Higher Structures in Homotopy Type Theory

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Joint work with Eric Finster

Plan

1. Algebra in HoTT
2. A universe of polynomial monads
3. Opetopic types and their applications

Sets

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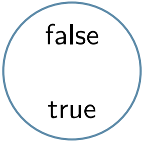

Figure: the set of booleans

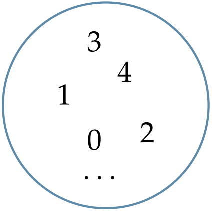

Figure: the set of natural numbers

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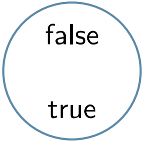

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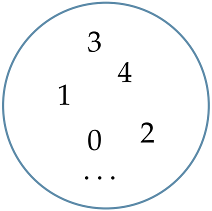

Figure: the set of natural numbers

Two elements of a set are *equal* if they have the same *definition*.

Principle of equivalence

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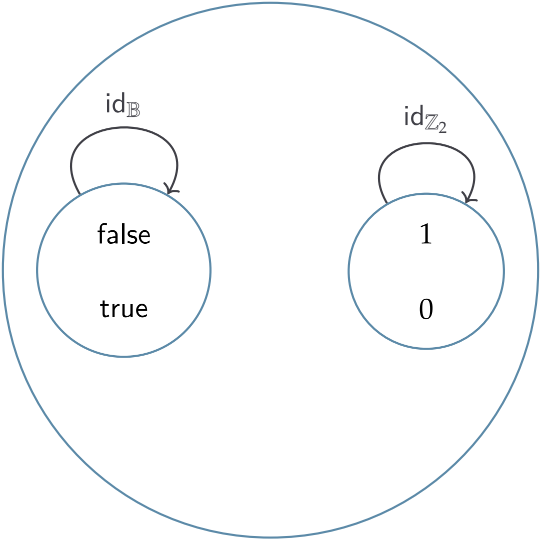
Set theory does not respect this principle (e.g., two bijective sets are not necessarily equal).

Principle of equivalence

Equality in sets does not account for all equivalences.



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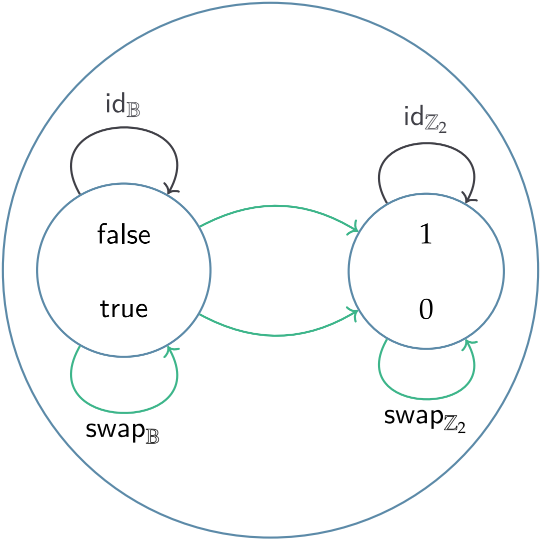

Homotopy type theory

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Type theory is a language rich enough to unify mathematical constructions and logical propositions.

Proposition-as-types paradigm

The correspondence goes as follows.

Logic	Type theory
$\bot$	$\mathbf{0}$
$A \wedge B$	$A \times B$
$A \vee B$	$A + B$
$A \implies B$	$A \to B$
$\exists (x \in A).B(x)$	$(x : A) \times B(x)$
$\forall (x \in A).B(x)$	$(x : A) \to B(x)$

Identity types

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Formal version of Leibniz's identity of indiscernibles.

Univalence

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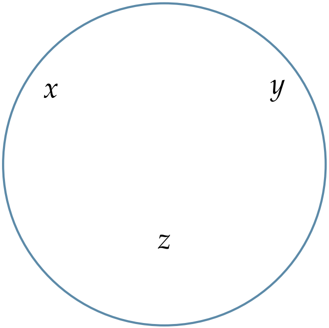
In general, no uniqueness of identity proofs (UIP).

Types

Types contain elements—like sets.



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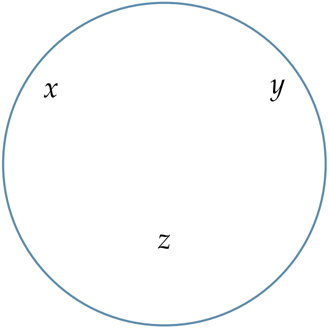
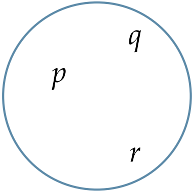


The proposition $x = y$ is a type.



