ERGODIC ACTIONS OF MAPPING CLASS GROUPS ON MODULI SPACES OF REPRESENTATIONS OF NON-ORIENTABLE SURFACES

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ABSTRACT. Let M be a non-orientable surface with Euler characteristic $\chi(M) \leq -2$. We consider the moduli space of flat SU(2)-connections, or equivalently the space of conjugacy classes of representations

$$\mathfrak{X}(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))/\,\mathrm{SU}(2).$$

There is a natural action of the mapping class group of M on $\mathfrak{X}(M)$. We show here that this action is ergodic with respect to a natural measure. This measure is defined using the push-forward measure associated to a map defined by the presentation of the surface group. This result is an extension of earlier results of Goldman for orientable surfaces (see [8]).

1. Introduction

Let M be a closed surface with $\chi(M) < 0$ and let π denote its fundamental group. Let G be a Lie group and consider the space of homomorphisms of π into G, denoted $\mathrm{Hom}(\pi, G)$, and called a *representation variety.* The space of G-conjugacy classes of such homomorphisms, called the *character variety* is denoted $\mathfrak{X}(M) = \mathrm{Hom}(\pi, G)/G$. Geometrically, $\mathfrak{X}(M)$ is the moduli space of flat principal G-bundles over M.

The mapping class group, denoted Γ_M, is defined as the group of isotopy classes of orientation-preserving diffeomorphisms of M when the surface is orientable, and as the whole group of isotopy classes of diffeomorphisms when the surface is non-orientable. A classical result of Nielsen [20] tells us that the mapping class group of an orientable surface is isomorphic to the group $\mathrm{Out}^+(\pi)$ of positive outer automorphisms of π. When the surface is non-orientable, Mangler [18] proved

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that the mapping class group is isomorphic to the full group $\mathrm{Out}(\pi)$. Hence, in both cases, there is a natural action of Γ_M on $\mathfrak{X}(M)$ induced by the action of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)$ on $\mathrm{Hom}(\pi, G)$ by left and right composition.

The dynamics of these actions have been extensively studied in the orientable case for various Lie groups (see [10] for a survey on the subject). In the following, we will focus on the case of a non-orientable surface with $G = \mathrm{SU}(2)$, which allows explicit calculations with trace coordinates.

When M is orientable, there is a natural Γ_M-invariant symplectic structure on $\mathfrak{X}(M)$ (see [5, 9]) which induces a volume form and hence a measure. In [8] Goldman uses this symplectic structure and a certain Hamiltonian $\mathbb{R}^n$-action defined on the character variety to show that the Γ_M-action is ergodic on $\mathfrak{X}(M)$ with respect to this measure. However, when the surface M is non-orientable, a symplectic structure may not exist on the character variety as the dimension of this space might be odd. So another approach is necessary to define a measure on $\mathfrak{X}(M)$ in the non-orientable case. In [27] Witten defines and computes a volume on $\mathfrak{X}(M)$ using the Reidemeister-Ray-Singer torsion (see e.g. [1]). In the case of an orientable surface, Witten proves that this volume equals the symplectic volume on the moduli space. In [12] Jeffrey and Ho prove that Witten's volume arises from the Haar measure, in the case of a non-orientable surface. In [19] Mulase and Penkava compute the volume of the representation space using a certain volume distribution given by the push-forward measure associated to a presentation map of $\pi_1(M)$ and their formula also agreed with Witten's result. Using this point of view, we define a Γ_M-invariant measure on the moduli space, denoted ν.

The main result of this paper is the following:

Theorem 1. *Let M be a closed non-orientable surface such that $\chi(M) \leq -2$ and let $G = \mathrm{SU}(2)$. Then the mapping class group Γ_M acts ergodically on $\mathfrak{X}(M)$ with respect to ν.*

The analogous result for an orientable surface was proved by Goldman in [8]. In order to prove Theorem 1 we need to consider subsurfaces (with boundary) of M and it will be useful to consider a more general version of this result for surfaces with boundary. Assume M has m boundary components denoted $\partial_1 M, ..., \partial_m M$. The inclusion maps $\partial_i M \hookrightarrow M$ induce the application

$$\partial^\# : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(\partial M) := \prod_{i=1}^{m} \mathfrak{X}(\partial_i M).$$

Then $\mathfrak{X}(M)$ can be viewed as a family of *relative character varieties* over $\mathfrak{X}(\partial M)$. As each $\mathfrak{X}(\partial_i M)$ is identified to the set $[G]$ of conjugacy classes in G, the base of this family is a product of copies of $[G]$. Specifically, let $\{C_1, C_2, ..., C_m\}$ be a set of elements of the fundamental group π corresponding to the m boundary components. Let $\mathcal{C} = (c_1, ..., c_m)$ be an element of $[G]^m$, and define the *relative character variety* over $\mathcal{C}$ as

$$\mathfrak{X}_{\mathcal{C}}(M) = \partial_{\#}^{-1}(\mathcal{C}) = \{[\rho] \in \mathfrak{X}(M) \mid [\rho(C_i)] = c_i, 1 \leq i \leq m\}.$$

The disintegration of the measure ν on $\mathfrak{X}(M)$ with respect to $\partial_{\#}$ is a measure $\nu_{\mathcal{C}}$ on the submanifold $\partial_{\#}^{-1}(\mathcal{C})$.

For a surface with boundary, the mapping class group Γ_M is identified with the group $\mathrm{Out}(\pi, \partial M)$ of outer automorphisms of π (respectively $\mathrm{Out}^+(\pi, \partial M)$, if the surface is orientable) which preserve the conjugacy class of every cyclic subgroup corresponding to a boundary component. Then Γ_M acts on $\mathfrak{X}(M)$ by outer automorphisms of π which preserve the function $\partial_{\#}$. Hence Γ_M acts on $\mathfrak{X}_{\mathcal{C}}(M)$, for every $\mathcal{C} \in [G]^m$. The generalization of Theorem 1 is the following:

Theorem 2. *Let M be a compact non-orientable surface with m boundary components such that $\chi(M) \leq -2$ and let $\mathcal{C} = (c_1, ..., c_m) \in [G]^m$. Then the action of the mapping class group Γ_M on $\mathfrak{X}_{\mathcal{C}}(M)$ is ergodic with respect to the measure $\nu_{\mathcal{C}}$.*

This theorem includes Theorem 1 as the special case where M has no boundary. The similar result for orientable surfaces was also proved by Goldman in [8].

Remark 3. For surfaces of Euler characteristic -1, the behavior of the Γ_M-action depends on the orientability:

- If M is orientable, namely a three-holed sphere or a one-holed torus, then the action of the mapping class group is ergodic on the relative character variety.
- If M is non-orientable, namely the two-holed projective plane, the one-holed Klein bottle or the connected sum of three projective planes, then the action of Γ_M is not ergodic. In each of these cases, there is an essential curve which is invariant under the action of the mapping class group (see [4]).

Remark 4. For an orientable surface, the analog of Theorem 2 was extended to the general case of a compact Lie group G by Pickrell and Xia in [21, 22]. Their approach relies on the study of the infinitesimal transitivity in the case of the one-holed torus and afterwards using sewing techniques on the representation variety. For non-orientable surfaces, one can expect that a similar result holds. However, as we can not have ergodicity for surfaces of Euler characteristic -1, we would have to study this infinitesimal transitivity in the case of the two-holed Klein bottle and the three-holed projective plane, which involve much more technical complications.

Remark 5. The topological dynamics of these actions are more delicate as we do not ignore the subsets of null measure. We can hope that if a representation $\rho \in \mathrm{Hom}(\pi, \mathrm{SU}(2))$ has dense image in $\mathrm{SU}(2)$, then the Γ_M-orbit of $[\rho]$ is dense in $\mathfrak{X}(M)$. This result is true if the surface M is orientable and the genus of M is strictly positive ([23, 24]). However in genus 0, there are representations ρ with dense image but whose orbit $\Gamma_M \cdot [\rho]$ consists only of two points (see [25]).

Summary. This paper is organized as follows.

In Section 2, we review some basic knowledge about non-orientable surfaces, their mapping class groups and moduli spaces. In Section 3, we define the Γ_M-invariant measure on $\mathfrak{X}(M)$ using a certain volume distribution and its character expansion.

In Section 4, we define the *Goldman flow* on non-orientable surfaces following Klein [13]. This is a circle action on a dense open subspace of the character variety of M. This action corresponds to the circle action defined by L. Jeffrey and J. Weitsman in [11] in the case of an orientable closed surface. This flow is related to a particular decomposition of the surface along a curve. In particular, the Dehn twist along this curve acts as a rotation on the orbit of the flow.

In Section 5 we study the case where M is a non-orientable surface of even genus. In this case we split M along a non-separating 2-sided curve X to obtain an orientable surface A with two additional boundary components. The orbits of the Goldman flow associated to X are the fibers of the map $\mathfrak{X}(M) \to \mathfrak{X}(A)$. The Dehn twist about X acts as a rotation on this fiber, and for almost all representation this rotation is irrational and hence ergodic. We infer that a Γ_M-invariant function on $\mathfrak{X}(M)$ depends only on its value on $\mathfrak{X}(A)$. Then the ergodicity in the case of an orientable surface proves that the Γ_M-invariant function depends only on its value at X. Then consider an embedding of a two-holed Klein bottle inside M, such that X cuts open the two-holed Klein bottle into a four-holed sphere. We can find trace coordinates on the character variety of the two-holed Klein bottle. The explicit calculations for the action of a certain Dehn twist in these coordinates, allow us to settle the Theorem 2 in the case of a two-holed Klein bottle. In particular, this shows that a Γ_M-invariant function on $\mathfrak{X}(M)$ does not depend on its value at X, which proves the theorem.

If M is a non-orientable surface of odd genus, then it is impossible to cut open M along a 2-sided curve into one or two orientable surfaces. Instead of that, we split M along a separating curve C into two parts denoted A and B, such that A is an orientable surface and B is a non-orientable surface of Euler characteristic -2. The surface B can be of two kind, a three-holed projective plane or a one-holed non-orientable surface of genus 3. For these surfaces, we use trace cordinates to make explicit calculations for the action of Dehn twists. These calculations are contained in Section 6 and setlle the Theorem 2 in the case of a non-orientable surface of odd genus with Euler characteristic -2.

In Section 7, we use the Goldman flow associated to the separating curve C to show that the Dehn twist about C acts as a rotation on the fiber of the map $\mathfrak{X}(M) \to \mathfrak{X}(A) \times \mathfrak{X}(B)$. For almost all representation, this rotation is ergodic. Hence, a Γ_M-invariant depends only on its value at C. Finally we consider an embedding of a four-holed sphere into M such that C is a separating non-trivial curve in it. The ergodicity for the four-holed sphere allows us to prove the theorem.

2. Preliminaries

2.1. Non-orientable surfaces.

We summarize some basic notions and results about on non-orientable surfaces and their mapping class groups. For more details and proofs, we refer to [14, 15, 18, 26].

Let M be a compact non-orientable surface of genus $g \geq 1$ and with m boundary components, denoted $N_{g,m}$. The boundary components of M are denoted

$$\partial M = C_1 \sqcup ... \sqcup C_m.$$

Recall that $N_{g,0}$ is a connected sum of g projective planes, and that $N_{g,m}$ is obtained by removing m open disks of $N_{g,0}$. The fundamental group $\pi_1(N_{g,0})$ admits two important presentations that we recall here. The first is the natural presentation which exhibits the fact that N_g is a connected sum of projective planes.

$$\pi_1(N_{g,0}) = \langle A_1, ..., A_g \mid A_1^2 \cdots A_g^2 \rangle.$$

Another presentation can be obtained by making use of the homeomorphism between the connected sum of three projective planes and the connected sum of a torus with one projective plane. The presentation depends on the parity of g:

$$\pi_1(N_{2k+1,0}) = \langle A_1, B_1, ..., A_k, B_k, C \mid [A_1, B_1]...[A_k, B_k]C^2 \rangle$$

$$\pi_1(N_{2k+2,0}) = \langle A_1, B_1, ..., A_k, B_k, C, D \mid [A_1, B_1]...[A_k, B_k]C^2 D^2 \rangle.$$

A simple closed curve on a surface M is called *two-sided* if a regular neighborhood of it within M is homeomorphic to an annulus. A simple closed curve is called *one-sided* if a regular neighborhood of it within M is homeomorphic to a Möbius strip. A *circle* on M is a closed connected one-dimensional submanifold of M. We denote by $M|X$ the surface obtained by cutting open M along a circle X, defined as the surface with boundary for which there is an identification map $i_X : M|X \to M$ satisfying

- the restriction of i_X to $i_X^{-1}(M - X)$ is a diffeomorphism;
- $i_X^{-1}(X)$ consists of two components $X_+, X_- \subset \partial(M|X)$, to each of which the restriction of i_X is a diffeomorphism onto X.

A circle X is called *non-separating* if $M|X$ is connected, and *separating* otherwise. A separating circle is *trivial* if one of the two components is either a disk, a cylinder or a Möbius strip.

