_s\chi_{(0,R)} \in \mathscr{I}_1((0,+\infty);\mathrm{Leb}).$$

The lemma now follows from the unique extension property of the Bessel point process. □

We now take $s \leqslant -1$ and recall that the number $n_s \in \mathbb{N}$ is defined by the relation

$$\frac{s}{2} + n_s \in \left(-\frac{1}{2}, \frac{1}{2}\right].$$

Put

$$\check{V}^{(s)} = \mathrm{span}\left(y^{s/2}, y^{s/2+1}, \dots, \frac{J_{s+2n_s-1}(\sqrt{y})}{\sqrt{y}}\right).$$

Proposition 2.21. *We have $\dim \check{V}^{(s)} = n_s$ and, for every $R > 0$,*

$$\chi_{(0,R)}\widetilde{V}^{(s)} \cap L_2((0,+\infty);\mathrm{Leb}) = 0.$$

Proof. The following argument was suggested by Yanqi Qiu. By definition of the Bessel kernel, every function in L^{s+2n_s} is the restriction to $\mathbb{R}_+$ of a harmonic function defined on the half-plane $\{z\colon \mathrm{Re}(z) > 0\}$. The desired claim now follows from the uniqueness theorem for harmonic functions. □

Proposition 2.21 immediately yields the existence of the infinite Bessel point process $\widetilde{\mathbb{B}}^{(s)}$, which completes the proof of Proposition 1.1.

By making the change of variable $y = 4/x$, we establish the existence of the modified infinite Bessel point process $\mathbb{B}^{(s)}$. Moreover, using the characterization (described in Proposition 2.14 and Corollary 2.19) of multiplicative functionals of infinite determinantal measures, we arrive at the proof of Propositions 1.5–1.7.

Appendix A. The Jacobi orthogonal polynomial ensemble

A.1. Jacobi polynomials. For $\alpha, \beta > -1$ let $P_n^{(\alpha,\beta)}$ be the standard Jacobi polynomials, namely, polynomials on the closed interval $[-1,1]$ that are orthogonal with the weight

$$(1-u)^\alpha (1+u)^\beta$$

and normalized by the condition

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

We recall that the leading coefficient $k_n^{(\alpha,\beta)}$ of the polynomial $P_n^{(\alpha,\beta)}$ is given by the following formula (see, for example, [32], formula (4.21.6)):

$$k_n^{(\alpha,\beta)} = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n \Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

and for the squared norm we have

$$\begin{aligned} h_n^{(\alpha,\beta)} &= \int_{-1}^{1} (P_n^{(\alpha,\beta)}(u))^2 (1-u)^\alpha (1+u)^\beta \, du \\ &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \, \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \end{aligned}$$

Let $\widetilde{K}_n^{(\alpha,\beta)}(u_1,u_2)$ be the nth Christoffel–Darboux kernel of the Jacobi orthogonal polynomial ensemble:

$$\widetilde{K}_n^{(\alpha,\beta)}(u_1,u_2) = \sum_{l=0}^{n-1} \frac{P_l^{(\alpha,\beta)}(u_1) P_l^{(\alpha,\beta)}(u_2)}{h_l^{(\alpha,\beta)}} (1-u_1)^{\alpha/2}(1+u_1)^{\beta/2}(1-u_2)^{\alpha/2}(1+u_2)^{\beta/2}.$$

The Christoffel—Darboux formula gives an equivalent representation for the kernel $\widetilde{K}_n^{(\alpha,\beta)}$:

$$\begin{aligned} \widetilde{K}_n^{(\alpha,\beta)}(u_1,u_2) &= \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta} \, \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha)\Gamma(n+\beta)} (1-u_1)^{\alpha/2}(1+u_1)^{\beta/2} \\ &\quad \times (1-u_2)^{\alpha/2}(1+u_2)^{\beta/2} \frac{P_n^{(\alpha,\beta)}(u_1) P_{n-1}^{(\alpha,\beta)}(u_2) - P_n^{(\alpha,\beta)}(u_2) P_{n-1}^{(\alpha,\beta)}(u_1)}{u_1-u_2}. \end{aligned} \tag{26}$$

A.2. The recurrence relations between Jacobi polynomials. We have the following recurrence relation between the Christoffel–Darboux kernels $\widetilde{K}_{n+1}^{(\alpha,\beta)}$ and $\widetilde{K}_n^{(\alpha+2,\beta)}$.

Proposition A.1. *For all* $\alpha, \beta > -1$,

$$\begin{aligned} &\widetilde{K}_{n+1}^{(\alpha,\beta)}(u_1,u_2) \\ &\quad = \frac{\alpha+1}{2^{\alpha+\beta+1}} \, \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)} P_n^{(\alpha+1,\beta)}(u_1)(1-u_1)^{\alpha/2}(1+u_1)^{\beta/2} \\ &\qquad \times P_n^{(\alpha+1,\beta)}(u_2)(1-u_2)^{\alpha/2}(1+u_2)^{\beta/2} + \widetilde{K}_n^{(\alpha+2,\beta)}(u_1,u_2). \end{aligned} \tag{27}$$

Remark. Taking the scaling limit of (27), we obtain a similar recurrence relation for Bessel kernels: the Bessel kernel with parameter s is a rank-one perturbation of the Bessel kernel with parameter $s+2$. This can also be easily established directly. Using the recurrence relation

$$J_{s+1}(x)=\frac{2s}{x}J_s(x)-J_{s-1}(x)$$

for the Bessel functions, we immediately obtain the desired relation

$$\tilde{J}_s(x,y)=\tilde{J}_{s+2}(x,y)+\frac{s+1}{\sqrt{xy}}J_{s+1}(\sqrt{x})J_{s+1}(\sqrt{y})$$

for the Bessel kernels.