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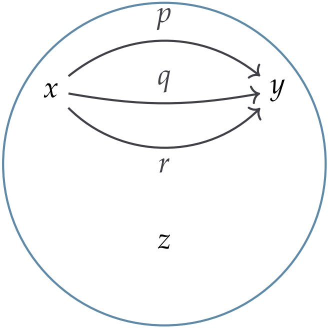

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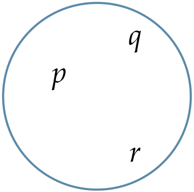

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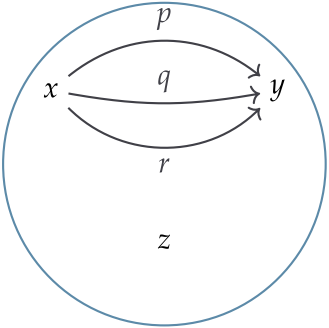

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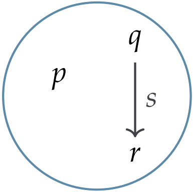

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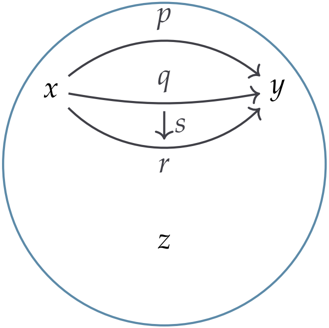

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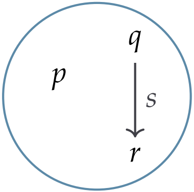
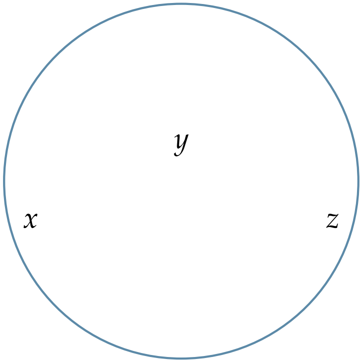


Any element comes with a distinguished loop called *reflexivity*.



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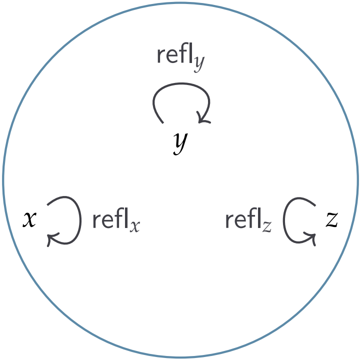
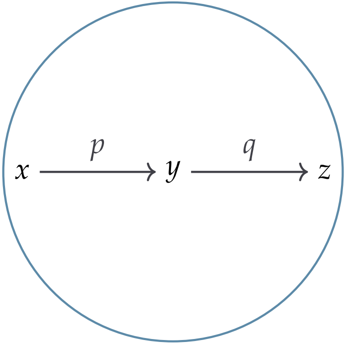


Identities can be composed (transitivity of equality).



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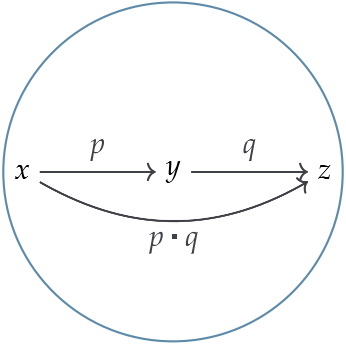
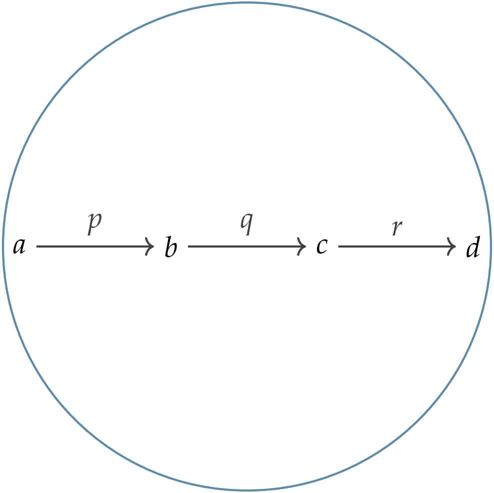


Algebra of paths

Composition of identities is associative up to a *higher identity*.



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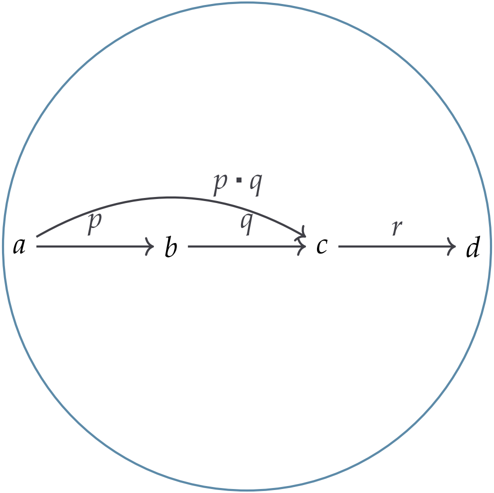

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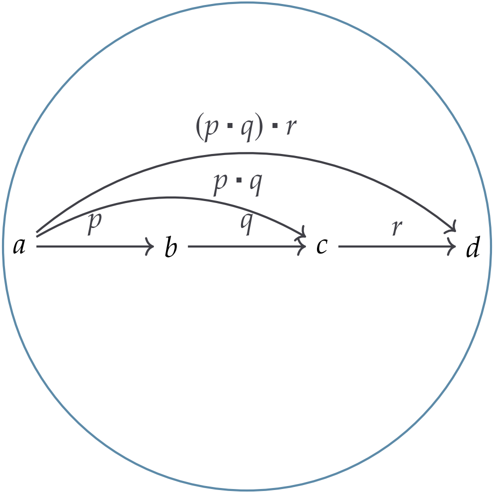