2.2. **Mapping class groups of non-orientable surfaces.** The mapping class group Γ_M is defined to be the group of isotopy classes of diffeomorphisms $\phi : M \longrightarrow M$ which restrict to the identity on each boundary component, *i.e.* $\phi_{|C_i} = Id_{|C_i}$ for all i. Let X be a two-sided circle on M, and let U be a regular neighborhood of X within M. The annulus U is homeomorphic to $\mathbb{S}^1 \times [0,1]$, and we chose coordinates (s,t) on this annulus. Let f be the diffeomorphism of M that is the identity outside of U, and that is defined inside U as

$$f(s,t) = (se^{2i\pi t}, t).$$

The isotopy class of this map is called the *Dehn twist* about X, denoted τ_X. Observe that this definition does not make sense for a one-sided curve.

For an orientable surface S, the mapping class group Γ_S is generated by Dehn twists, and the number of generators can be chosen to be finite (see e.g. [16, 20]). For a non-orientable surface M, the Dehn twists generate an index 2 subgroup of Γ_M, called the *twist subgroup* of M. Henceforth in this case, we need to define another family of diffeomorphisms of M to find a generating set for Γ_M.

Consider a Möbius strip M with one hole, or equivalently a projective plane from which the interiors of two disks have been removed. Attach another Möbius strip N along one of the boundary components. The resulting surface K is a Klein bottle with one hole. By sliding N once along the core of M, we get a diffeomorphism y_K of K fixing the boundary of K (cf. the Figure 1 below). Assume that this diffeomorphism is the identity in a neighborhood of the boundary of K. If K is embedded in a surface S, we define y as the diffeomorphism of S that is the identity outside of K and is given by y_K inside K. The isotopy class of y is called a *crosscap slide*. The mapping class y^2 is equal to a Dehn twist about the boundary of K.

We represent crosscaps as shaded disks in the picture.

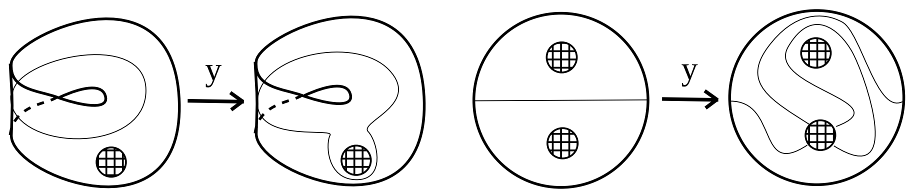

FIGURE 1. Crosscap Slide

For M a non-orientable surface, the mapping class group Γ_M is generated by Dehn twists and crosscap slides (see [15]). Moreover, the number of generators can be chosen to be finite (see [14]).

3. THE CONSTRUCTION OF AN INVARIANT MEASURE ON THE MODULI SPACE

The aim of this section is to define a natural measure on $\mathfrak{X}(M)$ that is invariant under the action of the mapping class group. First, we define a measure on the representation space $\mathrm{Hom}(\Pi, G)$ in a more general context using ideas from [19].

3.1. **Measure on** $\mathrm{Hom}(\Pi, G)$**.** Let G be a compact Lie group. Let

$$\Pi = \langle a_1, \dots, a_k | q_1(a_1, \dots, a_k), \dots, q_r(a_1, dots, a_k)\rangle$$

be a finitely presented group generated by k elements with r relations. We associate the *presentation map*

$$\begin{aligned} q : G^k &\longrightarrow G^r \\ x &\longmapsto (q_1(x), \dots, q_r(x)) \end{aligned}$$

For $x = (x_1, \dots, x_k)$, the element $q_j(x) = q_j(x_1, \dots, x_k)$ of G is obtained when we replace in the word $q_j(a_1, \dots, a_k)$ the letters a_i by elements x_i of G.

There is a canonical identification between $\mathrm{Hom}(\pi, G)$ and the fiber $q^{-1}(1, \dots, 1)$ of the presentation map provided by:

$$p(\mathrm{Hom}(\Pi, G)) = q^{-1}(1, \dots, 1) \tag{3.1}$$

where the map p is

$$p : \mathrm{Hom}(\Pi, G) \longrightarrow G^k$$
$$\phi \longmapsto (\phi(a_1), \dots, \phi(a_k)).$$

Let dx be a Haar measure on G. The group G being compact, the measure is left and right invariant. The Dirac distribution on G is the linear continuous functional $\delta : C^\infty(G) \to \mathbb{R}$ given by:

$$g \mapsto \int_G \delta(x) g(x) dx = g(1), \qquad \text{for any } g \in \mathcal{C}^\infty(G)$$

The Dirac distribution on G^r is defined by $\delta_r(w_1, \dots, w_r) = \delta(w_1) \dots \delta(w_r)$.

Let f_q be the *volume distribution* defined as

$$f_q(w) = \int_{G^k} \delta_r(q(x) \cdot w^{-1}) dx_1 \cdots dx_k, \qquad w \in G^r$$

This distribution equals the linear continuous functional $C^\infty(G^r) \to \mathbb{R}$,

$$g \mapsto \int_{G^k} g(q(x)) dx_1 \cdots dx_k = \int_{G^r} f_q(w) g(w) dw_1 \cdots dw_r.$$

Distributions cannot be evaluated in a meaningful way in general. However, a distribution f is said to be *regular* at $w \in G^r$ if there is an open neighborhood U of w such that the restriction of f to U is a C^∞ function on U.

Assume that the volume distribution f_q is regular at $(1, \dots, 1) \in G^r$. Let μ be the borelian measure on $\mathrm{Hom}(\Pi, G)$ defined by

$$\mu_q(U) = \int_{G^k} \delta_r(q(x)) \mathbf{1}_{p(U)}(x) dx_1 \cdots dx_k \tag{3.2}$$

for any borelian $U \subset \mathrm{Hom}(\Pi, G)$, where $\mathbf{1}_E$ is the characteristic function of E. The total volume $\mu_q(\mathrm{Hom}(\Pi, G)) = f_q(1)$ is well-defined, and hence μ_q is a finite measure on $\mathrm{Hom}(\Pi, G)$.

3.2. **Invariance of the measure.** The measure μ_q is defined using the presentation q of the group Π. The following proposition shows that, under certain hypotheses, the measure does not depend on the choice of the presentation of Π.

Proposition 3.2.1. *Let q and s be two presentations of the same group Π*

$$\Pi = \langle a_1, \dots, a_k | q_1, \dots, q_r \rangle = \langle b_1, \dots, b_l | s_1, \dots, s_t \rangle.$$

Assume that $k - r = l - t$. If the volume distributions f_q and f_s associated to the presentation maps q and s are regular at $(1, \dots, 1) \in G^r$ and $(1, \dots, 1) \in G^l$ respectively, then the measures μ_q and μ_s coincide.

Proof. First, assume that $k = l$ and $r = t$. In this case, the proof of this proposition is deeply related to the two following lemmas, whose proofs can be found in [19].

Lemma 3.2.2. *Let q and s be two presentations of the same group Π*

$$\Pi = \langle a_1, \dots, a_k | q_1, \dots, q_r \rangle = \langle b_1, \dots, b_k | s_1, \dots, s_r \rangle.$$

Then for every $a_i, i = 1, \dots, k$ there is a word $a_i(b)$ in the generators $b_1, \dots, b_k$, and for every $b_j, j = 1, \dots, k$ there is a word $b_j(a)$ in the generators $a_1, \dots, a_k$, such that the maps $a : G^k \to G^k$ and $b : G^k \to G^k$ associated to these words are bijective and the following diagram:

$$\begin{array}{ccccc} G^k & = & G^k & \xrightarrow{q} & G^r \\ {\scriptstyle a}\uparrow\wr & & \wr\downarrow{\scriptstyle b} & & \\ G^k & = & G^k & \xrightarrow{s} & G^r \end{array}$$

is commutative. Moreover,

$$q^{-1}(1) = b^{-1}(s^{-1}(1)), \qquad \textit{and} \qquad s^{-1}(1) = a^{-1}(q^{-1}(1)).$$

The isomorphisms a and b are real analytic automorphisms of the real analytic manifold G^k.

Lemma 3.2.3. *Suppose that the map,*

$$b : G^k \longrightarrow G^k$$
$$(x_1, \dots, x_k) = x \longmapsto (b_1(x), \dots, b_k(x))$$

is an analytic automorphism of the real analytic manifold G^k given by k words $b_1, \dots, b_k$ in $x_1, \dots, x_k$ such that its inverse has the same form. Then the volume form ω^k of G^k corresponding to the product of Haar measures $dx_1 \dots dx_k$ is invariant under the automorphism b up to a sign, i.e. $b^*\omega^k = \pm\omega^k$.

Now let the two presentation q and s satisfy the hypotheses of Lemma 3.2.2, and let a and b be the analytic automorphisms of G^k given by this lemma. Let p_q and p_s be the maps $\mathrm{Hom}(\Pi_i, G) \to G^k$. By Lemma 3.2.2, we have $p_s(V) = b(p_q(V))$ and $\delta_r(q(x)) = \delta_r(s(b(x)))$ and by Lemma 3.2.3 the measure $dx_1 \cdots dx_k$ on G^k is invariant by the automorphism

b. Hence, for V a borelian of $\mathrm{Hom}(\Pi, G)$ we have

$$\begin{aligned}\mu_s(V) &= \int_{G^k} \delta_r(s(x))\mathbf{1}_{p2(V)}(x)dx_1\cdots dx_k\\ &= \int_{G^k} \delta_r(s(b(b^{-1}(x))))\mathbf{1}_{b(p_1(V))}(x)dx_1\cdots dx_k\\ &= \int_{G^k} \delta_r(q(b^{-1}(x)))\mathbf{1}_{p_1(V)}(b^{-1}(x))dx_1\cdots dx_k\\ &= \mu_q(V)\end{aligned}$$

In the general case, we can assume without loss of generality that $k < l$. We add the new generators $a_{k+1}, \ldots, a_l$ and relators $q_{r+1} = a_{k+1}, \ldots, q_{r+l-k} = a_l$ to the presentation q of the group Π. The new presentation q' given by

$$\langle a_1, \ldots, a_l | q_1, \ldots, q_{r+l-k}\rangle$$

is also a presentation of the group Π. If f_q is regular at $(1, \ldots, 1) \in G^r$ then the volume distribution $f_{q'}$ of the new presentation q' is also regular at $(1, \ldots, 1) \in G^l$. Consider the following commutative diagram

$$\begin{array}{ccc} G^k \times G & \xrightarrow{(q,id)} & G^r \times G \\ {\scriptstyle i_k}\uparrow & & \uparrow{\scriptstyle i_r} \\ G^k & \xrightarrow[q]{} & G^r \end{array}$$

where i_k and i_r are the canonical injection of G^k in $G^k \times \{1\} \subset G^k \times G$. The identity

$$\int_{G^k\times G} \delta_r(q(x))\delta(x)dx_1\cdots dx_k \cdot dx = \int_{G^k} \delta_r(q(x))dx_1\cdots dx_k,$$

implies that the measure μ_q and $\mu_{q'}$ defined by the presentations q and q' coincide. The two presentations q' and s have the same number of generators and relations and hence the measure $\mu_{q'}$ and μ_s coincide. This ends the proof of Proposition 3.2.1. □

The natural action of $\mathrm{Aut}(\Pi) \times \mathrm{Aut}(G)$ on the representation space $\mathrm{Hom}(\Pi, G)$ is given by :

$$(\tau, \alpha) \cdot \rho \longmapsto \alpha \circ \rho \circ \tau^{-1}$$

for any $\rho \in \mathrm{Hom}(\Pi, G)$ and $(\tau, \alpha) \in \mathrm{Aut}(\Pi) \times \mathrm{Aut}(G)$. The group of inner automorphism of G, denoted $\mathrm{Inn}(G)$, is the subgroup of $\mathrm{Aut}(G)$ consisting of elements of the form $L_g : G \to G, L_g(h) = ghg^{-1}$ for all $h \in G$, with $g \in G$. We have the following proposition :

Proposition 3.2.4. *Let q be a presentation of Π such that the distribution f_q is regular at $(1,\ldots,1)\in G^r$. The measure μ_q on the representation space* $\mathrm{Hom}(\Pi,G)$ *is invariant under the action of the group of inner automorphisms of G.*

Proof. Let g be an element of G, and V be a borelian of the representation space $\mathrm{Hom}(\Pi,G)$. The Dirac distribution δ is invariant by conjugation, and so is the distribution δ_r. The Haar measure dx is also invariant by conjugation as a left and right invariant measure. Hence, we have

$$\begin{aligned}\mu_q(g\cdot V) &= \int_{G^k}\delta_r(q(x))\mathbf{1}_{p(g\cdot V)}(x)dx_1\cdots dx_k\\ &= \int_{G^k}\delta_r(g^{-1}q(x)g)\mathbf{1}_{gp(V)g^{-1}}(x)dx_1\cdots dx_k\\ &= \int_{G^k}\delta_r(q(g^{-1}xg))\mathbf{1}_{p(V)}(g^{-1}xg)dx_1\cdots dx_k\\ &= \mu_q(V)\end{aligned}$$

□

An automorphism of $\Pi=\langle a_1,\ldots,a_k|q_1,\ldots,q_r\rangle$ is given by k words in $a_1,\ldots,a_k$, and its inverse is of the same form. Hence, we have the immediate corollary to Proposition 3.2.1

Corollary 3.2.5. *Let q be a presentation of Π such that the distribution f_q is regular at $(1,\ldots,1)\in G^r$. The measure μ_q on* $\mathrm{Hom}(\Pi,G)$ *is invariant under the action of* $\mathrm{Aut}(\Pi)$.