Proof of Proposition A.1. This routine calculation is included for completeness. We use the standard recurrence relations for Jacobi polynomials. First, we use the relation

$$\left(n+\frac{\alpha+\beta}{2}+1\right)(u-1)P_n^{(\alpha+1,\beta)}(u)=(n+1)P_{n+1}^{(\alpha,\beta)}(u)-(n+\alpha+1)P_n^{(\alpha,\beta)}(u)$$

to obtain that

$$\begin{aligned}\frac{P_{n+1}^{(\alpha,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-P_{n+1}^{(\alpha,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}&=\frac{2n+\alpha+\beta+2}{2(n+1)}\\ \times\frac{(u_1-1)P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-(u_2-1)P_n^{(\alpha+1,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}&.\end{aligned}\tag{28}$$

Then we use the relation

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(u)=(n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)}(u)-(n+\beta)P_{n-1}^{(\alpha+1,\beta)}(u)$$

to arrive at the equality

$$\begin{aligned}&\frac{(u_1-1)P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-(u_2-1)P_n^{(\alpha+1,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}\\ &\quad=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha+1,\beta)}(u_2)+\frac{n+\beta}{2n+\alpha+\beta+1}\\ &\quad\times\frac{(1-u_1)P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+1,\beta)}(u_2)-(1-u_2)P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+1,\beta)}(u_1)}{u_1-u_2}.\end{aligned}\tag{29}$$

Next using the recurrence relation

$$\left(n+\frac{\alpha+\beta+1}{2}\right)(1-u)P_{n-1}^{(\alpha+2,\beta)}(u)=(n+\alpha+1)P_{n-1}^{(\alpha+1,\beta)}(u)-nP_n^{(\alpha+1,\beta)}(u),$$

we arrive at the equality

$$\begin{aligned}&\frac{(1-u_1)P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+1,\beta)}(u_2)-(1-u_2)P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+1,\beta)}(u_1)}{u_1-u_2}\\&\quad=-\frac{n}{n+\alpha+1}P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha+1,\beta)}(u_2)+\frac{2n+\alpha+\beta+1}{2(n+\alpha+1)}(1-u_1)(1-u_2)\\&\qquad\times\frac{P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}.\end{aligned}\tag{30}$$

Combining (29) and (30), we obtain

$$\begin{aligned}&\frac{(u_1-1)P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-(u_2-1)P_n^{(\alpha+1,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}\\&\quad=\frac{(\alpha+1)(2n+\alpha+\beta+1)}{(n+\alpha+1)(2n+\alpha+\beta+1)}P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha+1,\beta)}(u_2)+\frac{n+\beta}{2(n+\alpha+1)}\\&\qquad\times(1-u_1)(1-u_2)\frac{P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}.\end{aligned}\tag{31}$$

Using the recurrence relation

$$(2n+\alpha+\beta+2)P_n^{(\alpha+1,\beta)}(u)=(n+\alpha+\beta+2)P_n^{(\alpha+2,\beta)}(u)-(n+\beta)P_{n-1}^{(\alpha+2,\beta)}(u),$$

we now arrive at the equality

$$\begin{aligned}&\frac{P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}=\frac{n+\alpha+\beta+2}{2n+\alpha+\beta+2}\\&\qquad\times\frac{P_n^{(\alpha+2,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+2,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}.\end{aligned}\tag{32}$$

Combining (28), (31), (32) and using the definition (26) of the Christoffel–Darboux kernels, we conclude the proof of Proposition A.1. □

As above, given a finite family of functions $f_1,\dots,f_N$ on the interval $[-1,1]$ or on the real line, we write $\operatorname{span}(f_1,\dots,f_N)$ for the vector space spanned by these functions. For $\alpha,\beta\in\mathbb{R}$ we introduce the subspaces

$$\begin{aligned}L_{\mathrm{Jac}}^{(\alpha,\beta,n)}=\operatorname{span}\big(&(1-u)^{\alpha/2}(1+u)^{\beta/2},(1-u)^{\alpha/2}(1+u)^{\beta/2}u,\\&\dots,(1-u)^{\alpha/2}(1+u)^{\beta/2}u^{n-1}\big).\end{aligned}$$

Proposition A.1 yields the following orthogonal direct sum decomposition for $\alpha,\beta>-1$:

$$L_{\mathrm{Jac}}^{(\alpha,\beta,n)}=\mathbb{C}P_n^{(\alpha+1,\beta)}\oplus L_{\mathrm{Jac}}^{(\alpha+2,\beta,n-1)}.\tag{33}$$

The relation (33) remains valid for $\alpha\in(-2,-1]$ although the corresponding spaces are no longer subspaces of L_2. It is convenient to restate (33) shifting α by 2.

Proposition A.2. *For all* $\alpha > 0,\ \beta > -1,\ n \in \mathbb{N}$ *we have*

$$L_{\mathrm{Jac}}^{(\alpha-2,\beta,n)} = \mathbb{C}P_n^{(\alpha-1,\beta)} \oplus L_{\mathrm{Jac}}^{(\alpha,\beta,n-1)}.$$

Proof. Let $Q_n^{(\alpha,\beta)}$ be functions of the second kind corresponding to the Jacobi polynomials $P_n^{(\alpha,\beta)}$. By formula (4.62.19) in [32], for all $u \in (-1,1)$ and $\nu > 1$ we have

$$\begin{aligned}\sum_{l=0}^{n} \frac{(2l+\alpha+\beta+1)}{2^{\alpha+\beta+1}} \frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)} P_l^{(\alpha)}(u) Q_l^{(\alpha)}(\nu) \\ = \frac{1}{2}\frac{(\nu-1)^{-\alpha}(\nu+1)^{-\beta}}{(\nu-u)} + \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2} \\ \times \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \frac{P_{n+1}^{(\alpha,\beta)}(u)Q_n^{(\alpha,\beta)}(\nu) - Q_{n+1}^{(\alpha,\beta)}(\nu)P_n^{(\alpha,\beta)}(u)}{\nu-u}.\end{aligned}$$

We pass to the limit as $\nu \to 1$, using the following asymptotic expansion as $\nu \to 1$ for Jacobi functions of the second kind (see [32], formula (4.62.5)):

$$Q_n^{(\alpha)}(v) \sim \frac{2^{\alpha-1}\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}(\nu-1)^{-\alpha}.$$

Recalling the recurrence formula (22.7.19) in [33]:

$$P_{n+1}^{(\alpha-1,\beta)}(u) = (n+\alpha+\beta+1)P_{n+1}^{(\alpha,\beta)} - (n+\beta+1)P_n^{(\alpha,\beta)}(u),$$

we arrive at the relation

$$\frac{1}{1-u} + \frac{\Gamma(\alpha)\Gamma(n+2)}{\Gamma(n+\alpha+1)} P_{n+1}^{(\alpha-1,\beta)} \in L_{\mathrm{Jac}}^{(\alpha,\beta,n)},$$

which immediately yields Proposition A.2. □

We now take $s > -1$ and, for brevity, write $P_n^{(s)} = P_n^{(s,0)}$.