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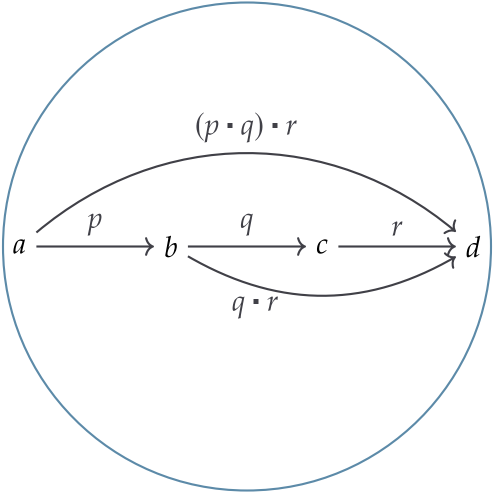

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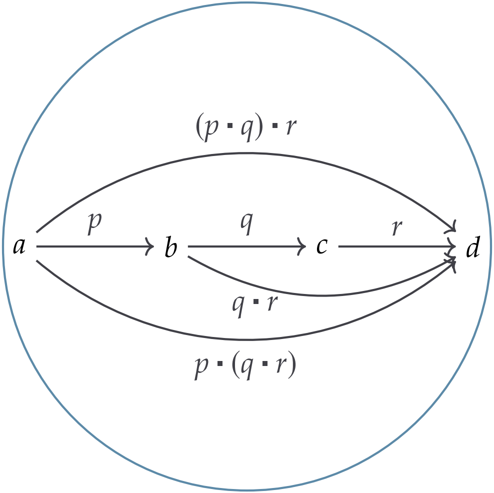

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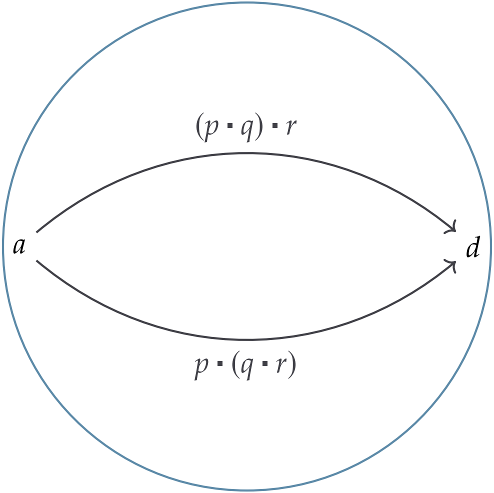

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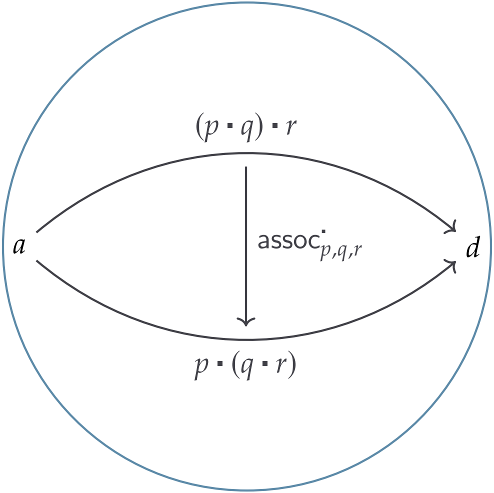

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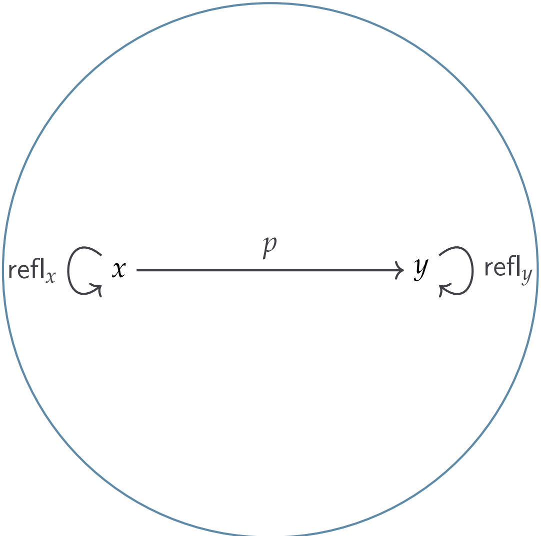

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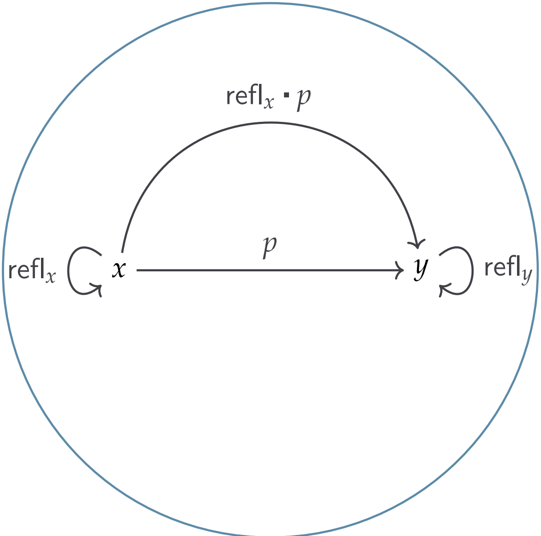