3.3. **Regularity of volume distributions for non-orientable surface groups in** $\mathrm{SU}(2)$**.** For a closed non-orientable surface M of genus k, we take the usual presentation:

$$\pi_1(M)=\langle a_1,\ldots,a_k|a_1^2\cdots a_k^2\rangle.$$

The volume distribution f_k becomes

$$f_k(w)=\int_{G^k}\delta_r(x_1^2\cdots x_k^2\cdot w^{-1})dx_1\cdots dx_k.$$

To show that this distribution is regular at the identity element, we compute its character expansion. We first have to set some notations.

Let $\widehat{G}$ denote the set of isomorphism classes of complex irreducible representations of G and let χ_λ be the character of the irreducible representation $\lambda\in\widehat{G}$. Using the Frobenius-Schur indicator of irreducible characters (see [2]), we decompose $\widehat{G}$ into the disjoint union of the three following subsets:

$$\begin{aligned}\widehat{G}_1 &= \left\{\lambda \in \widehat{G} \,\middle|\, \frac{1}{|G|}\int_G \chi_\lambda(w^2)dw = 1\right\},\\ \widehat{G}_2 &= \left\{\lambda \in \widehat{G} \,\middle|\, \frac{1}{|G|}\int_G \chi_\lambda(w^2)dw = 0\right\},\\ \widehat{G}_4 &= \left\{\lambda \in \widehat{G} \,\middle|\, \frac{1}{|G|}\int_G \chi_\lambda(w^2)dw = -1\right\}.\end{aligned}$$

With these notations we state the following proposition, whose detailed proof can be found in [19].

Proposition 3.3.1. *The character expansion of the volume distribution f_k is given by*

$$f_k(w) = \sum_{\lambda\in\widehat{G}_1}\left(\frac{|G|}{\dim\lambda}\right)^{k-1}\chi_\lambda(w) - \sum_{\lambda\in\widehat{G}_4}\left(-\frac{|G|}{\dim\lambda}\right)^{k-1}\chi_\lambda(w) \tag{3.3}$$

If the right-hand side sum is absolutely convergent for $w = 1$, then it is uniformly and absolutely convergent on G, and the volume distribution f_k is a C^∞ function.

When $G = \mathrm{SU}(2)$, the series associated to $f_k(1)$ are absolutely convergent for $k \geq 4$.

Proof. The proof of Proposition 3.3.1 relies on the convolution property of the δ-function. Namely, let q_k be the word in $a_1, \dots, a_k$ given by $q_k(a_1, \dots, a_k) = a_1^2 \cdots a_k^2$. We have that

$$\delta(q_{g+h}w^{-1}) = \int_G \delta(q_h w^{-1}u^{-1})\delta(uq_g)du$$

and hence

$$f_k = \overbrace{f_1 * \cdots * f_1}^{k-\text{times}}.$$

Here $f * g$ denotes the usual convolution product, given by

$$(f * g)(x) = \int_G f(xw^{-1})g(w)dw.$$

The irreducible characters are real analytic functions on G and form a orthonormal basis for the L^2 class function on G. The character expansion in terms of irreducible characters of the class distribution f_1 is given by:

$$f_1(w) = \sum_{\lambda\in\widehat{G}} Z_\lambda \chi_\lambda(w)$$

where

$$Z_\lambda = \frac{1}{|G|}\int_G \chi_\lambda(x^2)dx.$$

Hence using the decomposition $\widehat{G} = \widehat{G}_1 \sqcup \widehat{G}_2 \sqcup \widehat{G}_4$ given by the Frobenius-Schur indicator, we obtain

$$f_1(w) = \sum_{\lambda\in\widehat{G}_1} \chi_\lambda(w) - \sum_{\lambda\in\widehat{G}_4} \chi_\lambda(w)$$

The convolution property of irreducible characters states that

$$\chi_\lambda * \chi_\mu = \frac{|G|}{\dim\lambda}\delta_{\lambda\mu}\chi_\lambda.$$

This formula applied $k-1$ times to the convolution $f_k = f_1 * \cdots * f_1$ gives us the formula (3.3).

Moreover, for $G = \mathrm{SU}(2)$, we know that the dimension of the representation in $\widehat{G}_1$ consists of odd integers and in $\widehat{G}_4$ of even integers. Hence

$$f_k(1) = |G|^{k-1}\left(\sum_{n=1}^{\infty}(2n-1)^{2-k} + (-1)^{2-k}\sum_{n=1}^{\infty}(2n)^{2-k}\right)$$

which is absolutely convergent for $k \geq 4$. □

Let M be a closed non-orientable surface of genus $k \geq 4$. We have defined a measure μ on $\mathrm{Hom}(\pi, \mathrm{SU}(2))$. Consider the quotient map

$$Q : \mathrm{Hom}(\pi, \mathrm{SU}(2)) \longrightarrow \mathrm{Hom}(\pi, \mathrm{SU}(2))/\,\mathrm{SU}(2) = \mathfrak{X}(M)$$

and define a measure ν on $\mathfrak{X}(M)$ as the push-forward measure of the measure μ through Q, given by:

$$\nu(V) = \mu(Q^{-1}(V)) \tag{3.4}$$

Then Corollary 3.2.5 together with Proposition 3.2.4 show that the measure ν is $\mathrm{Out}(\Pi)$-invariant on the quotient $\mathrm{Hom}(\Pi, \mathrm{SU}(2))/\,\mathrm{SU}(2)$. Moreover, this measure is independent on the choice of the presentation of π. The ergodicity result of Theorem 1 will be proved with respect to this measure.

Remark 6. The fundamental group of a surface M with boundary is a free group on $l = 1 - \chi(M)$ generators. The representation space $\mathrm{Hom}(\pi, G)$ is isomorphic to G^l with the natural presentation of the free group with l generators and no relation. Hence, the measure obtained by the above construction is simply the Haar measure on G^l.

4. Surfaces decompositions and Goldman's flow

4.1. **Goldman's flow.** Let $f : G \to \mathbb{R}$ be a C^1 function invariant under inner automorphisms of G, namely satisfying $f(PAP^{-1}) = f(A)$, for all $A, P \in G$. For the rest of this paper G will denote the group $\mathrm{SU}(2)$ and we will consider henceforth $f(A) = \cos^{-1}\left(\frac{\mathrm{tr}(A)}{2}\right) \in [0, \pi]$, where tr denotes the usual trace in $\mathrm{SU}(2)$.

Let $\mathfrak{g}$ denote the Lie algebra of G and let $\langle X, Y\rangle$ denote the inner product on $\mathfrak{g}$ (i.e. the Killing form) defined by $< X, Y >= -\mathrm{Tr}(XY)$, for all X, Y in $\mathfrak{g}$. The *variation* of the function f is the G-equivariant function $F : G \to \mathfrak{g}$ defined by the equation:

$$< X, F(A) >= df_A(X) = \frac{d}{dt} f(A\exp(tX)), \qquad \text{for any } A \in G, \text{ and } X \in \mathfrak{g}.$$

If A is a matrix of the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ with $\theta \in]0, \pi[$, then $F(A) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. For $B = gAg^{-1}$ with $g \in G$ and A of the above form, we have

$$F(B) = F(gAg^{-1}) = g \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} g^{-1}. \tag{4.5}$$

The function F is defined on $G \setminus \{\pm I\}$, but is not defined for the extremal values $\theta \in \{0, \pi\}$. However, for our purpose it will be sufficient to define the flow on an open dense subset of full-measure of the character variety.

Let M be a compact non-orientable surface and γ a two-sided circle on M. Let $f_\gamma : \mathrm{Hom}(\pi_1(M), G)/G \longrightarrow \mathbb{R}$ be the function defined on $\mathfrak{X}(M)$ by:

$$f_\gamma([\rho]) = f(\rho(\gamma)).$$

We define the open dense subset $\widetilde{\mathcal{S}_\gamma}$ of full measure of $\mathrm{Hom}(\pi, G)$ as:

$$\widetilde{\mathcal{S}_\gamma} = \{\rho \in \mathrm{Hom}(\pi, G) | \rho(\gamma) \neq \pm I\}.$$

Let $\mathcal{S}_\gamma$ denote its image in $\mathfrak{X}(M)$. For $\rho \in \widetilde{\mathcal{S}_\gamma}$, let $\zeta_t(\rho) = \exp(tF(\rho(\gamma)))$. This defines a path in the centralizer $Z(\rho(\gamma))$ of $\rho(\gamma)$ in G.

We construct a flow on $\mathcal{S}_\gamma$ called a *generalized twist flow* or *Goldman flow* (see [11, 13]). We define the flow $\Xi_t(\rho)$ in the two cases of interest for us, namely when γ is a separating circle, and when γ is a non-separating circle such that $M|\gamma$ is orientable. The other situation, corresponding to a non-separating circle such that $M|\gamma$ is non-orientable, will not be used in the sequel but can be treated in the same way than the case when $M|\gamma$ is orientable.

4.2. **The flow associated to a separating circle.** Let γ be a separating circle on M. Then $M|\gamma$ is the disjoint union of two subsurfaces A and B. Without loss of generality, we can assume that A is non-orientable. We place a base point p on the circle γ. The surface M is obtained by gluing A and B along the circle γ. Hence, the Seifert-Van Kampen theorem shows that the fundamental group $\pi_1(M)$ can be reconstructed from $\pi_1(A)$ and $\pi_1(B)$ as

$$\pi_1(M,p) = \pi_1(A,p) *_{\pi_1(\gamma,p)} \pi_1(B,p).$$

The fundamental group $\pi_1(\gamma,p)$ is isomorphic to the cyclic group $\mathbb{Z}$. We also denote by γ the class of the curve γ in $\pi_1(M)$. Hence we have $\pi_1(\gamma) = \langle\gamma\rangle$.

The flow on $\widetilde{\mathcal{S}_\gamma}$ is defined by:

$$\widetilde{\Xi}_t\rho(\delta) = \begin{cases} \rho(\delta), & \text{if } \delta \in \pi_1(A), \\ \zeta_t(\rho)\rho(\delta)\zeta_t(\rho)^{-1}, & \text{if } \delta \in \pi_1(B) \end{cases} \tag{4.6}$$

where ρ is an element of $\mathrm{Hom}(\pi_1(M), G)$, and t is a real number. The element $\zeta_t(\rho)$ is in the centralizer of $\rho(\gamma)$, hence $\rho(\gamma) = \zeta_t(\rho)\rho(\gamma)\zeta_t(\rho)^{-1}$, and the element $\widetilde{\Xi}_t\rho(\gamma)$ is well-defined.

We define the flow $\{\Xi_t\}_{t\in\mathbb{R}}$ on $\mathcal{S}_\gamma$, such that it is covered by $\{\widetilde{\Xi}_t\}_{t\in\mathbb{R}}$. For any representation ρ in $\widetilde{\mathcal{S}_\gamma}$, the formula (4.5) gives

$$\zeta_\pi(\rho) = \exp(\pi F(\rho(\gamma))) = \exp\left(\pi g \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} g^{-1}\right) = -I.$$

Similarly $\zeta_{2\pi}(\rho) = I$. So we have $\widetilde{\Xi}_\pi\rho = \rho$, and thus the flow $\{\Xi_t\}$ is π-periodic and defines a circle action on an open dense subset of full measure of $\mathfrak{X}(M)$.

We then define $\mathfrak{X}(M; A, B, \gamma)$ as the pull-back in the diagram:

$$\begin{array}{ccc} \mathfrak{X}(\gamma) & \longleftarrow & \mathfrak{X}(A) \\ \uparrow & & \uparrow \\ \mathfrak{X}(B) & \longleftarrow & \mathfrak{X}(M; A, B, \gamma) \end{array}$$

Namely, $\mathfrak{X}(M; A, B, \gamma)$ is the set of pairs $([\alpha], [\beta]) \in \mathfrak{X}(A) \times \mathfrak{X}(B)$ such that

$$[\alpha_{|_{\pi_1(\gamma)}}] = [\beta_{|_{\pi_1(\gamma)}}] \in \mathfrak{X}(\gamma)$$

We have a natural map $j : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M; A, B, \gamma)$ given by:

$$j([\rho]) = ([\rho_{|_{\pi_1(A)}}], [\rho_{|_{\pi_1(B)}}]).$$

Proposition 4.2.1. *The generic fibers of the map*

$$j \,:\, \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M; A, B, \gamma)$$

are the orbits of the circle action $\{\Xi_t\}$.

Proof. Let $([\alpha],[\beta])$ be an element of $\mathfrak{X}(M; A, B, \gamma)$ and let α and β be representatives of $[\alpha]$ and $[\beta]$ respectively. By definition, there exists an element g of G such that $\alpha_{|_{\pi_1(\gamma)}} = g \cdot \beta_{|_{\pi_1(\gamma)}} \cdot g^{-1}$. Without loss of generality, we can choose a representative β such that $\alpha_{|_{\pi_1(\gamma)}} = \beta_{|_{\pi_1(\gamma)}}$. Let $[\rho]$ be a conjugacy class of representation in $\mathfrak{X}(M)$ and let ρ be a representative of $[\rho]$. The class $[\rho]$ is in the fiber $j^{-1}([\alpha],[\beta])$ if and only if there exists h_1, h_2 in G such that $\rho_{|_{\pi_1(A)}} = h_1 \alpha h_1^{-1}$ and $\rho_{|_{\pi_1(B)}} = h_2 \beta h_2^{-1}$. Without loss of generality, we can choose a representative ρ of $[\rho]$ such that $h_1 = 1$. Then we obtain $\alpha(\gamma) = h_2\beta(\gamma)h_2^{-1} = h_2\alpha(\gamma)h_2^{-1}$. It follows that h_2 is in the centralizer $Z(\alpha(\gamma))$ of $\alpha(\gamma)$. Henceforth the fiber is identified with the centralizer $Z(\alpha(\gamma))$.