The leading coefficient $k_n^{(s)}$ and the squared norm $h_n^{(s)}$ of the polynomial $P_n^{(s)}$ are given by the formulae

$$\begin{aligned}k_n^{(s)} &= \frac{\Gamma(2n+s+1)}{2^n n!\,\Gamma(n+s+1)}, \\ h_n^{(s)} &= \int_{-1}^{1} (P_n^{(s)}(u))^2(1-u)^s\,du = \frac{2^{s+1}}{2n+s+1}.\end{aligned}$$

We denote the corresponding nth Christoffel–Darboux kernel by $\widetilde{K}_n^{(s)}(u_1,u_2)$:

$$\widetilde{K}_n^{(s)}(u_1,u_2) = \sum_{l=1}^{n-1} \frac{P_l^{(s)}(u_1)P_l^{(s)}(u_2)}{h_l^{(s)}}(1-u_1)^{s/2}(1-u_2)^{s/2}. \tag{34}$$

The Christoffel–Darboux formula gives an equivalent representation of the kernel $\widetilde{K}_n^{(s)}$:

$$\widetilde{K}_n^{(s)}(u_1,u_2) = \frac{n(n+s)}{2^s(2n+s)}(1-u_1)^{s/2}(1-u_2)^{s/2}\frac{P_n^{(s)}(u_1)P_{n-1}^{(s)}(u_2)-P_n^{(s)}(u_2)P_{n-1}^{(s)}(u_1)}{u_1-u_2}. \tag{35}$$

A.3. The Bessel kernel. Consider the half-line $(0,+\infty)$ with the standard Lebesgue measure Leb. Take $s>-1$ and consider the standard Bessel kernel

$$\tilde{J}_s(y_1,y_2) = \frac{\sqrt{y_1}J_{s+1}(\sqrt{y_1})J_s(\sqrt{y_2})-\sqrt{y_2}J_{s+1}(\sqrt{y_2})J_s(\sqrt{y_1})}{2(y_1-y_2)}$$

(see, for example, [5], p. 295).

An alternative integral representation for the kernel $\tilde{J}_s$ is given by

$$\tilde{J}_s(y_1,y_2) = \frac{1}{4}\int_0^1 J_s\bigl(\sqrt{ty_1}\bigr)J_s\bigl(\sqrt{ty_2}\bigr)\,dt \tag{36}$$

(see, for example, formula (2.2) on p. 295 in [5]).

It follows from (36) that the kernel $\tilde{J}_s$ induces on $L_2((0,+\infty);\mathrm{Leb})$ an operator of orthogonal projection onto the subspace of functions whose Hankel transform vanishes everywhere outside $[0,1]$ (see [5]).

Proposition A.3. *For every $s>-1$, as $n\to\infty$, the kernel $\widetilde{K}_n^{(s)}$ converges to $\tilde{J}_s$ uniformly in all variables on compact subsets of $(0,+\infty)\times(0,+\infty)$.*

Proof. This follows immediately from the classical Heine–Mehler asymptotics for Jacobi orthogonal polynomials (see, for example, [32], Ch. VIII). Note that the uniform convergence actually holds on arbitrary simply connected compact subsets of $(\mathbb{C}\setminus\{0\})\times(\mathbb{C}\setminus\{0\})$. □

Appendix B. Spaces of configurations and determinantal point processes

B.1. Spaces of configurations. Let E be a locally compact complete metric space.

A configuration X on E is an unordered family of points called *particles*. The main assumption is that the particles do not accumulate anywhere in E or, equivalently, that every bounded subset of E contains only finitely many particles of the configuration.

With each configuration X we associate a Radon measure

$$\sum_{x\in X}\delta_x,$$

where the sum is taken over all particles in X. Conversely, every purely atomic Radon measure on E is given by a configuration. Thus the space $\mathrm{Conf}(E)$ of all configurations on E can be identified with a closed subset in the set of all integer-valued Radon measures on E. This enables us to endow $\operatorname{Conf}(E)$ with the structure of a complete metric space, which is, however, not locally compact.

The Borel structure on $\operatorname{Conf}(E)$ can be defined equivalently as follows. For every bounded Borel subset $B \subset E$ we introduce the function

$$\#_B \colon \operatorname{Conf}(E) \to \mathbb{R}$$

sending every configuration X to the number of its particles lying in B. The family of functions $\#_B$ over all bounded Borel subsets B of E determines a Borel structure on $\operatorname{Conf}(E)$. In particular, to define a probability measure on $\operatorname{Conf}(E)$, it is necessary and sufficient to determine the joint distributions of the random variables $\#_{B_1}, \dots, \#_{B_k}$ for all finite tuples of disjoint bounded Borel subsets $B_1, \dots, B_k \subset E$.

B.2. The weak topology on the space of probability measures on the space of configurations. The space $\operatorname{Conf}(E)$ is endowed with the natural structure of a complete metric space and, therefore, the space $\mathfrak{M}_{\mathrm{fin}}(\operatorname{Conf}(E))$ of finite Borel measures on the space of configurations is also a complete metric space with respect to the weak topology.

Let $\varphi \colon E \to \mathbb{R}$ be a compactly supported continuous function. We define a measurable function $\#_\varphi \colon \operatorname{Conf}(E) \to \mathbb{R}$ by the formula

$$\#_\varphi(X) = \sum_{x \in X} \varphi(x).$$

For a bounded Borel set $B \subset E$ we have $\#_B = \#_{\chi_B}$.