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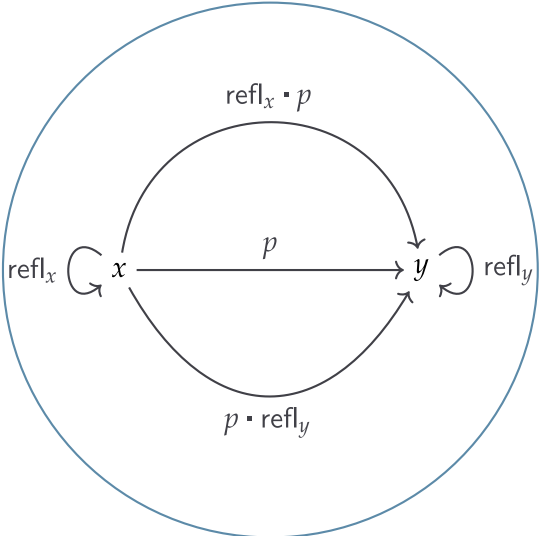

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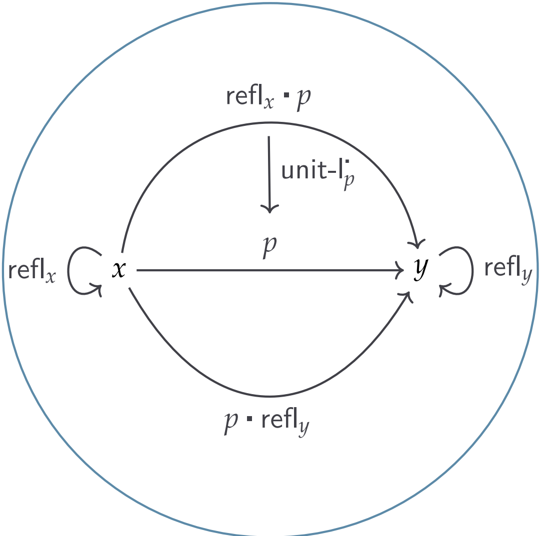

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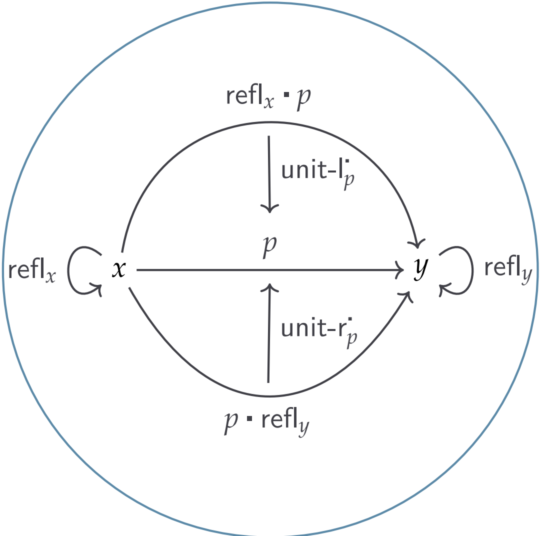
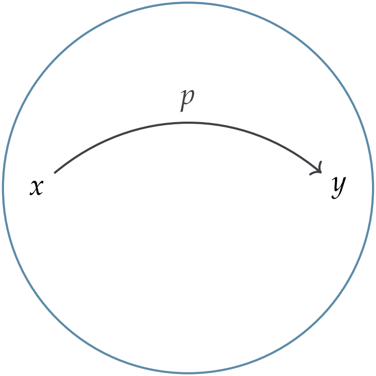


Identities can be inverted.



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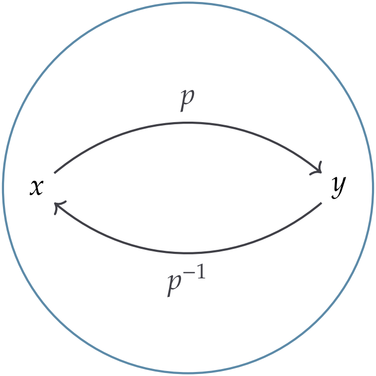

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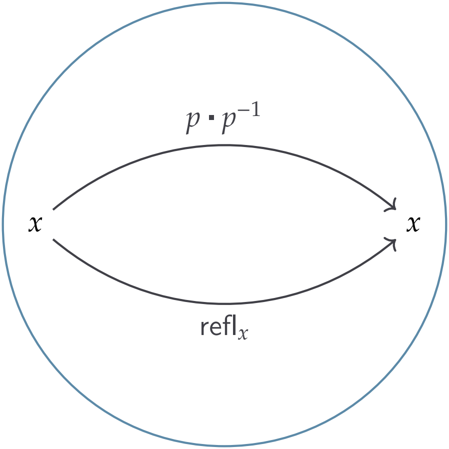

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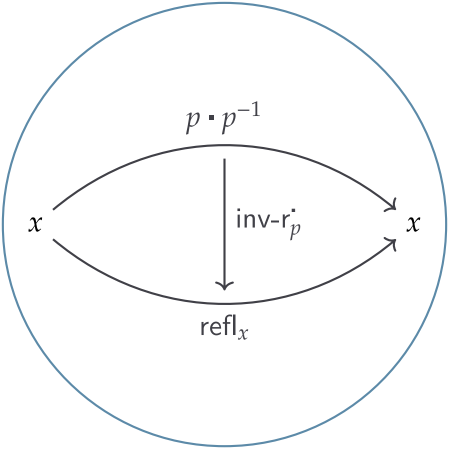

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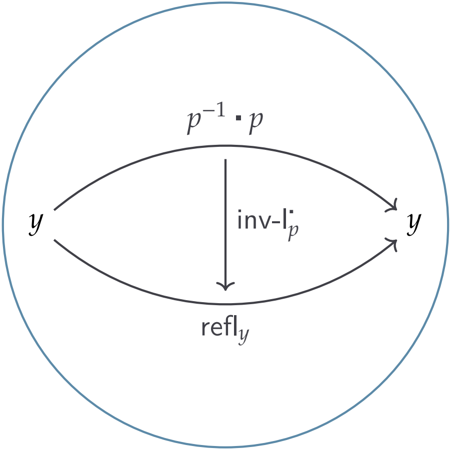
Similarly, there is an identity



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This structure we just described is the one of ∞-groupoid.