If $\alpha(\gamma) \neq \pm Id$ then $Z(\alpha(\gamma))$ is a maximal torus in $SU(2)$ which acts simply transitively on itself by left multiplication. Therefore, we identify the maximal torus $Z(\alpha(\gamma))$ with the space $\{\zeta_t(\rho) | t \in [0, 2\pi]\}$ when $\alpha(\gamma) \neq \pm I$. Finally, for a generic element of $\mathfrak{X}(M; A, B, \gamma)$, we have $\alpha(\gamma) \neq \pm Id$, which ends the proof of the proposition. □

Relation with the measure. We can choose a presentation of M as

$$\pi_1(M) = \langle A_1, \ldots, A_k, B_1, \ldots B_l | q_k(A_1, \ldots, A_k)(q_l(B_1, \ldots, B_l))^{-1} \rangle$$

where q_n is the word defined for any n in $\mathbb{N}$ by $q_n(x_1, \ldots, x_n) = x_1^2 \cdots x_n^2$. The fundamental group of A and B are given by;

$$\pi_1(A) = \langle A_1, \ldots, A_k, C | q_k(A_1, \ldots, A_k)C^{-1} \rangle$$
$$\pi_1(B) = \langle B_1, \ldots, B_l, C | q_l(B_1, \ldots, B_l)C^{-1} \rangle.$$

The circle γ on M is represented by $q(A_1, \ldots, A_k)$. In this setting, the flow on $\mathrm{Hom}(\pi, G)$ acts by left and right multiplication on the k first generators, and hence is measure preserving.

Moreover, we can see the fundamental group of the surface with boundary A as a free group with generators $A_1, \ldots, A_k$. The restriction map $\mathfrak{X}(M) \longrightarrow \mathfrak{X}(A)$ is defined by the image of these generators. Hence, the measure on $\mathfrak{X}(A)$ defined as the push-forward measure through this restriction, is in the class of the Haar measure on G^k/G. And the same result holds with the restriction $\mathfrak{X}(M) \longrightarrow \mathfrak{X}(B)$.

Finally the decomposition measure $\nu_{[\alpha],[\beta]}$, with respect to the map j, on the fiber $j^{-1}([\alpha],[\beta])$ is the Haar measure on the maximal torus

$Z(\alpha(\gamma))$, which is a circle. Thus the Haar measure is in the Lebesgue class on $\mathbb{S}^1$.

The action of the Dehn twist. With the identification $\pi_1(M) = \pi_1(A) *_{\pi_1(\gamma)} \pi_1(B)$, the Dehn twist τ_γ about the curve γ acts on an element ρ in $\mathrm{Hom}(\pi_1(M), G)$ as

$$(\tau_\gamma \cdot \rho)(\delta) = \begin{cases} \rho(\delta), & \text{if } \delta \in \pi_1(A), \\ \rho(\gamma) \cdot \rho(\delta) \cdot \rho(\gamma)^{-1}, & \text{if } \delta \in \pi_1(B) \end{cases}$$

For any $g \in G$ we have $g = \exp(f(g) \cdot F(g))$, so we obtain

$$\rho(\gamma) = \exp(f(\rho(\gamma)) \cdot F(\rho(\gamma))) = \zeta_{f(\rho(\gamma))}(\rho).$$

Therefore, the Dehn twist on $\mathfrak{X}(M)$ can be expressed in terms of the Goldman flow :

$$\tau_\gamma = \Xi_{f(\rho(\gamma))}. \tag{4.7}$$

The Goldman flow is a π-periodic circle action on a generic fiber of the application j, which is homeomorphic to a circle. So the twist τ_γ acts on the fiber $j^{-1}([\alpha],[\beta])$ of the application $j : \mathfrak{X}(M) \to \mathfrak{X}(M; A, B, \gamma)$ as the rotation of angle $2f(\rho(\gamma))$ on this circle with $[\rho] \in j^{-1}([\alpha],[\beta])$.

4.3. **The flow associated to a non-separating circle.** Let $M = N_{2g+2,m}$ be the orientable surface of genus g with m boundary components, with two crosscaps attached. The circle γ is a two-sided curve passing through the two crosscaps (see Figure 4.3). The surface $A = M|\gamma$ is an orientable surface of genus g with $m+2$ boundary components. The two additional boundary components that correspond to the two sides of γ are denoted γ_+ and γ_-. Recall that crosscaps are drawn as shaded disks.

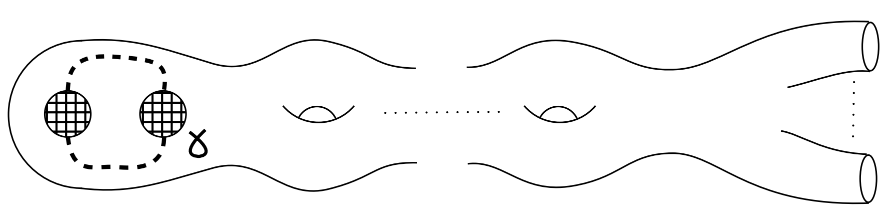

FIGURE 2. A non orientable surface of even genus

The surface M is obtained from A by gluing the two boundary components γ_+, γ_- with an orientation-reversing homeomorphism. So $\pi_1(M)$ can be constructed from $\pi_1(A)$ by an HNN construction. The group $\pi_1(M)$ is the quotient of the free product of $\pi_1(A)$ with a cyclic group $\langle\beta\rangle \cong \mathbb{Z}$, by the normal subgroup generated by the set

$$N = \left\{ i_-(\tau) \cdot \beta \cdot i_+(\tau) \cdot \beta^{-1} \mid \tau \in \pi_1(\gamma) \right\},$$

here $i_\pm$ are the embeddings induced by inclusion $\gamma \hookrightarrow \gamma_\pm \hookrightarrow M$.

$$i_\pm : \pi_1(\gamma) \longrightarrow \pi_1(\gamma_\pm) \xrightarrow{i_\gamma} \pi_1(M).$$

Namely, we obtain

$$\pi_1(M) = (\pi_1(A) * \langle\beta\rangle) \, / \, N,$$

The new generator β corresponds to a one-sided circle on M which crosses γ exactly once.

The flow on $\widetilde{\mathcal{S}_\gamma}$ is defined by:

$$\widetilde{\Xi}_t\rho(\delta) = \begin{cases} \rho(\delta), & \text{if } \delta \in \pi_1(A), \\ \zeta_t(\rho)\rho(\delta), & \text{if } \delta \in \pi_1(\beta). \end{cases} , \tag{4.8}$$

where ρ is an element of $\mathrm{Hom}(\pi_1(M), G)$, and t is a real number.

We define a flow $\{\Xi_t\}_{t\in\mathbb{R}}$ on $\mathcal{S}_\gamma$, that is covered by $\{\widetilde{\Xi}_t\}_{t\in\mathbb{R}}$. For any representation ρ in $\widetilde{\mathcal{S}_\gamma}$, the formula (4.5) gives $\zeta_{2\pi}(\rho) = I$. So we have $\widetilde{\Xi}_{2\pi}\rho = \rho$, and thus the flow $\{\Xi_t\}$ is 2π-periodic and defines a circle action on an open dense subset of full measure of $\mathfrak{X}(M)$.

We have a natural map $\phi : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(A)$ given by

$$\phi([\rho]) = [\rho_{|_{\pi_1(A)}}]$$

Proposition 4.3.1. *The generic fibers of the map ϕ are the orbits of the circle action $\{\Xi_t\}$.*

Proof. We denote by β_0 the preimage of β in A, which is an arc with one endpoint on γ_+ and one endpoint on γ_-. Let x_0 be the endpoint of β_0 on γ_-. For convenience we also denote by γ_- the corresponding element of $\pi_1(A, x_0)$. Let γ_+ be the element of $\pi_1(A, x_0)$ corresponding to the loop $\beta_0^{-1} \star \widetilde{\gamma}_+ \star \beta_0$ where $\widetilde{\gamma}_+$ is the loop following γ_+ based on the endpoint of β_0 on γ_+. A representation $\rho_A : \pi_1(A) \to G$ extends to a representation of $\pi_1(M)$ if and only if there exists $b \in G$ such that $\rho_A(\gamma_-)^{-1}$ is conjugate to $\rho_A(\gamma_+)$, i.e.

$$\rho_A(\gamma_-)^{-1} = b\rho_A(\gamma_+)b^{-1}. \tag{4.9}$$

The choice of the element b corresponds to the choice of the image of the new generator β. Two elements of $G = \mathrm{SU}(2)$ are conjugate if and only if they have the same trace, and an element of G and its inverse have the same trace. We infer that

$$\phi(\mathfrak{X}(M)) = \{[\rho_A] \in \mathfrak{X}(A) \mid \mathrm{tr}(\rho_A(\gamma_-)) = \mathrm{tr}(\rho_A(\gamma_+))\}.$$

Let $[\rho_A]$ be an element of $\phi(\mathfrak{X}(M))$, and $\rho_A \in \mathrm{Hom}(\pi, G)$ a representative. Let g be the element of G such that $\rho_A(\gamma_-) = g\rho_A(\gamma_+)g^{-1}$. Then the fiber $\phi^{-1}([\rho_A])$ is identified with the set of those $b \in G$ satisfying (4.9). Thus we can see that

$$\phi^{-1}([\rho_A]) = \{b \cdot g \mid b \in Z(\rho_A(\gamma_-))\}\,.$$

So the fiber $\phi^{-1}([\rho_A])$ is a right-coset of the centralizer $Z(\rho_A(\gamma_-))$ of $\rho_A(\gamma_-)$. If $\rho_A(\gamma_-) \neq \pm Id$ then $Z(\rho_A(\gamma_-))$ is a maximal torus in $SU(2)$ which acts transitively on the fiber by left multiplication. Therefore, we identify the maximal torus $Z(\rho_A(\gamma_-))$ with the space $\{\zeta_t(\rho_A(\gamma_-))|t \in [0, 2\pi]\}$ when $\rho_A(\gamma_-) \neq \pm I$. Finally, the set of all $[\rho_A] \in \phi(\mathfrak{X}(M))$ such that $\rho_A(\gamma) \neq \pm Id$ is an open dense subset of full measure of $\phi(\mathfrak{X}(M))$, which ends the proof of the proposition. ☐

Relation with the measure. We can choose a presentation of $\pi_1(M)$ as

$$\pi_1(M) = \langle A_1, \ldots, A_k, \gamma, \beta | q_k(A_1, \ldots, A_k)\beta\gamma\beta^{-1}\gamma\rangle$$

such that the elements $A_1, \ldots, A_k, \gamma$ generates $\pi_1(A)$ as a free group with $k+1$ generators. In this setting, the flow on $\mathrm{Hom}(\pi, G)$ acts on a representation ρ by left multiplication on $\rho(\gamma)$ and let $\rho(A_1), \ldots, \rho(A_k)$ invariant. Hence the flow is measure preserving.

Moreover, the map $\phi : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(A)$ is defined by the image of the first $k+1$ generators of $\pi_1(M)$. Hence the push-forward of the measure ν on $\mathfrak{X}(A)$ equals the quotient of Haar measure on G^{k+1}/G. Finally, the decomposition measure $\nu_{[\rho_A]}$, with respect to the map ϕ, on the fiber $\phi^{-1}([\rho_A])$ is the Haar measure on the maximal torus $Z(\rho_A(\gamma))$, and hence is in the Lebesgue class.

The action of the Dehn twist. With the identification of $\pi_1(M) = (\pi_1(A) * \langle\beta\rangle) \,/\, N$, the Dehn twist τ_γ about the curve γ acts on $\mathrm{Hom}(\pi_1(M), G)$ as

$$(\tau_\gamma \cdot \rho)(\delta) = \begin{cases} \rho(\delta), & \text{if } \delta \in \pi_1(A), \\ \rho(\gamma) \cdot \rho(\delta), & \text{if } \delta \in \langle\beta\rangle \end{cases}.$$

For any $\rho \in \mathrm{Hom}(\pi, G)$, we have $\rho(\gamma) = \exp(f(\rho(\gamma)) \cdot F(\rho(\gamma))) = \zeta_{f(\rho(\gamma))}(\rho)$. Then as in the previous case we express the Dehn twist in the form

$$\tau_\gamma = \Xi_{f(\rho(\gamma))}. \tag{4.10}$$

The Goldman flow is a 2π-periodic circle action on a generic fiber which is a circle. So the twist τ_γ acts on a generic fiber $\phi^{-1}([\rho_A])$ of the application $\phi : \mathfrak{X}(M) \to \mathfrak{X}(A)$ as the rotation of angle $f(\rho_A(\gamma)) = f(\rho(\gamma))$ on this circle, with $\rho \in \phi^{-1}([\rho_A])$.