The Borel σ-algebra on $\operatorname{Conf}(E)$ coincides with the σ-algebra generated by the integer-valued random variables $\#_B$ over all bounded Borel subsets $B \subset E$. Therefore it also coincides with the σ-algebra generated by the random variables $\#_\varphi$ over all compactly supported continuous functions $\varphi \colon E \to \mathbb{R}$. Thus the following proposition holds.

Proposition B.1. *Every Borel probability measure $\mathbb{P} \in \mathfrak{M}_{\mathrm{fin}}(\operatorname{Conf}(E))$ is uniquely determined by the joint distributions of the random variables*

$$\#_{\varphi_1}, \#_{\varphi_2}, \dots, \#_{\varphi_l}$$

over all possible finite tuples of continuous functions $\varphi_1, \dots, \varphi_l \colon E \to \mathbb{R}$ with disjoint compact supports.

The weak topology on $\mathfrak{M}_{\mathrm{fin}}(\operatorname{Conf}(E))$ admits the following characterization in terms of these finite-dimensional distributions (see [34], vol. 2, Theorem 11.1.VII). Let $\mathbb{P}_n$, $n \in \mathbb{N}$, and $\mathbb{P}$ be Borel probability measures on $\operatorname{Conf}(E)$. Then the measures $\mathbb{P}_n$ converge weakly to $\mathbb{P}$ as $n \to \infty$ if and only if for every finite tuple $\varphi_1, \dots, \varphi_l$ of continuous functions with disjoint compact supports, the joint distributions of the random variables $\#_{\varphi_1}, \dots, \#_{\varphi_l}$ with respect to $\mathbb{P}_n$ converge as $n \to \infty$ to the joint distribution of $\#_{\varphi_1}, \dots, \#_{\varphi_l}$ with respect to $\mathbb{P}$ (the convergence of joint distributions is understood in the sense of the weak topology on the space of Borel probability measures on $\mathbb{R}^l$).

B.3. Spaces of locally trace-class operators. Let μ be a σ-finite Borel measure on E. We write $\mathscr{I}_1(E,\mu)$ for the ideal of all trace-class operators $\widetilde{K}\colon L_2(E,\mu)\to L_2(E,\mu)$ (a precise definition is given, for example, in vol. 1 of [35] and in [36]) and denote the $\mathscr{I}_1$-norm of an operator $\widetilde{K}$ by $\|\widetilde{K}\|_{\mathscr{I}_1}$. We also write $\mathscr{I}_2(E,\mu)$ for the ideal of Hilbert–Schmidt operators $\widetilde{K}\colon L_2(E,\mu)\to L_2(E,\mu)$ and denote the $\mathscr{I}_2$-norm of $\widetilde{K}$ by $\|\widetilde{K}\|_{\mathscr{I}_2}$.

Let $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be the space of operators $K\colon L_2(E,\mu)\to L_2(E,\mu)$ such that for every bounded Borel set $B\subset E$ we have

$$\chi_B K\chi_B\in\mathscr{I}_1(E,\mu).$$

We endow the space $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ with a countable family of seminorms

$$\|\chi_B K\chi_B\|_{\mathscr{I}_1},$$

where, as above, B ranges over an exhausting family $\{B_n\}$ of bounded sets.

B.4. Determinantal point processes. A Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ is said to be *determinantal* if there is an operator $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ such that the following equality holds for every bounded measurable function g with $g-1$ supported on a bounded set B:

$$\mathbb{E}_{\mathbb{P}}\Psi_g=\det(1+(g-1)K\chi_B).\tag{37}$$

The Fredholm determinant in (37) is well defined since $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$. The equation (37) determines the measure $\mathbb{P}$ uniquely. For every tuple of disjoint bounded Borel sets $B_1,\dots,B_l\subset E$ and all $z_1,\dots,z_l\in\mathbb{C}$ it follows from (37) that

$$\mathbb{E}_{\mathbb{P}}z_1^{\#_{B_1}}\cdots z_l^{\#_{B_l}}=\det\biggl(1+\sum_{j=1}^{l}(z_j-1)\chi_{B_j}K\chi_{\bigsqcup_i B_i}\biggr).$$

For further results and background on determinantal point processes see, for example, [7], [24], [31], [37]–[43].

For every operator K in $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ we denote the corresponding determinantal measure throughout by $\mathbb{P}_K$. Note that $\mathbb{P}_K$ is uniquely determined by K, but different operators may yield the same measure. By the Macchi–Soshnikov theorem (see [44], [31]) every Hermitian positive contraction belonging to $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ induces a determinantal point process.

B.5. Change of variables. Every homeomorphism $F\colon E\to E$ induces a homeomorphism of the space $\mathrm{Conf}(E)$, which by a slight abuse of notation will again be denoted by F. For every $X\in\mathrm{Conf}(E)$ the particles of the configuration $F(X)$ are of the form $F(x)$, $x\in X$. Let μ be a σ-finite measure on E, and let $\mathbb{P}_K$ be the determinantal measure induced by an operator $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$. We define an operator F_*K by the formula $F_*K(f)=K(f\circ F)$.

Assume that the measures $F_*\mu$ and μ are equivalent and consider the operator

$$K^F=\sqrt{\frac{dF_*\mu}{d\mu}}F_*K\sqrt{\frac{dF_*\mu}{d\mu}}.$$

Note that if K is self-adjoint, then so is K^F. If K is given by a kernel $K(x,y)$, then K^F is given by the kernel

$$K^F(x,y)=\sqrt{\frac{dF_*\mu}{d\mu}(x)}K(F^{-1}x,F^{-1}y)\sqrt{\frac{dF_*\mu}{d\mu}(y)}.$$

The following proposition is obtained directly from the definitions.