Algebra on types

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An algebraic structure on a type is an operation acting on the elements of this type satisfying *coherent* laws expressed in terms of identities.

Algebra on a type

Example

An associative magma on a type X is the data of a binary operation

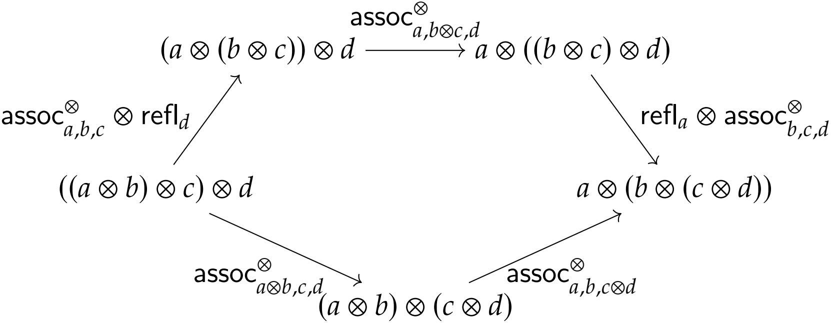
$$\_ \otimes \_ : X \times X \to X$$

along with an identity

$$(a \otimes b) \otimes c \xrightarrow{\mathsf{assoc}^{\otimes}_{a,b,c}} a \otimes (b \otimes c)$$

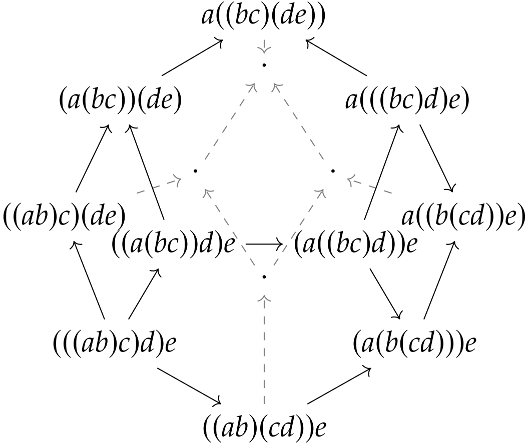
for any $a, b, c : X$ witnessing that the multiplication is associative.

In addition, we require this identity to be such that the following diagram commutes up to a higher identity:



Coherence

In turn, this new data has to satisfy its own coherence conditions leading to an infinite tower of data described by Stasheff's associahedra K_n.



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Sets are degenerate types whose only identities are reflexivities. Laws of algebraic structures on sets are therefore trivially coherent.

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Spaces are primitive in type theory, any algebraic structure must be stated coherently in the first place.

A theory of structures

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Type theory seems to be missing a theory of structures.

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This approach is compatible with univalence.

Applications

In this type theory, the following results have been established:

- Fibrant opetopic types are equivalent to Baez-Dolan coherent algebras whose morphisms are invertible.
- The internal ∞-groupoid associated to a type.
- The $(\infty, 1)$-category of types.
- Adjunctions between $(\infty, 1)$-categories.
- Fibrant opetopic types are closed under dependent sums.

Setting

The base type theory is book HoTT with Agda's features (coinductive records, inductive-recursive types, …).

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Most of this work has been formalised in Agda using postulates and rewrite rules to define the universe of polynomial monads.

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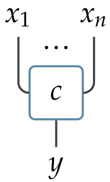
Constructors depicted as corollas:



Polynomial monads

The structure of cartesian polynomial monad is defined by a unit operation η and a multiplication operation μ:

$$\eta_M : (i : \mathsf{Idx}_M) \to \mathsf{Cns}_M(i)$$

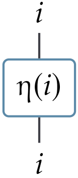
$$\mu_M : \{i : \mathsf{Idx}_M\}\ (c : \mathsf{Cns}_M(i)) \to \overrightarrow{\mathsf{Cns}_M}(c) \to \mathsf{Cns}_M(i)$$

Polynomial monads

The unit

$$\eta_M : (i : \mathsf{Idx}_M) \to \mathsf{Cns}_M(i)$$

Units $\eta(i)$ are *unary* constructors whose source and target have the same sort:

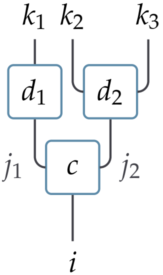


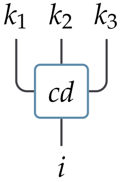
Polynomial monads

The multiplication

$$\mu_M : \{i : \mathsf{Idx}_M\}\ (c : \mathsf{Cns}_M(i))\ (d : \overrightarrow{\mathsf{Cns}_M}(c)) \longrightarrow \mathsf{Cns}_M(i)$$

The multiplication "contracts" a tree of constructors while preserving the type of positions and their typing.