Remark 7. When the surface M is oriented and compact, the flow defined by (4.6) or (4.8) covers the flow of the Hamiltonian vector field on $\mathfrak{X}(M)$ associated to the function f_α with respect to the natural symplectic structure on the space $\mathfrak{X}(M)$ (see [6]).

5. Surfaces of even genus

In this section, we prove theorem 2 in the case of a non-orientable surface of even genus. Let M be the non-orientable surface $N_{2g+2,m}$ and let X be a non-separating curve such that the surface $A = M|X$ is orientable, as the curve γ in Figure 4.3.

5.1. **Action of the Dehn twist about X.** According to (4.10) the Dehn twist τ_X about the curve X acts on a generic fiber of the application $\phi : \mathfrak{X}(M) \to \mathfrak{X}(A)$ as the rotation of angle $f(\rho(X))$. Let $\mathfrak{X}_{\mathbb{Q}}(M)$ be the set of representations $[\rho]$ in $\mathfrak{X}(M)$ such that $f(\rho(X))$ is a rational multiple of π. Then $\mathfrak{X}_{\mathbb{Q}}(M)$ has zero measure. Specifically, we have

$$\mathfrak{X}_{\mathbb{Q}}(M) = \bigcup_{q\in\mathbb{Q}} \mathrm{tr}_X^{-1}(2\cos(q\pi)).$$

where tr_X is the function $\mathrm{tr}_X([\rho]) \mapsto \mathrm{tr}(\rho(X))$. The trace function is a non-constant algebraic function on $\mathfrak{X}(M)$ which is an irreducible algebraic variety. Hence the set $\mathfrak{X}_{\mathbb{Q}}(M)$ is a countable union of lower-dimensional subvarieties (which have zero measure). So on the full-measure subset $\mathfrak{X}'(M)$ defined as $\mathfrak{X}(M) \setminus \mathfrak{X}_{\mathbb{Q}}(M)$, the angle $f(\rho(X))$ is irrational. A rotation by an irrational angle on the circle is ergodic with respect to its Lebesgue measure. So we have a measurable map $\phi : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(A)$ such that ϕ is τ_X-invariant. Moreover the action of τ_X on the fiber $\phi^{-1}([\alpha])$ is ergodic, with respect to the decomposition measure $\nu_{[\rho_A]}$, for almost all $[\alpha] \in \mathfrak{X}(A)$.

We recall the following classical result of measure theory and refer to ([3] Theorem 5.8) for a proof:

Lemma 5.1.1. of ergodic decomposition:
Let $(X, \mathfrak{B}, \mu)$ a measured space, Y, Z Borel spaces, and $F : X \longrightarrow Y$ a measurable map. Suppose that Γ is a group of automorphisms of $(X, \mathfrak{B}, \mu)$ such that F is Γ-invariant. Let μ_y be the measures on $F^{-1}(y)$ obtained by disintegrating μ over F. Let $h : X \longrightarrow Z$ be a measurable Γ-invariant function.

Suppose that the action of Γ is ergodic on the fiber $(F^{-1}(y), \mu_y)$ for almost all $y \in Y$.

Then there exists a measurable function $H : Y \longrightarrow Z$ such that $h = H \circ F$ almost everywhere.

Let $h : \mathfrak{X}(M) \longrightarrow \mathbb{R}$ be a τ_X-invariant measurable function. By the Lemma of ergodic decomposition, there exists a function $H : \mathfrak{X}(A) \longrightarrow \mathbb{R}$ such that $h = H \circ \phi$ almost everywhere. So a Γ_M-invariant function is almost everywhere equal to a function depending only on $\mathfrak{X}(A)$.

5.2. **The ergodicity of the mapping class group action on $\mathfrak{X}(A)$.** The surface A is an orientable surface with boundary. Let g_A be an element of the mapping class group Γ_A of A. The mapping class g_A is an element of $\mathrm{Out}(\pi_1(A))$. Using the identity $\pi_1(M) = \pi_1(A) * \langle\tau\rangle/N$, we define an element g of the mapping class group Γ_M of M. First, let $\widetilde{g}$ be an element acting on the free product $\pi_1(M) = \pi_1(A) * \langle\tau\rangle$ such that the restriction of $\widetilde{g}$ on $\pi_1(A)$ equals g_A, and $\widetilde{g}$ acts identically on $\langle\tau\rangle$. The element $\widetilde{g}$ leaves X_- and X_+ invariants, and hence we can define an element g on the quotient $\pi_1(M) = \pi_1(A) * \langle\tau\rangle/N$. This construction embeds Γ_A as a subgroup of Γ_M.

We recall that in [8] Goldman showed that for the natural symplectic measure on $\mathfrak{X}(A)$, a measurable function $f : \mathfrak{X}(A) \longrightarrow \mathbb{R}$ that is Γ_A-invariant is almost everywhere equal to a function depending only on the traces of the boundary components $\{X_+, X_-, C_1, \ldots, C_m\}$. In our case, an element $[\rho_A]$ in $\phi(\mathfrak{X}(M))$ satisfies $\mathrm{tr}(\rho_A(X_+)) = \mathrm{tr}(\rho_A(X_-))$. For a representation $[\rho]$ in $\mathfrak{X}(M)$ we denote by $x, c_1, \ldots, c_m$ the traces of the elements $\rho(X), \rho(C_1), \ldots, \rho(C_m)$ respectively. So we infer from previous results, the following:

Proposition 5.2.1. *Let $f : \mathfrak{X}(M) \to \mathbb{R}$ be a Γ_M-invariant function. There exists a function $G : [-2,2]^{m+1} \to \mathbb{R}$ such that $f([\rho]) = G(x, c_1, ..., c_m)$ almost everywhere.*

To conclude the proof of Theorem 2 in the case of a non-orientable surface of even genus, we have to "eliminate" the coordinate x. At this point the surface needs to have a sufficiently large mapping class group in order to be able to find a Dehn twist that acts non-trivially on the coordinate x corresponding to the trace of X. Thanks to the hypothesis on $\chi(M)$, there is a two-holed Klein bottle $N_{2,2}$ embedded in M, such that X is a non-separating two-sided curve in $N_{2,2}$. Now we study the particular case of $N_{2,2}$.

5.3. **The two-holed Klein Bottle.**

5.3.1. *The character variety.* Let M be a two-holed Klein bottle.

Its fundamental group is as follows

$$\pi = \pi_1(M) = \langle A, B, C, K \mid A^2B^2CK^{-1}\rangle,$$

where A, B, C, K are the curves drawn in Figure 5.3.1. So π is a free group in three generators A, B, C. According to Magnus [17], we have

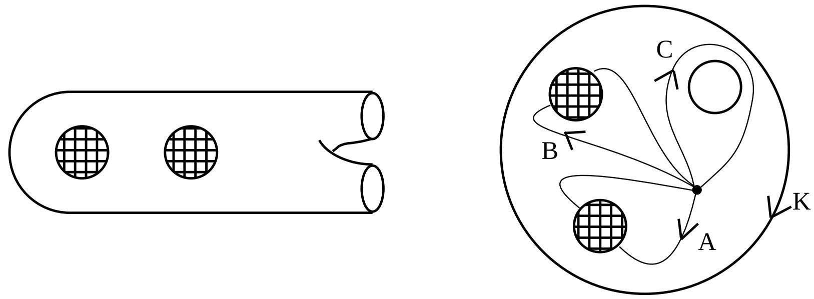

FIGURE 3. Two-holed Klein bottle

trace coordinates on the space $\mathfrak{X}(M) = \mathrm{Hom}(\pi, \mathrm{SU}(2))/\mathrm{SU}(2)$ given by the seven functions a, b, c, d, x, y, z defined on $\mathfrak{X}(M)$ by

$$a = \mathrm{tr}(\rho(A)); \quad b = \mathrm{tr}(\rho(B)); \quad c = \mathrm{tr}(\rho(C)); \quad x = \mathrm{tr}(\rho(AB));$$

$$y = \mathrm{tr}(\rho(BC)); \quad z = \mathrm{tr}(\rho(CA)); \quad d = \mathrm{tr}(\rho(ABC)). \text{ for any } [\rho] \in \mathfrak{X}(M)$$

These seven coordinates satisfy the Fricke relation

$$a^2 + b^2 + c^2 + d^2 + x^2 + y^2 + z^2 - \tag{5.1}$$

$$((ab + cd)x + (bc + da)y + (ca + bd)z) + xyz + abcd - 4 = 0$$

For any element $W \in \pi_1(M)$ written as a word in $A, B, C, A^{-1}, B^{-1}, C^{-1}$, it is possible to compute the value of $\mathrm{tr}(\rho(W))$ in function of the seven coordinates using the simple formulas $\mathrm{tr}(PQ^{-1}) = \mathrm{tr}(P)\cdot\mathrm{tr}(Q) - \mathrm{tr}(PQ)$ and $\mathrm{tr}(PQP^{-1}) = \mathrm{tr}(Q)$. These computations can be done algorithmically on a computer using a recursive program.

The elements $C, K \in \pi$ correspond to the two boundary components of the surface. The character variety of the boundary of M is

$$\mathfrak{X}(\partial M) = \{(c, k) \in [-2, 2]^2\}$$

where $k = \mathrm{Tr}(K) = \mathrm{tr}(A^2B^2C) = adb - az - by + c$.

Let X be the non-separating two-sided curve represented by AB in π. The surface $M|X$ is a four-holed sphere. According to Proposition 5.2.1, a measurable function $f : \mathfrak{X}(N_{2,2}) \longrightarrow \mathbb{R}$ that is Γ_M-invariant, is almost everywhere equal to a function depending only on the coordinates (x, c, k).

5.3.2. *The action of the twist about the curve BBC.* The curve U shown in Figure 5.3.2 is represented by the element $U = BBC$ of $\pi_1(M)$. It is a simple two-sided curve, so the Dehn twist about U can be defined.

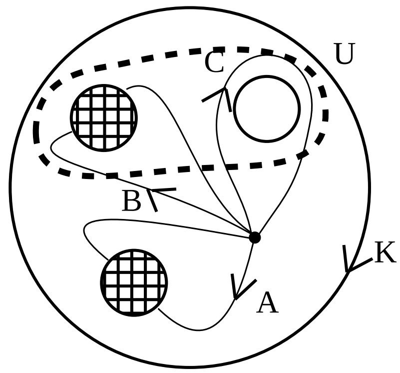

FIGURE 4. The curve U

The Dehn twist τ_U about U is given by the automorphism of $\pi_1(M)$

$$\begin{aligned} A &\mapsto A \\ B &\mapsto BBCBC^{-1}B^{-1}B^{-1} \\ C &\mapsto BBCB^{-1}B^{-1} \end{aligned}$$

The elements corresponding to X, Y, Z, D and K are transformed by τ_U as follows

$$\begin{aligned} X = AB &\mapsto ABBCBC^{-1}B^{-1}B^{-1} \\ Y = BC &\mapsto BBCB^{-1} \\ Z = CA &\mapsto BBCB^{-1}B^{-1}A \\ D = ABC &\mapsto ABBCB^{-1} \\ K = AABBC &\mapsto AABBC \end{aligned}$$

This automorphism of $\pi_1(M)$ is a lift of $\tau_U \in \mathrm{Out}(\pi_1(M))$ in $\mathrm{Aut}(\pi_1(M))$. Two lifts differ by an inner automorphism, which leaves invariant the conjugacy class and hence the trace coordinates of an element $[\rho] \in \mathfrak{X}(M)$. The transformation leaves invariant the conjugacy class of the elements corresponding to the boundary components, namely C and K.

The twist τ_U induces an action on $\mathfrak{X}_{\mathcal{C}}(M)$ which can be seen on the coordinates $(a,b,x,y,z,d) \in [-2,2]^6$ as :

$$\begin{aligned}
a \mapsto & a \\
b \mapsto & b \\
x \mapsto & b^2xy^2 + b^2yz - b^3dy - aby^2 + b^2cd + c^2x \\
& - 2bcxy + 2bdy - bcz + acy - yz - cd + ab - x \\
y \mapsto & y \\
d \mapsto & b^2d - bxy - bz + cx + ay - d \\
z \mapsto & b(b^2d - bxy - bz + cx + ay - d) - bd + z
\end{aligned}$$

For non-zero a, b , we can write

$$d = \frac{az + by - c + k}{ab}. \tag{5.1}$$

We replace in (5.1) the coordinate d with its expression in function of k which is τ_U-invariant. Then the equation (5.1) becomes:

$$x^2 + (by - c)x\left(\frac{z}{b}\right) + \left(\frac{z}{b}\right)^2 + 2Dx + 2Ez + F = 0 \tag{5.2}$$

with

$$D = \frac{a^2b^2 - c^2 + ck + bcy}{-2ab}, E = \frac{2c - 2k - 2by - b^2c + b^2k + b^3y + a^2by}{-2ab^2},$$

$$F = \frac{1}{a^2b^2}(k^2 + (by-c)^2 - (a^2-2)(by-c)k + a^2b^2(a^2+b^2+ck-4) - a^2ck + a^2bcy).$$

Remark 8. The set of representations $[\rho]$ such that a and b are zero, is a null measure subset of $\mathfrak{X}(N_{2,2})$. So it suffices to prove ergodicity in the complementary of this subset to have ergodicity on the whole space.