Proposition B.2. *The action of the homeomorphism F on the determinantal measure $\mathbb{P}_K$ is given by the formula*

$$F_*\mathbb{P}_K=\mathbb{P}_{K^F}.$$

Note that if K is the operator of orthogonal projection onto a closed subspace $L\subset L_2(E,\mu)$, then, by definition, K^F is the operator of orthogonal projection onto the closed subspace

$$\left\{\varphi\circ F^{-1}(x)\sqrt{\frac{dF_*\mu}{d\mu}(x)}\right\}\subset L_2(E,\mu).$$

B.6. Multiplicative functionals on spaces of configurations. Let g be an arbitrary non-negative measurable function on E. We introduce the *multiplicative functional* $\Psi_g\colon \operatorname{Conf}(E)\to\mathbb{R}$ by the formula

$$\Psi_g(X)=\prod_{x\in X}g(x).$$

If the infinite product $\prod_{x\in X}g(x)$ converges absolutely to 0 or ∞, then we put $\Psi_g(X)=0$ or $\Psi_g(X)=\infty$ respectively. If the product on the right-hand side does not converge absolutely, then the multiplicative functional is not defined at the point considered.

B.7. Determinantal point processes and multiplicative functionals. The construction of infinite determinantal measures is based on the results of [8], [9], which may informally be stated as follows: the product of a determinantal measure and a multiplicative functional is again a determinantal measure. In other words, if $\mathbb{P}_K$ is the determinantal measure on $\operatorname{Conf}(E)$ induced by an operator K on $L_2(E,\mu)$, then it was shown under certain additional assumptions in [8], [9] that the measure $\Psi_g\mathbb{P}_K$ yields a determinantal point process after normalization.

As above, let g be a non-negative measurable function on E. If the operator $1+(g-1)K$ is invertible, then we put

$$\mathfrak{B}(g,K)=gK(1+(g-1)K)^{-1},\qquad \widetilde{\mathfrak{B}}(g,K)=\sqrt{g}\,K(1+(g-1)K)^{-1}\sqrt{g}.$$

By definition, $\mathfrak{B}(g,K),\widetilde{\mathfrak{B}}(g,K)\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ since $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$. If K is self-adjoint, then so is $\widetilde{\mathfrak{B}}(g,K)$.

We now recall some propositions from [9].

Proposition B.3. *Let $K \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be a positive self-adjoint contraction, $\mathbb{P}_K$ the corresponding determinantal measure on* $\mathrm{Conf}(E)$*, and g a non-negative bounded measurable function on E such that*

$$\sqrt{g-1}\,K\sqrt{g-1}\in \mathscr{I}_1(E,\mu) \tag{38}$$

and the operator $1+(g-1)K$ is invertible. Then the following assertions hold.

(1) *We have $\Psi_g \in L_1(\mathrm{Conf}(E),\mathbb{P}_K)$ and*

$$\int \Psi_g\, d\mathbb{P}_K = \det\bigl(1+\sqrt{g-1}\,K\sqrt{g-1}\,\bigr) > 0.$$

(2) *The operators $\mathfrak{B}(g,K), \widetilde{\mathfrak{B}}(g,K)$ induce a determinantal measure $\mathbb{P}_{\mathfrak{B}(g,K)} = \mathbb{P}_{\widetilde{\mathfrak{B}}(g,K)}$ on* $\mathrm{Conf}(E)$ *satisfying*

$$\mathbb{P}_{\mathfrak{B}(g,K)} = \frac{\Psi_g \mathbb{P}_K}{\int_{\mathrm{Conf}(E)} \Psi_g\, d\mathbb{P}_K}.$$

Remark. Suppose that the condition (38) holds and K is self-adjoint. Then the operator $1+(g-1)K$ is invertible if and only if $1+\sqrt{g-1}\,K\sqrt{g-1}$ is.

If Q is a projection operator, then the operator $\tilde{\mathfrak{B}}(g,Q)$ admits the following description.

Proposition B.4. *Let $L\subset L_2(E,\mu)$ be a closed subspace, Q the operator of orthogonal projection onto L, and g a bounded measurable non-negative function such that the operator $1+(g-1)Q$ is invertible. Then $\tilde{\mathfrak{B}}(g,Q)$ is the operator of orthogonal projection onto the closure of $\sqrt{g}\,L$.*

We now consider the particular case when g is the characteristic function of a Borel subset. If $E'\subset E$ is a Borel subset such that the subspace $\chi_{E'}L$ is closed (we recall that Proposition 2.18 gives a sufficient condition for this), then we write $Q^{E'}$ for the operator of orthogonal projection onto $\chi_{E'}L$.

We obtain the following corollary of Proposition B.3.

Corollary B.5. *Let $Q\in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be the operator of orthogonal projection onto a closed subspace $L\in L_2(E,\mu)$, and let $E'\subset E$ be a Borel subset such that $\chi_{E'}Q\chi_{E'}\in \mathscr{I}_1(E,\mu)$. Then*

$$\mathbb{P}_Q(\mathrm{Conf}(E,E')) = \det(1-\chi_{E\setminus E'}Q\chi_{E\setminus E'}).$$

Assume further that for every function $\varphi\in L$ the equality $\chi_{E'}\varphi=0$ implies that $\varphi=0$. Then the subspace $\chi_{E'}L$ is closed and we have

$$\mathbb{P}_Q(\mathrm{Conf}(E,E'))>0,\qquad Q^{E'}\in \mathscr{I}_{1,\mathrm{loc}}(E,\mu),$$

and

$$\frac{\mathbb{P}_Q|_{\mathrm{Conf}(E,E')}}{\mathbb{P}_Q(\mathrm{Conf}(E,E'))} = \mathbb{P}_{Q^{E'}}.$$

Thus the measure induced from a determinantal measure on the set of configurations all of whose particles lie in E' is again a determinantal measure. In the case of a discrete phase space, the related induced processes were considered by Lyons [39] and Borodin and Rains [45].

We now state a sufficient condition for a multiplicative functional to be *positive* on almost all configurations.

Proposition B.6. *If*

$$\mu(\{x \in E\colon g(x)=0\})=0, \qquad \sqrt{|g-1|}\,K\sqrt{|g-1|}\,\in \mathscr{I}_1(E,\mu),$$

then

$$0<\Psi_g(X)<+\infty$$

for $\mathbb{P}_K$-almost all configurations $X \in \mathrm{Conf}(E)$.

Proof. Our assumptions imply that for $\mathbb{P}_K$-almost all $X\in\mathrm{Conf}(E)$ we have

$$\sum_{x\in X}|g(x)-1|<+\infty,$$

which is in its turn sufficient for the absolute convergence of the infinite product $\prod_{x\in X} g(x)$ to a finite non-zero limit. □

We state a version of Proposition B.3 in the particular case when the function g assumes no values less than 1. In this case the multiplicative functional Ψ_g is automatically non-zero and we obtain the following result.