The operation μ_M is associative and unital with units η_M:

$$\mu_M(c, \lambda\, p \to \eta_M(\mathsf{Typ}_M(c, p))) \equiv c$$

$$\mu_M(\eta_M(i), d) \equiv d(\eta\text{-}\mathsf{pos}\,(i))$$

$$\mu_M(\mu_M(c, d), e) \equiv \mu_M(c, (\lambda\, p \to \mu_M(d(p), (\lambda\, q \to e(\mathsf{pair}^{\mu}(p, q))))))$$

Identity monad

The identity monad $\mathsf{Id} : \mathcal{M}$ has a single unary constructor.

Its monad structure is trivial.

Baez-Dolan slice construction

The universe $\mathcal{M}$ is closed under the Baez-Dolan slice construction. For any monad $M : \mathcal{M}$ and family $X : \mathsf{Fam}_M$ with

$$\mathsf{Fam}_M :\equiv \mathsf{Idx}_M \to \mathcal{U}$$

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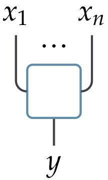
Iterating this construction, we capture the combinatorics of

- the composition of X-cells
- its laws
- the coherences satisfied by the laws
- …

Baez-Dolan slice construction

The indices of M/X are *frames*: constructors of M decorated with elements in X:



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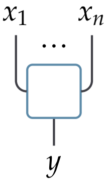
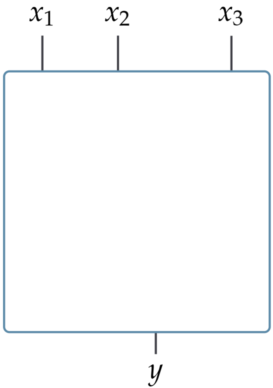


Defined as quadruplets $(i, y) \triangleleft (c, x)$ of type

$$\mathsf{Idx}_{M/X} :\equiv (i : \mathsf{Idx}_M) \times (y : X(i)) \times (c : \mathsf{Cns}_M(i)) \times (x : \overrightarrow{X}(c))$$

Baez-Dolan slice construction

Constructors of M/X are well-founded trees of frames which multiply to their indexing frame.



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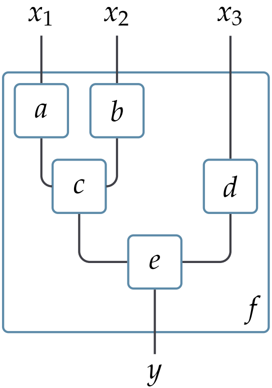

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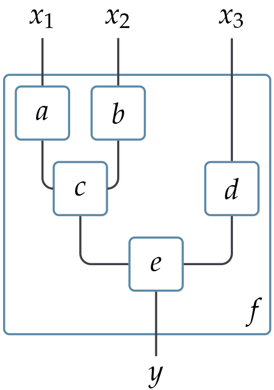

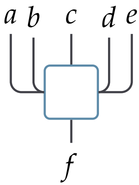

Baez-Dolan slice construction

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A lot of proofs in this framework go by induction on pasting diagrams.

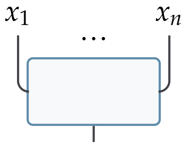
Particularly suited to type theory.

0-ALGEBRAS

A 0-algebra for a monad M is

- a family $X_0 : \mathsf{Fam}_M$,
- a family $X_1 : \mathsf{Fam}_{M/X_0}$.

such that for any constructor $c : \mathsf{Cns}_M(i)$ and values $x : \overrightarrow{X_0}(c)$, there exists is a unique pair composed of



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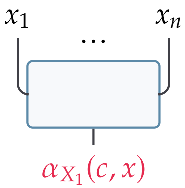

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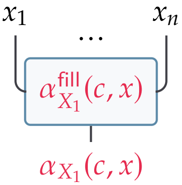
- a filler $\alpha^{\mathsf{fill}}_{X_1}(c, x) : X_1((i, \alpha_{X_1}(c, x)) \triangleleft (c, x))$.



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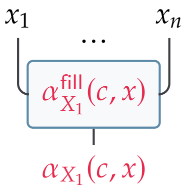

X_1 is an *entire* and *functional* relation.

Fundamental thm. of identity types

Theorem (Fundamental thm. of identity types)

Let $A : \mathcal{U}$ *and* $B : A \to \mathcal{U}$ *such that* $(x : A) \times B(x)$ *is contractible with centre of contraction* (x, p)*, then for any* $y : A$*,*

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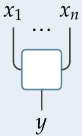
$$B(y) \simeq (x = y)$$

Corollary

Let (X_0, X_1) *be a* M*-0-algebra, for any constructor* $c : \mathsf{Cns}_M(i)$*, values* $x : \overrightarrow{X_0}(c)$*, and value* $y : X_0(i)$*,*

$X_1($ 
$) \simeq (\alpha_{X_1}(c, x) = y)$

Opetopic types

A M-opetopic type is the data of

- a family $X : \mathsf{Fam}_M$
- a M/X-opetopic type

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A M-opetopic type X is *fibrant* if it satisfies the following coinductive property:

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- (X_0, X_1) is an algebra.
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$\mathcal{O}_M$ denotes the type of M-opetopic types.