We make a change of variable $z' = \frac{z}{b}$, and the equation (5.2) becomes:

$$x^2 + (by - c)xz' + z'^2 + 2D'x + 2E'z + F = 0, \tag{5.3}$$

with

$$D' = D, E' = \frac{(b^2 - 2)(by + k - c) + a^2by}{-2ab}.$$

We denote by $u \in [-2,2]$ the trace $\mathrm{Tr}(\rho(U)) = by - c$. When $u \notin \{-2,2\}$, we rewrite (5.2) as

$$\frac{2+u}{4}((x+z') - (x_0(u) + z_0'(u))^2 + \frac{2-u}{4}((x-z') - (x_0(u) - z_0'(u))^2 = R$$

where x_0, z_0 and R are functions in a, b, c, y, k, and therefore are τ_U-invariants. The expression of R is the following:

$$R = \frac{(b^2 + c^2 + y^2 - bcy - 4)((a^2 - 2)^2 + u^2 + k^2 - (a^2 - 2)uk - 4)}{a^2(4 - u^2)}.$$

For a particular value of $-2 < u < 2$, the left term of the equation is a quadratic function on x, z' with positive coefficients. The positivity of the right term is given by the following fact concerning representations of the free group in two generators, and we refer to [7, 20, 17] for proofs.

Lemma 5.3.1. *Let $\mathbb{F}_2$ be the free group in two generators P and Q. Let $\mathcal{X}(M)$ be the space of conjugacy classes of representations* $\mathrm{Hom}(\mathbb{F}_2, \mathrm{SU}(2))/\mathrm{SU}(2)$. *Then the trace map*

$$\begin{array}{ccc} \mathcal{X}(M) & \longrightarrow & \mathbb{R}^3 \\ [\rho] & \longmapsto & \begin{pmatrix} \mathrm{tr}(\rho(P)) \\ \mathrm{tr}(\rho(Q)) \\ \mathrm{tr}(\rho(PQ)) \end{pmatrix} \end{array}$$

identifies $\mathcal{X}(M)$ with the set

$$\mathfrak{B} = \left\{(p, q, r) \in [-2, 2]^3 \,\middle|\, p^2 + q^2 + r^2 - pqr - 4 \leq 0\right\}$$

In the right term R we recognize (b, c, y) and $(a^2 - 2, u, k)$ as the characters of representations in $\mathrm{SU}(2)$ of the free groups in two generators $\langle B, C\rangle$ and $\langle AA, BBC\rangle$ respectively, as we have $BC = Y$ and $AABBC = K$. So according to Lemma 5.3.1, we have the following inequalities:

$$(b^2 + c^2 + y^2 - bcy - 4) \leq 0, \tag{5.4}$$

$$((a^2 - 2)^2 + u^2 + k^2 - (a^2 - 2)uk - 4) \leq 0. \tag{5.5}$$

Moreover $4 - u^2 > 0$, so R is non-negative. Hence the set of all (x, z') satisfying the equation (5.3) corresponds to an ellipse. This exhibits $\mathfrak{X}_{\mathcal{C}}(M)$ as a family of ellipses $E_{\mathcal{C}}(M)(a, b, y)$ that are parametrized by (c, k, a, b, y). Now we express the action of τ_U on (x, z') using (x_0, z_0'), as follows:

$$\begin{bmatrix} x \\ z' \end{bmatrix} \mapsto \begin{bmatrix} x_0(u) \\ z_0'(u) \end{bmatrix} + \begin{bmatrix} u^2 - 1 & u \\ -u & -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ z' \end{bmatrix} - \begin{bmatrix} x_0(u) \\ z_0'(u) \end{bmatrix} \right)$$

This transformation is a rotation of angle $2\theta_U = 2\cos^{-1}(\mathrm{tr}(\rho(U))/2)$ on the ellipse $E_{\mathcal{C}}(M)(a, b, y)$ for fixed (c, k, a, b, y). For all boundary traces (c, k) and for almost all (a, b, y), the angle θ_U is an irrational multiple of π. So for almost all (a, b, y), the action of τ_U is ergodic on the ellipse $E_{\mathcal{C}}(M)(a, b, y)$.

Let $f : \mathfrak{X}(M) \to \mathbb{R}$ be a Γ_M-invariant measurable function. The function f is τ_U-invariant, and by the Lemma of ergodic decomposition, there exists a function $H : [-2,2]^5 \longrightarrow \mathbb{R}$ such that $f([\rho]) = H(c,k,a,b,y)$ almost everywhere. On the other hand, according to Proposition 5.2.1, there exists $G : [-2,2]^3 \to \mathbb{R}$ such that $f([\rho]) = G(c,k,x)$ almost everywhere. Therefore the function f depends only on the traces (c,k) of the boundary components . This ends the proof of the Theorem 2 in the case of a two-holed Klein bottle.

5.4. **Conclusion.** We can now prove the case of a non-orientable surface of even genus. Let M be the surface $N_{g,m}$, with $\chi(M) \leq -2$. We can find an embedding of a two-holed Klein bottle $S = N_{2,2}$ in M. Let X be the non-separating two-sided circle on S such that the surface $M|X$ is an orientable surface. The Proposition 5.2.1 states that for any Γ_M-invariant function $f : \mathfrak{X}(M) \to \mathbb{R}$ there is a function $G : [-2,2]^{m+1} \to \mathbb{R}$ such that $f([\rho]) = G(x, c_1, ..., c_m)$ almost everywhere.

The mapping class group Γ_S of the two-holed Klein bottle can be seen as a subgroup of Γ_M. The restriction map $\mathfrak{X}(M) \longrightarrow \mathfrak{X}(S)$ is Γ_S-equivariant. The ergodicity of Γ_S on the relative character variety of S proves that the function f is almost everywhere equal to a function that does not depend on the function $x = \mathrm{tr}(\rho X)$. The two arguments combine to prove that a Γ_M-invariant function is almost everywhere equal to a function depending only on the traces of the boundaries $\mathcal{C} = (c_1, ..., c_m)$. Hence, the action of Γ_M is ergodic on $\mathfrak{X}_{\mathcal{C}}(M)$ and this ends the proof of the theorem 2 in the case of a non-orientable surface of even genus.

6. Non-orientable surfaces of odd genus with Euler characteristic -2

In this section, we study the case where M is a three-holed projective plane $N_{1,3}$ or the one-holed non-orientable surface of genus three $N_{3,1}$. In both cases, the fundamental group is isomorphic to the free group in three generators. Hence, we will use the trace coordinates defined in the previous section.

6.1. **The character variety of $N_{1,3}$.** Let M be the surface $N_{1,3}$, and let B, C and K be its three boundary components.

Its fundamental group admits the following presentation

$$\pi = \pi_1(M) = \langle A, B, C, K \mid A^2BCK^{-1} \rangle.$$

where A, B, C, K are the curves drawn in Figure 6.1. We see that π is a free group on three generators A, B, C, so the coordinates on the

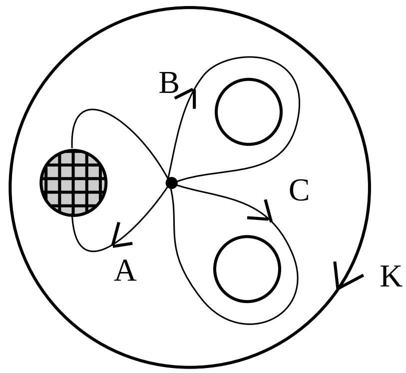

FIGURE 5. The three-holed projective plane

space $\mathfrak{X}(M)$ are given by the seven functions a, b, c, d, x, y, z defined as previously. These seven functions satisfy the Fricke relation (5.1). The character of the boundary of M is

$$\mathfrak{X}(\partial M) = (b, c, k)\,,$$

where $k = \mathrm{tr}(\rho(K)) = \mathrm{tr}(\rho(A^2BC)) = ad - y$. We replace y in the equation (5.1) with its expression in function of a, d and k. The equation then becomes:

$$a^2 + b^2 + c^2 + d^2 + x^2 + z^2 + k^2 -$$

$$((ab + cd)x + (xz - bc - da)k + (ca + bd)z) + adxz - 4 = 0. \tag{6.6}$$

For a fixed character of the boundary $\mathcal{C} = (b, c, k)$, the character variety relative to $\mathcal{C}$ is given by

$$\mathfrak{X}_{\mathcal{C}}(M) = \{(a, x, z, d) \in [-2, 2]^4 \mid (a, x, z, d) \text{ satisfies (6.6)}\}.$$

The complete character variety can be expressed as

$$\mathfrak{X}(M) = \bigcup_{-2 \leq b,c,k \leq 2} \mathfrak{X}_{\mathcal{C}}(M).$$

The Theorem 2 becomes in this particular case

Proposition 6.1.1. *For all boundary components $\mathcal{C} = (b, c, k) \in]-2, 2[^3$, the action of Γ_M on $\mathfrak{X}_{\mathcal{C}}(M)$ is ergodic.*

6.2. **The action of Dehn twists.**

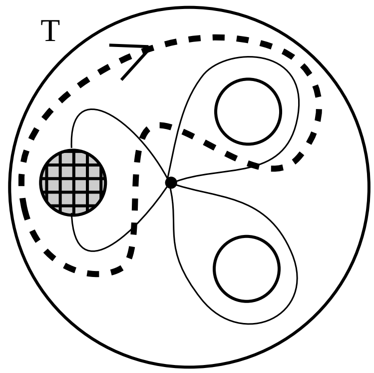

FIGURE 6. The curve T

6.2.1. *The twist about the curve $T = AAB$.* The curve T shown in Figure 6 is represented by the element $T = AAB$ in $\pi_1(M)$. It is a two-sided circle, so the Dehn twist about T can be defined.

The Dehn twist τ_T about T is given by the following automorphism of $\pi_1(M)$

$$\begin{aligned} A &\mapsto AABAB^{-1}A^{-1}A^{-1} \\ B &\mapsto AABA^{-1}A^{-1} \\ C &\mapsto C \end{aligned}$$

The curves corresponding to X, K, Z and D are transformed by τ_T as follows

$$\begin{aligned} X = AB &\mapsto AABA^{-1} \\ Z = CA &\mapsto CAABAB^{-1}A^{-1}A^{-1} \\ D = ABC &\mapsto AABA^{-1}C \\ K = AABC &\mapsto AABC \end{aligned}$$

This transformation leaves invariant the boundary character $\mathcal{C}$. So τ_T induces an action on $\mathfrak{X}_{\mathcal{C}}(M)$ which can be seen on the coordinates $(a, x, z, d) \in]-2, 2[^4$ as:

$$\begin{aligned} a \mapsto & a \\ x \mapsto & x \\ z \mapsto & a^2x^2z - a^2kx - a^2cx + b^2z - 2abxz \\ & + axd + bcx + abk + kx - bc + ac - z \\ d \mapsto & ak - axz + cx + bz - d \end{aligned}$$

The coordinates a and x are τ_T-invariant. We denote $t = \mathrm{Tr}(\rho(T)) = ax - b$. When $t \neq \pm 2$, we rewrite (6.6) as:

$$\frac{2+t}{4}\left((d+z) - \frac{(a+x)(c+k)}{t+2}\right)^2 + \frac{2-t}{4}\left((d-z) - \frac{(a-x)(k-c)}{t-2}\right)^2 = R_T \tag{6.1}$$

with

$$R_T := \frac{(t^2+c^2+k^2-tck-4)(a^2+b^2+x^2-abx-4)}{4-t^2}.$$

The function R_T is also τ_T-invariant. For a fixed value of t in $]-2,2[$, the left term of the equation (6.1) is a quadratic function of d and z with positive coefficients. In the right term R_T we recognize (a,b,x) and (t,c,k) as the characters of representations in SU(2) of the free groups in two generators $\langle A,B\rangle$ and $\langle T,C\rangle$ respectively, as we have $AB = X$ and $TC = K$. So according to Lemma 5.3.1, we have the following inequalities:

$$(a^2+b^2+x^2-abx-4) \leq 0, \tag{6.2}$$

$$(t^2+c^2+k^2-tck-4) \leq 0. \tag{6.3}$$

Moreover $4-t^2 > 0$, so that $R_T \geq 0$. So the set of coordinates (d,z) satisfying the equation (6.1) corresponds to an ellipse. For fixed values of b,c,k,a,x, the intersection

$$\begin{aligned} E_{\mathcal{C}} &:= \mathfrak{X}_{\mathcal{C}}(M) \cap (\{a,x\} \times \mathbb{R}^2) \\ &= \{a,x\} \times \left\{(d,z) \in \mathbb{R}^2 \mid (d,z) \text{ satisfies } 6.1\right\} \end{aligned}$$

is an ellipse preserved by τ_T. This exhibits $\mathfrak{X}_{\mathcal{C}}(M)$ as a family of ellipses $E_{\mathcal{C}}(M)(a,x)$ parametrized by (b,c,k,a,x). We can rewrite the equation in the following way :

$$Q_t(d-d_0(t), z-z_0(t)) = R_T$$

with

$$d_0(\nu) = ac + xk - \nu(ak+cx) \tag{6.4}$$

$$z_0(\nu) = ak + xc \tag{6.5}$$

$$Q_\nu(\eta,\zeta) = \frac{\eta^2+\zeta^2-\nu\eta\zeta}{4-\nu^2} \tag{6.6}$$

Now we express the action of τ_T on d,z in terms of d_0, z_0, which gives us (after simplification) :

$$\begin{bmatrix} d \\ z \end{bmatrix} \mapsto \begin{bmatrix} d_0(t) \\ z_0(t) \end{bmatrix} + \begin{bmatrix} -1 & -t \\ t & t^2-1 \end{bmatrix} \cdot \left(\begin{bmatrix} d \\ z \end{bmatrix} - \begin{bmatrix} d_0(t) \\ z_0(t) \end{bmatrix} \right). \tag{6.7}$$

This transformation is the rotation of angle $-\theta_T = -2\cos^{-1}(t/2)$ on the ellipse $E_{\mathcal{C}}(M)(a,x)$ defined for (b,c,k,a,x) fixed. In particular, for fixed boundary traces (b,c,k) and for almost all (a,x), the angle θ_T is an irrational multiple of π . So for almost all (a,x), the action of τ_T is ergodic on the ellipse $E_{\mathcal{C}}(M)(a,x)$.