Proposition B.7. *Let $\Pi \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be the operator of orthogonal projection onto a closed subspace H, and let g be a bounded Borel function on E such that $g(x)\geqslant 1$ for all $x\in E$. Assume that*

$$\sqrt{g-1}\,\Pi\sqrt{g-1}\,\in\mathscr{I}_1(E,\mu).$$

Then the following assertions hold.

(1) *We have $\Psi_g\in L_1(\mathrm{Conf}(E),\mathbb{P}_\Pi)$ and*

$$\int\Psi_g\,d\mathbb{P}_\Pi=\det\bigl(1+\sqrt{g-1}\,\Pi\sqrt{g-1}\,\bigr).$$

(2) *We have*

$$\frac{\Psi_g\mathbb{P}_\Pi}{\int\Psi_g\,d\mathbb{P}_\Pi}=\mathbb{P}_{\Pi^g},$$

where Π^g is the operator of orthogonal projection onto $\sqrt{g}\,H$.

Appendix C. Construction of Pickrell measures and proof of Proposition 1.8

We recall that the Pickrell measures are naturally defined on the space of $m\times n$ *rectangular* matrices.

Let $\mathrm{Mat}(m \times n, \mathbb{C})$ be the space of complex $m \times n$ matrices:

$$\mathrm{Mat}(m \times n, \mathbb{C}) = \{z = (z_{ij}),\ i = 1, \dots, m,\ j = 1, \dots, n\}.$$

We write dz for the Lebesgue measure on $\mathrm{Mat}(m \times n, \mathbb{C})$.

Take $s \in \mathbb{R}$ and $m_0, n_0 \in \mathbb{N}$ such that $m_0 + s > 0$, $n_0 + s > 0$. Following Pickrell, we take $m > m_0$, $n > n_0$ and introduce a measure $\mu^{(s)}_{m,n}$ on $\mathrm{Mat}(m \times n, \mathbb{C})$ by the formula

$$\mu^{(s)}_{m,n} = \mathrm{const}^{(s)}_{m,n} \det(1 + z^* z)^{-m-n-s}\, dz,$$

where

$$\mathrm{const}^{(s)}_{m,n} = \pi^{-mn} \prod_{l=m_0}^{m} \frac{\Gamma(l+s)}{\Gamma(n+l+s)}.$$

For $m_1 \leqslant m$ and $n_1 \leqslant n$ we consider the natural projections

$$\pi^{m,n}_{m_1,n_1} \colon \mathrm{Mat}(m \times n, \mathbb{C}) \to \mathrm{Mat}(m_1 \times n_1, \mathbb{C}).$$

Proposition C.1. *Let $m, n \in \mathbb{N}$ be such that $s > -m-1$. Then for all $\tilde z \in \mathrm{Mat}(n, \mathbb{C})$ we have*

$$\int_{(\pi^{m+1,n}_{m,n})^{-1}(\tilde z)} \det(1+z^*z)^{-m-n-1-s}\, dz = \pi^n \frac{\Gamma(m+1+s)}{\Gamma(n+m+1+s)} \det(1+\tilde z^* \tilde z)^{-m-n-s}.$$

Proposition 1.8 is an immediate corollary of Proposition C.1.

Proof of Proposition C.1. As already mentioned in the introduction, the following computation goes back to the classical work of Hua Loo-Keng [10]. Take $z \in \mathrm{Mat}((m+1) \times n, \mathbb{C})$. Multiplying, if necessary, by a unitary matrix on the left and on the right, we represent the matrix $\pi^{m+1,n}_{m,n} z = \widetilde z$ in diagonal form with positive real diagonal entries: $\widetilde z_{ii} = u_i > 0$, $i = 1, \dots, n$, $\widetilde z_{ij} = 0$ for $i \neq j$.

Here we put $u_i = 0$ when $i > \min(n, m)$. Writing $\xi_i = z_{m+1,i}$ for $i = 1, \dots, n$, we transform the determinant as follows:

$$\det(1 + z^* z)^{-m-1-n-s} = \prod_{i=1}^{m} (1 + u_i^2)^{-m-1-n-s} \left(1 + \xi^* \xi - \sum_{i=1}^{n} \frac{|\xi_i|^2 u_i^2}{1 + u_i^2}\right)^{-m-1-n-s}.$$

Simplify the expression in parentheses:

$$1 + \xi^* \xi - \sum_{i=1}^{n} \frac{|\xi_i|^2 u_i^2}{1 + u_i^2} = 1 + \sum_{i=1}^{n} \frac{|\xi_i|^2}{1 + u_i^2}.$$

Integrating with respect to ξ, we obtain

$$\int \left(1 + \sum_{i=1}^{n} \frac{|\xi_i|^2}{1 + u_i^2}\right)^{-m-1-n-s} d\xi = \prod_{i=1}^{m} (1 + u_i^2) \frac{\pi^n}{\Gamma(n)} \int_0^{+\infty} r^{n-1} (1+r)^{-m-1-n-s}\, dr,$$

where

$$r = r^{(m+1,n)}(z) = \sum_{i=1}^{n} \frac{|\xi_i|^2}{1 + u_i^2}. \tag{39}$$

Using the Euler integral

$$\int_0^{+\infty} r^{n-1}(1+r)^{-m-1-n-s}\,dr = \frac{\Gamma(n)\Gamma(m+1+s)}{\Gamma(n+1+m+s)},$$

we arrive at the desired equality. □

We introduce a map

$$\widetilde{\pi}^{m+1,n}_{m,n}\colon \operatorname{Mat}((m+1)\times n,\mathbb{C}) \to \operatorname{Mat}(m\times n,\mathbb{C})\times\mathbb{R}_+$$

by the formula

$$\widetilde{\pi}^{m+1,n}_{m,n}(z) = (\pi^{m+1,n}_{m,n}(z), r^{(m+1,n)}(z)),$$

where $r^{(m+1,n)}(z)$ is given by the formula (39). Let $P^{(m,n,s)}$ be a probability measure on $\mathbb{R}_+$ such that

$$dP^{(m,n,s)}(r) = \frac{\Gamma(n+m+s)}{\Gamma(n)\Gamma(m+s)}\, r^{n-1}(1-r)^{-m-n-s}\,dr.$$

The measure $P^{(m,n,s)}$ is well defined for $m+s>0$.