Opetopic types

Some definitions of higher algebraic structures:

- $\infty\text{-}\mathsf{Grp} = (X : \mathcal{O}_{\mathsf{Id}}) \times \mathsf{is\text{-}fibrant}(X)$
- $(\infty, 1)\text{-}\mathsf{Cat} = (X : \mathcal{O}_{\mathsf{Id}}) \times \mathsf{is\text{-}fibrant}(X_{>0})$

Let X be a fibrant Id-opetopic type (i.e., an ∞-groupoid).

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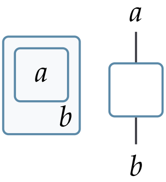
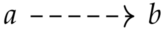
X_0 is the type of objects.

The family of 1-cells $X_1 : \mathsf{Fam}_{\mathsf{Id}/X_0}$ is a binary relation on X_0.





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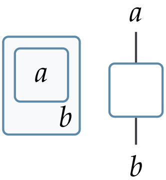
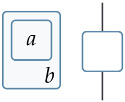
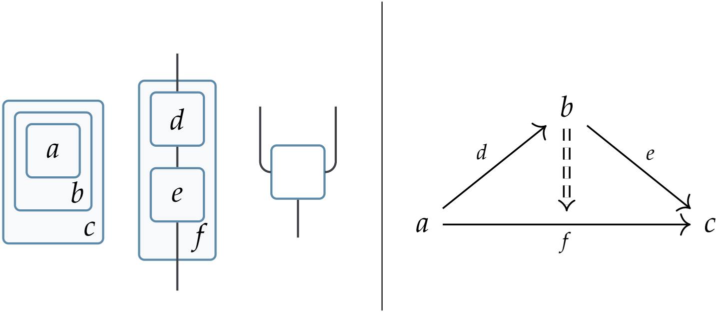


$$a \dashrightarrow b$$

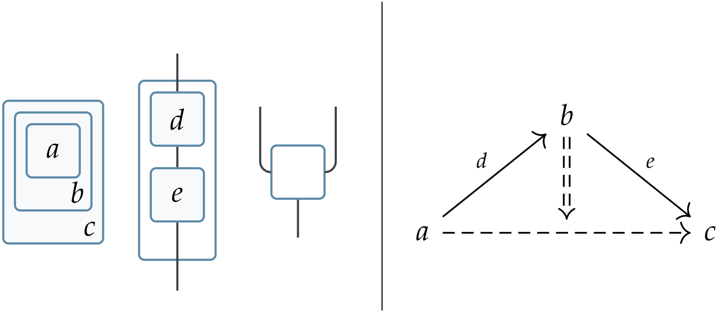
X being fibrant,

$X_1($  $) \simeq (a = b)$

The family of 2-cells $X_2 : \mathsf{Fam}_{\mathsf{Id}/X_0/X_1}$ relates a source pasting diagram of 1-cells with a target *parallel* 1-cell.



X being fibrant, pasting diagrams of 1-cells can be composed.



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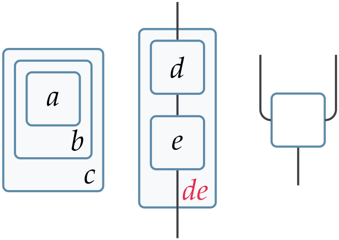

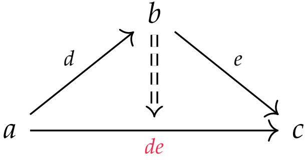

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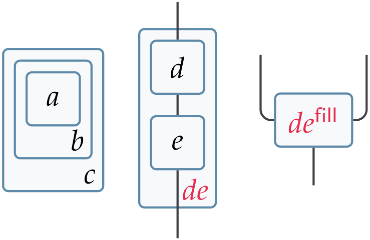
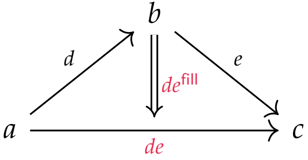




The family of 3-cells $X_3 : \mathsf{Fam}_{\mathsf{Id}/X_0/X_1/X_2}$ relates a source pasting diagram of 2-cells to a target 2-cell.

Fibrancy makes the of composition of 1-cells associative and unital.

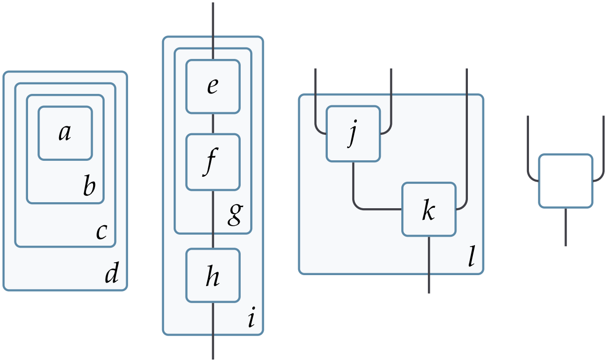
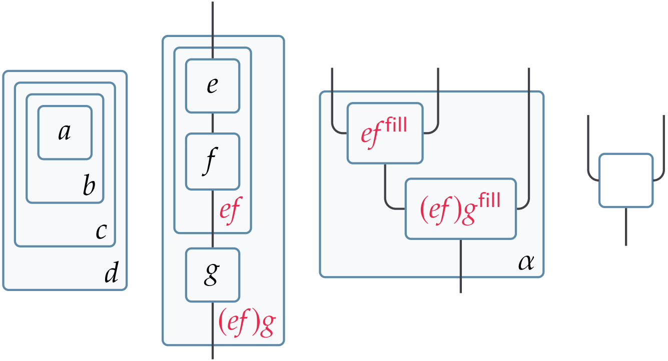