6.2.2. *The twist about the curve $U = CAA$.* The curve U shown in Figure 7 is represented by the element $U = CAA$ in $\pi_1(M)$. It is a simple two-sided circle, so the Dehn twist about U can be defined.

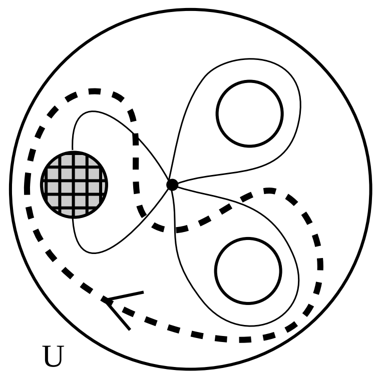

FIGURE 7. The curve U

The Dehn twist τ_U about U is given by the automorphism of $\pi_1(M)$

$$
\begin{aligned}
&A \mapsto CAAAA^{-1}A^{-1}C^{-1} = CAC^{-1}\\
&B \mapsto B\\
&C \mapsto CAACA^{-1}A^{-1}C^{-1}
\end{aligned}
$$

The curves corresponding to X, K, Z and D map to :

$$
\begin{aligned}
X = AB &\mapsto CAC^{-1}B\\
Z = CA &\mapsto CAACA^{-1}C^{-1}\\
D = ABC &\mapsto CAC^{-1}BCAACA^{-1}A^{-1}C^{-1}\\
K = AABC &\mapsto CAAC^{-1}BCAACA^{-1}A^{-1}C^{-1}
\end{aligned}
$$

We easily check that this transformation leaves invariant the boundary character $\mathcal{C}$. So τ_U induces an action on $\mathfrak{X}_{\mathcal{C}}(M)$ which can be seen on the coordinates $(a, x, z, d) \in]-2, 2[^4$

$$\begin{aligned} a \mapsto & a \\ x \mapsto & ab - zy + cd - x \\ z \mapsto & z \\ d \mapsto & ad - ay + abc - acx + abz - axz + acdz - acyz + \\ & a^2d + ac^2d - ayz^2 \end{aligned}$$

The coordinates a and z are τ_U-invariant. We denote $u = \mathrm{Tr}(\rho(U)) = az - c$ and when $u \neq \pm 2$, and we rewrite (6.6) as

$$Q_u(d - d_0(u), x - x_0(u)) = R_U$$

with $u := \mathrm{Tr}\, U = az - c$ which is τ_U-invariant and

$$d_0(\nu) = ab + zk - \nu(ak + bz) \tag{6.1}$$

$$x_0(\nu) = ak + zb \tag{6.2}$$

$$R_U = \frac{(u^2 + b^2 + k^2 - ubk - 4)(a^2 + c^2 + z^2 - acz - 4)}{4 - u^2} \tag{6.3}$$

The function R_U is also τ_U-invariant. For fixed value of $-2 < u < 2$, the left term of the equation is a quadratic function of d and x with positive coefficients. In the right term R_U we recognize (a, c, z) and (u, b, k) as the characters of representations in SU(2) of the free groups in two generators $\langle C, A\rangle$ and $\langle U, B\rangle$ respectively, as we have $CA = Z$ and $UB = CKC^{-1}$. So according to Lemma 5.3.1, we have:

$$(a^2 + c^2 + z^2 - acz - 4) \leq 0, \tag{6.4}$$

$$(u^2 + b^2 + k^2 - ubk - 4) \leq 0. \tag{6.5}$$

Moreover $4 - u^2 > 0$, so that $R_U \geq 0$. So the set of coordinates d and x satisfying the equation corresponds to an ellipse. This exhibits $\mathfrak{X}_{\mathcal{C}}(M)$ as a family of ellipses $E_{\mathcal{C}}(M)(a, z)$ parametrized by (b, c, k, a, z). Now we express the transformation τ_U on the coordinates d, x in terms of d_0, x_0, which gives us :

$$\begin{bmatrix} d \\ x \end{bmatrix} \mapsto \begin{bmatrix} d_0(u) \\ x_0(u) \end{bmatrix} + \begin{bmatrix} -1 & -u \\ u & u^2 - 1 \end{bmatrix} \cdot \left(\begin{bmatrix} d \\ x \end{bmatrix} - \begin{bmatrix} d_0(u) \\ x_0(u) \end{bmatrix} \right). \tag{6.6}$$

This transformation is the rotation of angle $-\theta_U = -2\cos^{-1}(u/2)$ on the ellipse $E_{\mathcal{C}}(M)(a, z)$ defined earlier for fixed (b, c, k, a, z). In particular, for fixed boundary traces (b, c, k) and for almost all (a, z), θ_U is an irrational multiple of π. So for almost all (a, z), the action of τ_U is ergodic on the ellipse $E_{\mathcal{C}}(M)(a, z)$.

6.2.3. *The twist about* W. The curve W shown in Figure 6.2.3 is represented by the element $W = CAB^{-1}A^{-1}$ in $\pi_1(M)$. It is a simple two-sided circle, so the Dehn twist about W can be defined.

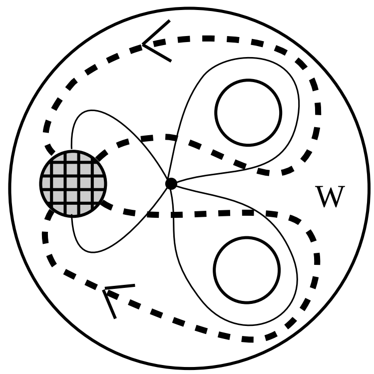

FIGURE 8. The curve W

The Dehn twist τ_U about U is given by the following automorphism of $\pi_1(M)$

$$\begin{aligned} A &\mapsto CAB^{-1}A^{-1}C^{-1}AB \\ B &\mapsto (B^{-1}A^{-1}CA)B(A^{-1}C^{-1}AB) \\ C &\mapsto CAB^{-1}A^{-1}CABA^{-1}C^{-1} = WCW^{-1} \end{aligned}$$

The curves corresponding to X, K, Z and D are transformed by τ_W as follows:

$$\begin{aligned} X = AB &\mapsto AB \\ Z = CA &\mapsto CA \\ D = ABC &\mapsto ABCAB^{-1}A^{-1}CABA^{-1}C^{-1} \\ K = AABC &\mapsto (CAB^{-1}A^{-1}C^{-1}AB)ABCA(B^{-1}A^{-1}CABA^{-1}C^{-1}) \end{aligned}$$

We easily check that this transformation leaves invariant the boundary character $\mathcal{C}$. So τ_W induces an action on $\mathfrak{X}_{\mathcal{C}}(M)$ which can be seen on the coordinates $(a, x, z, d) \in]-2, 2[^4$ as follows:

$$\begin{aligned} a &\mapsto w(xc-d) - (x(cw-b) - (zc-a)) \\ x &\mapsto x \\ z &\mapsto z \\ d &\mapsto w(dw - (zc-a)) - (x(wb-c) - (zb-d)), \end{aligned}$$

where $w = \operatorname{Tr} W = xz - k$. The coordinates x and z are τ_W-invariant and when $w \neq \pm 2$ we rewrite (6.6) as

$$Q_w(a - a_0(w), d - d_0(w)) = R_W$$

with

$$\begin{aligned} a_0(\nu) &= bx + cz - \nu(bz + cx) \\ d_0(\nu) &= bz + cx \\ R_W &= \frac{(x^2 + z^2 + k^2 - xzk - 4)(b^2 + c^2 + w^2 - bcw - 4)}{4 - w^2}. \end{aligned}$$

Moreover, R_W is τ_W-invariant. For a fixed value of $-2 < w < 2$, the left term of the equation is a quadratic function of a and d with positive coefficients. In the right term R_W we recognize (x, z, k) and (b, c, w) as characters of representations in SU(2) of the free groups $\langle X, Z\rangle$ and $\langle C, AB^{-1}A^{-1}\rangle$ respectively, because $XZ = ABCA = A^{-1}WA$ and $CAB^{-1}A^{-1} = W$. So according to Lemma 5.3.1, we have:

$$\begin{aligned} (x^2 + z^2 + k^2 - xzk - 4) \leq 0, \\ (b^2 + c^2 + w^2 - bcw - 4) \leq 0. \end{aligned}$$

Moreover $4 - w^2 > 0$, which implies $R_W \geq 0$. So the set of coordinates d and a satisfying the equation corresponds to an ellipse. This exhibits $\mathfrak{X}_{\mathcal{C}}(M)$ as a family of ellipses $E_{\mathcal{C}}(M)(b, c)$ parametrized by (b, c, k, x, z). Then we express the transformation τ_W on the coordinates a and d in terms of a_0 and d_0, which gives us :

$$(6.1)\quad \begin{bmatrix} a \\ d \end{bmatrix} \mapsto \begin{bmatrix} a_0(w) \\ d_0(u) \end{bmatrix} + \begin{bmatrix} -1 & -w \\ w & w^2 - 1 \end{bmatrix} \cdot \left(\begin{bmatrix} a \\ d \end{bmatrix} - \begin{bmatrix} a_0(w) \\ d_0(u) \end{bmatrix} \right).$$

This transformation is the rotation of angle $-\theta_W = -2\cos^{-1}(w/2)$ on the ellipse $E_{\mathcal{C}}(M)(x, z)$ defined earlier for fixed (b, c, k, x, z). In particular, for fixed boundary traces (b, c, k) and for almost all (x, z), the angle θ_W is an irrational multiple of π. So for almost all (x, z), the action of τ_W is ergodic on the set $E_{\mathcal{C}}(M)(x, z)$.

6.2.4. *Conclusion.* Let $f : \mathfrak{X}(M) \longrightarrow \mathbb{R}$ be a measurable Γ_M-invariant function. In particular f is τ_T, τ_U and τ_W-invariant. We deduce from the Lemma of ergodic decomposition, that a measurable function f that is τ_T-invariant, τ_U-invariant or τ_W-invariant, is almost everywhere equal to a function depending only on the coordinates (b, c, k, a, x), the coordinates (b, c, k, a, z) or the coordinates (b, c, k, x, z) respectively. Therefore a measurable Γ_M-invariant function f is almost everywhere equal to a function depending only on the traces of the boundary components $\mathcal{C} = (b, c, k)$. This proves the ergodicity of the Γ_M-action on $\mathfrak{X}_\mathcal{C}(M)$, and hence Proposition 6.1.1.

6.3. **The case of $N_{3,1}$.** In this section, we consider the case where M is the non-orientable surface of genus 3 with one boundary component $N_{3,1}$.

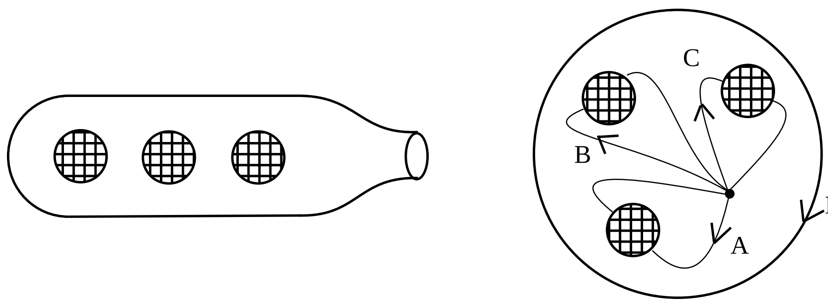

FIGURE 9. The surface $N_{3,1}$

Its fundamental group admits the following presentation

$$\pi = \pi_1(M) = \langle A, B, C, K \mid A^2B^2C^2K^{-1} \rangle,$$

where A, B, C, K are the curves drawn in Figure 6.3. We see that π is a free group on three generators A, B, C, so the coordinates on the space $\mathfrak{X}(M)$ are given by the seven trace functions a, b, c, d, x, y, z satisfying the Fricke relation (5.1). The character of the boundary of M is $\mathfrak{X}(\partial M) = \{k \in [-2, 2]\}$, where $k = \mathrm{tr}(\rho(K)) = \mathrm{tr}(\rho(A^2B^2C^2)) = abcd - bcy - acz - bax + a^2 + b^2 + c^2 - 2$.

Proposition 6.3.1. *For all boundary component $\mathcal{C} = k \in]-2, 2[$, the action of Γ_M on $\mathfrak{X}_\mathcal{C}(M)$ is ergodic.*

Proof. The curve U shown in Figure 9 is represented by $W = AACC \in \pi_1(M)$. It is a simple two-sided curve, so the Dehn twist can be defined about it.