Corollary C.2. *For all $m,n\in\mathbb{N}$ and $s>-m-1$ we have*

$$\big(\widetilde{\pi}^{m+1,n}_{m,n}\big)_*\mu^{(s)}_{m+1,n} = \mu^{(s)}_{m,n}\times P^{(m+1,n,s)}.$$

Indeed, this is precisely what was shown by our computations.

Removing a column is similar to removing a row:

$$\big(\pi^{m,n+1}_{m,n}(z)\big)^t = \pi^{m+1,n}_{m,n}(z^t).$$

We introduce the notation $\widetilde{r}^{(m,n+1)}(z) = r^{(n+1,m)}(z^t)$ and define a map

$$\widetilde{\pi}^{m,n+1}_{m,n}\colon \operatorname{Mat}(m\times(n+1),\mathbb{C}) \to \operatorname{Mat}(m\times n,\mathbb{C})\times\mathbb{R}_+$$

by the formula

$$\widetilde{\pi}^{m,n+1}_{m,n}(z) = \big(\pi^{m,n+1}_{m,n}(z), \widetilde{r}^{(m,n+1)}(z)\big).$$

Corollary C.3. *For all $m,n\in\mathbb{N}$ and $s>-m-1$ we have*

$$\big(\widetilde{\pi}^{m,n+1}_{m,n}\big)_*\mu^{(s)}_{m,n+1} = \mu^{(s)}_{m,n}\times P^{(n+1,m,s)}.$$

We now take an n such that $n+s>0$, and define the map

$$\widetilde{\pi}_n\colon \operatorname{Mat}(\mathbb{N}\times\mathbb{N},\mathbb{C}) \to \operatorname{Mat}(n\times n,\mathbb{C})$$

by the formula

$$\widetilde{\pi}_n(z) = \big(\pi^{\infty,\infty}_{n,n}(z), r^{(n+1,n)}, \widetilde{r}^{(n+1,n+1)}, r^{(n+2,n+1)}, \widetilde{r}^{(n+2,n+2)}, \dots\big).$$

Recalling the definition of the Pickrell measure $\mu^{(s)}$ on $\mathrm{Mat}(\mathbb{N}\times\mathbb{N},\mathbb{C})$ (see § 1.7.1), we can now restate the result of our computations as follows.

Proposition C.4. *If* $n+s>0$, *then*

$$(\widetilde{\pi}_n)_*\mu^{(s)} = \mu^{(s)}_{n,n} \times \prod_{l=0}^{\infty}\bigl(P^{(n+l+1,n+l,s)} \times P^{(n+l+1,n+l+1,s)}\bigr). \tag{40}$$

Bibliography

[1] D. Pickrell, "Measures on infinite dimensional Grassmann manifolds", *J. Funct. Anal.* **70**:2 (1987), 323–356.

[2] D. M. Pickrell, "Mackey analysis of infinite classical motion groups", *Pacific J. Math.* **150**:1 (1991), 139–166.

[3] G. Ol'shanskii and A. Vershik, "Ergodic unitarily invariant measures on the space of infinite Hermitian matrices", *Contemporary mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, vol. 175, Amer. Math. Soc., Providence, RI 1996, pp. 137–175.

[4] A. Borodin and G. Olshanski, "Infinite random matrices and ergodic measures", *Comm. Math. Phys.* **223**:1 (2001), 87–123.

[5] C. A. Tracy and H. Widom, "Level spacing distributions and the Bessel kernel", *Comm. Math. Phys.* **161**:2 (1994), 289–309.

[6] A. I. Bufetov, "Finiteness of ergodic unitarily invariant measures on spaces of infinite matrices", *Ann. Inst. Fourier* (*Grenoble*) **64**:3 (2014), 893–907; arXiv: 1108.2737.

[7] T. Shirai and Y. Takahashi, "Random point fields associated with certain Fredholm determinants. II. Fermion shifts and their ergodic and Gibbs properties", *Ann. Probab.* **31**:3 (2003), 1533–1564.

[8] A. I. Bufetov, "Multiplicative functionals of determinantal processes", *Uspekhi Mat. Nauk* **67**:1(403) (2012), 177–178; English transl., *Russian Math. Surveys* **67**:1 (2012), 181–182.

[9] A. I. Bufetov, "Infinite determinantal measures", *Electron. Res. Announc. Math. Sci.* **20** (2013), 12–30.

[10] L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, translated from the Chinese (Science Press, Peking 1958), Inostr. Lit., Moscow 1959 (Russian); English transl., Transl. Math. Monogr., vol. 6, Amer. Math. Soc., Providence, RI 1963.

[11] Yu. A. Neretin, "Hua-type integrals over unitary groups and over projective limits of unitary groups", *Duke Math. J.* **114**:2 (2002), 239–266.

[12] P. Bourgade, A. Nikeghbali, and A. Rouault, "Ewens measures on compact groups and hypergeometric kernels", *Séminaire de Probabilités* XLIII, Lecture Notes in Math., vol. 2006, Springer, Berlin 2011, pp. 351–377.

[13] A. I. Bufetov, "Infinite determinantal measures and the ergodic decomposition of infinite Pickrell measures. II", *Izv. Ross. Akad. Nauk Ser. Mat.* (to appear).

[14] A. I. Bufetov, "Infinite determinantal measures and the ergodic decomposition of infinite Pickrell measures. III", *Izv. Ross. Akad. Nauk Ser. Mat.* (to appear).

[15] A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergeb. Math. Grenzgeb., vol. 2, J. Springer, Berlin 1933; Russian transl., 2nd ed., Nauka, Moscow 1974.