Figure: A 3-dimensional frame

∞-GROUPOIDS

3-CELLS



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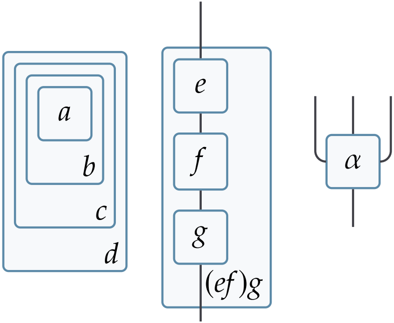

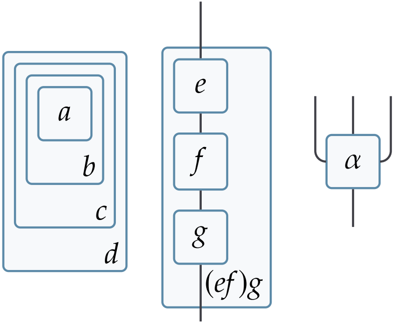

X is fibrant therefore

$$(ef)g = efg$$

The universe

0-cells

Types and their fibrant relations assemble into the $(\infty, 1)$-category

$$\mathcal{U}^o : \mathcal{O}_{\mathsf{Id}}$$

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$$\mathcal{U}^o : \mathcal{O}_{\mathsf{Id}}$$

Its family of objects $\mathcal{U}^o_0$ is the universe of types $\mathcal{U}$:

$$\mathcal{U}^o_0(*) \equiv \mathcal{U}$$

The family of 1-cells

$$\mathcal{U}_1^o : \mathsf{Idx}_{\,\mathsf{Id}/\mathcal{U}_0^o} \to \mathcal{U}$$

is a binary relation on $\mathcal{U}$.

The universe

1-cells

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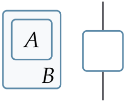
For example,

$\mathcal{U}_1^o($  $) \simeq (R : (a : A)\,(b : B) \to \mathcal{U}) \times \text{is-fibrant}(R)$

The universe

2-cells

The family of 2-cells

$$\mathcal{U}_2^o : \mathsf{Idx}_{\,\mathsf{Id}/\mathcal{U}_0^o/\mathcal{U}_1^o} \to \mathcal{U}$$

relates a *source* pasting diagram of 1-cells to a *target* 1-cell.

The universe
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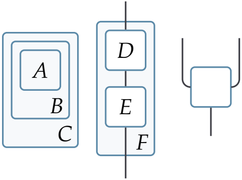

$) \simeq (R : (a : A)\,(b : B)\,(c : C)$
$\to (d : D(a,b))\,(e : E(b,c))\,(f : F(a,c)) \to \mathcal{U})$
$\times\ \text{is-fibrant}(R))$

Formally, the domain of our relations are frames of the universal fibration $\mathcal{U}^o_\bullet \to \mathcal{U}^o$.

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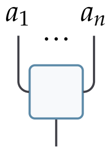

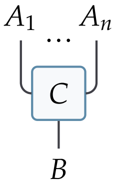

The universe

Fibrant relations

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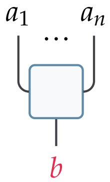

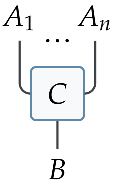

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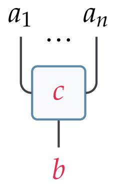

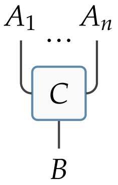

Conclusion

Fibrant opetopic types are internal presentations of types which enables the definition of higher algebraic structures on arbitrary types.

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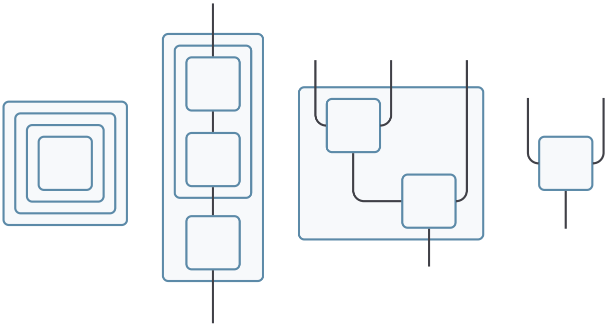
Paves the way for the development of higher category theory in univalent opetopic foundations.

Thank you for your attention.



Slice monad constructors

The type of constructors of the slice monad is an inductive type with two constructors:

$$\mathsf{lf} : (x : \mathsf{Idx}_M) \to \mathsf{Cns}_{M/}\ (x \triangleleft \eta_M\ x)$$

$$\mathsf{nd} : (x : \mathsf{Idx}_M)\ (y : \mathsf{Cns}_M\ x)\ \{z : \overrightarrow{\mathsf{Cns}_M}\ y\}$$
$$\to (t : \overrightarrow{\mathsf{Cns}_{M/}}\ (y \blacktriangleleft z))$$
$$\to \mathsf{Cns}_{M/}\ (x \triangleleft \mu_M\ y\ z)$$