[16] G. I. Olshanskii, "Unitary representations of infinite dimensional pairs (G, K) and the formalism of R. Howe", *Representation of Lie groups and related topics*, Adv. Stud. Contemp. Math., vol. 7, Gordon and Breach, New York 1990, pp. 269–463; http://www.iitp.ru/upload/userpage/52/HoweForm.pdf.

[17] G. I. Ol'shanskii, *Unitary representations of infinite-dimensional classical groups*, Diss. Doct. Phys.-Math. Sci., Institute of Geography, Academy of Sciences of the USSR, Moscow 1989, iitp.ru/upload/userpage/52/Olshanski_thesis.pdf. (Russian)

[18] A. I. Bufetov, "Ergodic decomposition for measures quasi-invariant under a Borel action of an inductively compact group", *Mat. Sb.* **205**:2 (2014), 39–70; English transl., *Sb. Math.* **205**:2 (2014), 192–219; *Ergodic decomposition for measures quasi-invariant under Borel actions of inductively compact groups*, arXiv: 1105.0664.

[19] D. Pickrell, "Separable representations of automorphism groups of infinite symmetric spaces", *J. Funct. Anal.* **90**:1 (1990), 1–26.

[20] A. M. Vershik, "Description of invariant measures for the actions of some infinite-dimensional groups", *Dokl. Akad. Nauk SSSR* **218**:4 (1974), 749–752; English transl., *Soviet Math. Dokl.* **15**:5 (1974), 1396–1400.

[21] M. Rabaoui, "Asymptotic harmonic analysis on the space of square complex matrices", *J. Lie Theory* **18**:3 (2008), 645–670.

[22] M. Rabaoui, "A Bochner type theorem for inductive limits of Gelfand pairs", *Ann. Inst. Fourier (Grenoble)* **58**:5 (2008), 1551–1573.

[23] V. I. Bogachev, *Measure theory*, vol. II, Springer-Verlag, Berlin 2007.

[24] T. Shirai and Y. Takahashi, "Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes", *J. Funct. Anal.* **205**:2 (2003), 414–463.

[25] J. Faraut, *Analyse sur les groupes de Lie. Une introduction*, Calvage et Mounet, Paris 2006.

[26] J. Faraut, *Analysis on Lie groups*, Cambridge Stud. Adv. Math., vol. 110, Cambridge Univ. Press, Cambridge 2008.

[27] S. Ghobber and P. Jaming, "Strong annihilating pairs for the Fourier–Bessel transform", *J. Math. Anal. Appl.* **377**:2 (2011), 501–515.

[28] S. Ghobber and P. Jaming, "Uncertainty principles for integral operators", *Studia Math.* **220**:3 (2014), 197–220; arXiv: 1206.1195v1.

[29] E. M. Baruch, *The classical Hankel transform in the Kirillov model*, arXiv: 1010.5184v1.

[30] A. Lenard, "States of classical statistical mechanical systems of infinitely many particles. I", *Arch. Rational Mech. Anal.* **59**:3 (1975), 219–239.

[31] A. Soshnikov, "Determinantal random point fields", *Uspekhi Mat. Nauk* **55**:5(335) (2000), 107–160; English transl., *Russian Math. Surveys* **55**:5 (2000), 923–975.

[32] G. Szegő, *Orthogonal polynomials*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI 1959; Russian transl., Fizmatlit, Moscow 1962.

[33] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions*, National Bureau of Standards Applied Mathematics Series, vol. 55, Superintendent of Documents, U.S. Government Printing Office, Washington, DC 1964; Russian transl., Nauka, Moscow 1979.

[34] D. J. Daley and D. Vere-Jones, *An introduction to the theory of point processes*, vols. I, II, 2nd ed., Probab. Appl. (N. Y.), Springer-Verlag, New York 2003, 2008.

[35] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. I, 2nd ed., Academic Press, New York–London 1980; vols. II–IV, 1st ed., 1975, 1979, 1978; Russian transl., vols. 1–4, Mir, Moscow 1977, 1978, 1982, 1982.

[36] B. Simon, *Trace ideals and their applications*, 2nd ed., Math. Surveys Monogr., vol. 120, Amer. Math. Soc., Providence, RI 2005.

[37] A. M. Borodin, "Determinantal point processes", *The Oxford handbook of random matrix theory*, Oxford Univ. Press, Oxford 2011, pp. 231–249.

[38] J. Ben Hough, M. Krishnapur, Y. Peres, and B. Virág, "Determinantal processes and independence", *Probab. Surv.* **3** (2006), 206–229.

[39] R. Lyons, "Determinantal probability measures", *Publ. Math. Inst. Hautes Études Sci.* **98**:1 (2003), 167–212.

[40] R. Lyons and J. E. Steif, "Stationary determinantal processes: phase multiplicity, Bernoullicity, entropy, and domination", *Duke Math. J.* **120**:3 (2003), 515–575.

[41] E. Lytvynov, "Fermion and boson random point processes as particle distributions of infinite free Fermi and Bose gases of finite density", *Rev. Math. Phys.* **14**:10 (2002), 1073–1098.

[42] T. Shirai and Y. Takahashi, "Random point fields associated with fermion, boson and other statistics", *Stochastic analysis on large scale interacting systems*, Adv. Stud. Pure Math., vol. 39, Math. Soc. Japan, Tokyo 2004, pp. 345–354.

[43] G. Olshanski, "The quasi-invariance property for the Gamma kernel determinantal measure", *Adv. Math.* **226**:3 (2011), 2305–2350.

[44] O. Macchi, "The coincidence approach to stochastic point processes", *Advances in Appl. Probability* **7**:1 (1975), 83–122.

[45] A. Borodin and E. M. Rains, "Eynard–Mehta theorem, Schur process, and their Pfaffian analogs", *J. Stat. Phys.* **121**:3-4 (2005), 291–317.

Alexander I. Bufetov
Steklov Mathematical Institute,
Russian Academy of Sciences, Moscow
Institute of information transmission problems,
Russian Academy of Sciences, Moscow
National Research University 'Higher
School of Economics', Moscow
Aix-Marseille Université, CNRS,
Centrale Marseille, I2M, UMR 7373
E-mail: bufetov@mi.ras.ru

Received 6/APR/15
Translated by P. P. NIKITIN