The Dehn twist τ_U about the curve U is given by the following automorphism of $\pi_1(M)$:

$$\begin{aligned} A &\mapsto A \\ B &\mapsto A^{-2}C^{-2}BC^2A^2 \\ C &\mapsto C \end{aligned}$$

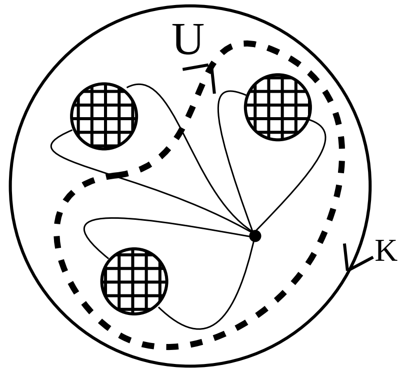

FIGURE 10. The curve U

The elements corresponding to X, Y, Z, D and K are transformed by τ_U as follows:

$$\begin{aligned} X = AB &\mapsto A^{-1}C^{-2}BC^2AA \\ Y = BC &\mapsto A^{-2}C^{-2}BC^2A^2C \\ Z = CA &\mapsto CA \\ D = ABC &\mapsto A^{-1}C^{-2}BC^2A^2C \\ K = AABBC &\mapsto AABBC \end{aligned}$$

This transformation leaves invariant the boundary component K. So the Dehn twist τ_U induces an action on $\mathfrak{X}_{\mathcal{C}}(M)$ which can be seen on the coordinates $(a,b,c,x,y,z,d) \in]-2,2[^7$ as:

$$\begin{aligned} a &\mapsto a \\ b &\mapsto b \\ c &\mapsto c \\ z &\mapsto z \\ x &\mapsto c^3d - c^2yz - c^2x - 2cd + cbz + cay + x \\ y &\mapsto \tau_U(y) \\ d &\mapsto \tau_U(d) \end{aligned}$$

where
$\tau_U(y) = -a^4bc^3 + a^4bc + a^4c^2y + a^3bc^4z - a^3bc^2z + a^3c^3x - a^3c^3yz - a^3c^2d - 2a^3cx + a^3d - a^2bc^5 - a^2bc^3z^2 + 4a^2bc^3 + a^2bcz^2 - 2a^2bc - a^2c^4xz + a^2c^4y + a^2c^3dz + 3a^2c^2xz + a^2c^2yz^2 - 3a^2c^2y - 2a^2cdz - a^2xz - a^2y + abc^4z - 3abc^2z + abz + ac^5x - ac^4d - 5ac^3x - ac^3yz + 4ac^2d + 5acx + 2acyz - 2ad + y$
and

$\tau_U(d) = -a^3bc^3 + a^3bc + a^3c^2y + a^2bc^4z - a^2bc^2z + a^2c^3x - a^2c^3yz - a^2c^2d - 2a^2cx + a^2d - abc^5 - abc^3z^2 + 3abc^3 + abcz^2 - abc - ac^4xz + ac^4y + ac^3dz + 3ac^2xz + ac^2yz^2 - 2ac^2y - 2acdz - axz - ay + bc^4z - 2bc^2z + bz + c^5x - c^4d - 4c^3x - c^3yz + 3c^2d + 3cx + cyz - d.$

We define new coordinates on $\mathfrak{X}(M)$ by replacing d with its expression in function of a, b, c, x, y, z and k.

$$d = \frac{bcy + acz + abx - a^2 - b^2 - c^2 + 2 + k}{abc}. \tag{6.1}$$

The equation (5.1) becomes

$$\left(\frac{x}{c}\right)^2 + (acz - a^2 - c^2 + 2)\left(\frac{x}{c}\frac{y}{a}\right) + \left(\frac{y}{a}\right)^2 + 2Dx + 2Ey + F = 0, \tag{6.2}$$

with constants D, E, F depending on the τ_U-invariant coordinates a, b, c, z, k. We make a change of variable $y' = \frac{y}{a}$ and $x' = \frac{x}{c}$and the equation (6.2) becomes

$$x'^2 + (acz - a^2 - c^2 + 2)x'y' + y'^2 + 2D'x + 2E'y + F = 0, \tag{6.3}$$

with

$D' = \frac{1}{abc}(a^2c^2 - 2a^2 - ab^2cz - ac^3z + 2acz + b^2c^2 - 2b^2 + c^4 - c^2k - 4c^2 + 2k + 4)$

$E' = \frac{1}{abc}(a^4 - a^3cz + a^2b^2 + a^2c^2 - a^2k - 4a^2 - ab^2cz + 2acz - 2b^2 - 2c^2 + 2k + 4)$

$F = \frac{1}{c^2b^2a^2}(a^4 + a^3b^2cz - 2a^3cz + a^2b^2c^2k - 2a^2b^2c^2 + 2a^2b^2 + a^2c^2z^2 + 2a^2c^2 - 2a^2k - 4a^2 + ab^4cz + ab^2c^3z - ab^2ckz - 4ab^2cz - 2ac^3z + 2ackz + 4acz + b^4 + 2b^2c^2 - 2b^2k - 4b^2 + c^4 - 2c^2k - 4c^2 + k^2 + 4k + 4).$

We denote $u = \mathrm{tr}(\rho(U)) = acz - a^2 - c^2 + 2$ and when $u \neq \pm 2$ we rewrite 6.2 as

$$Q_u(x - x_0(u), y - y_0(u)) = R,$$

where
$x'_0 = \frac{1}{abc(u^2-4)}(-a^4u + a^3cuz - a^2b^2u - a^2c^2u + 2a^2c^2 + a^2ku + 4a^2u - 4a^2 + ab^2cuz - 2ab^2cz - 2ac^3z - 2acuz + 4acz + 2b^2c^2 + 2b^2u - 4b^2 + 2c^4 - 2c^2k + 2c^2u - 8c^2 - 2ku + 4k - 4u + 8)$

$$y_0' = \frac{1}{abc(u^2-4)}(2a^4 - 2a^3cz + 2a^2b^2 - a^2c^2u + 2a^2c^2 - 2a^2k + 2a^2u - 8a^2 + ab^2cuz - 2ab^2cz + ac^3uz - 2acuz + 4acz - b^2c^2u + 2b^2u - 4b^2 - c^4u + c^2ku + 4c^2u - 4c^2 - 2ku + 4k - 4u + 8)$$

and

$$R = \frac{(a^2+c^2+z^2-acz-4)((b^2-2)^2+u^2+k^2-(b^2-2)uk-4)}{b^2(4-u^2)}.$$

The term R is also τ_U-invariant. For a particular value of $-2 < u < 2$, the left term of the equation is a quadratic function on (x', z') with positive coefficients. In the right term R we recognize that (a, c, z) and (b^2-2, u, k) are the characters of representations in SU(2) of the free groups $\langle A, C\rangle$ and $\langle BB, CCAA\rangle$ respectively, as we have $AC = Z$ and $CCAABB = K$. So according to Fricke's relation, we have:

$$(a^2+c^2+z^2-acz-4) < 0, \tag{6.4}$$

$$((b^2-2)^2+u^2+k^2-(b^2-2)uk-4) < 0. \tag{6.5}$$

Moreover $4-u^2 > 0$, so we deduce that $R > 0$. So the set of coordinates (x', y') satisfying the equation corresponds to an ellipse. This exhibits $\mathfrak{X}_{\mathcal{C}}(M)$ as a family of ellipses $E_{\mathcal{C}}(M)(a,b,c,z)$ that are parametrized by (k, a, b, c, z). Now we express the action of τ_U on x', y' using x_0', y_0', which gives us :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \mapsto \begin{bmatrix} x_0'(u) \\ y_0'(u) \end{bmatrix} + \begin{bmatrix} -1 & -u \\ (u^2-1) & u \end{bmatrix} \cdot \left(\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} x_0'(u) \\ y_0'(u) \end{bmatrix} \right).$$

This transformation is a rotation of angle $2\theta_U = -2\cos^{-1}(U/2)$ on the ellipse $E_{\mathcal{C}}(M)(a,b,c,z)$ defined for fixed (k,a,b,c,z). In fact, for any boundary trace k and for almost all (a,b,c,z), θ_U is an irrational multiple of π. So for almost all (a,b,c,z), the action of τ_U is ergodic on the set $E_{\mathcal{C}}(M)(a,b,c,z)$. Finally, let $f : \mathfrak{X}(M) \to \mathbb{R}$ be a τ_U-invariant measurable function. We deduce from the Lemma of ergodic decomposition, that there exists a function $H : [-2,2]^5 \longrightarrow \mathbb{R}$ such that $f([\rho]) = H(k,a,b,c,z)$ almost everywhere.

Finally, we split the surface along the curve $X = AB$ to obtain a surface A which is a three holed projective plane whose boundary components are K and X_-, X_+ which correspond to the two sides of X. According to the proposition 6.1.1, a Γ_A-invariant measurable function $f : \mathfrak{X}(A) \longrightarrow \mathbb{R}$ which is , is almost everywhere equal to a function depending only on the boundary character (x, x, k). So we have $f([\rho]) = G(k, x)$ almost everywhere. Combining the two arguments proves that a Γ_M-invariant function f depends only on the traces of the boundary component k. This ends the proof of Proposition 6.3.1. □

7. Surface of odd genus

In this section, we prove Theorem 2 in the case of a non-orientable surface of odd genus k. We split the surface M along a separating curve, such that one of the subsurface is orientable, and the other is a non-orientable surface of Euler characteristic -2. Two cases occur depending on the genus.

7.1. **When the genus is greater than 3.** Let M be the non-orientable surface $N_{2g+1,m}$ with $g \geq 1$ and $\chi(M) < -2$. Let C be a separating circle such that one of the two subsurfaces is the surface $A = N_{3,1}$ and the other is the orientable surface $B = \Sigma_{g,m+1}$ of genus g with $m+1$ boundary components.

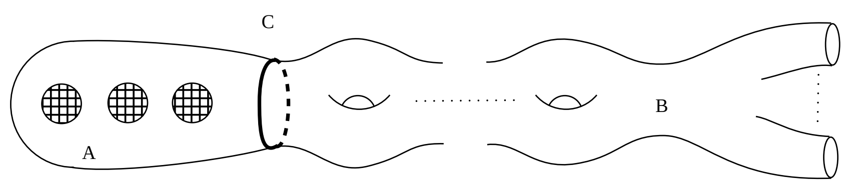

FIGURE 11. Decomposition of $N_{2g+1,m}$

Let $f : \mathfrak{X}(M) \to \mathbb{R}$ be a measurable Γ_M-invariant function. The Dehn twist τ_C about the curve C acts on the generic fiber of the application $j : \mathfrak{X}(M) \to \mathfrak{X}(M; A, B, C)$ as the rotation of angle $2\theta_C = 2\cos^{-1}(\mathrm{tr}(\rho(C)))$ for any representation ρ in the fiber $j^{-1}([\rho_A], [\rho_B])$. For almost all $([\rho_A], [\rho_B]) \in \mathfrak{X}(M; A, B, C)$, the angle is an irrational multiple of π and the τ_C-action is ergodic. By the lemma of ergodic decomposition, there exists a measurable function $h : \mathfrak{X}(M; A, B, C) \to \mathbb{R}$ such that $f = h \circ j$ almost everywhere. There are natural injective maps $\Gamma_A \hookrightarrow \Gamma_M$ and $\Gamma_B \hookrightarrow \Gamma_M$. Hence, the function h is Γ_A-invariant and Γ_B-invariant.

Next, consider the projection

$$\begin{aligned} \phi : \mathfrak{X}(M; A, B, C) &\longrightarrow \mathfrak{X}(B) \\ ([\rho_A], [\rho_B]) &\longmapsto [\rho_B]. \end{aligned}$$

The fiber $\phi^{-1}([\rho_B])$ can be identified with $\mathfrak{X}_c(A)$ where $c = \mathrm{tr}(\rho_B(C))$. According to Proposition 6.3.1, the mapping class group Γ_A acts ergodically on the fibers of ϕ. Thus, by the lemma of ergodic decomposition, there exists a measurable function $H : \mathfrak{X}(B) \to \mathbb{R}$ such that $h = H \circ \phi$ almost everywhere. Moreover, the function H is Γ_B-invariant.

We infer from the ergodicity result in the orientable case, that a Γ_B-invariant function is almost everywhere constant on almost every level set of the application

$$\begin{aligned} \partial^\sharp : \mathfrak{X}(B) &\longrightarrow [-2,2]^{m+1} \\ [\rho] &\longmapsto (\mathrm{tr}(\rho(k_1)), ..., \mathrm{tr}(\rho(k_m)), \mathrm{tr}(\rho(C))). \end{aligned}$$

So, there exists a measurable function $F : [-2,2]^{m+1} \to \mathbb{R}$ such that $f = F \circ \delta^\sharp \circ \phi \circ j = F(\mathrm{tr}(\rho(k_1)), ..., \mathrm{tr}(\rho(k_m)), \mathrm{tr}(\rho(C)))$ almost everywhere.

The orientable surface B has negative Euler characteristic $\chi(B) \leq -1$ and hence B can be decomposed into pants. The surface A can be decomposed into a pair of pants and a 2-holed projective plane such that the curve C is a boundary component of the pair of pants. Gluing the two pairs of pants containing C as a boundary component gives a 4-holed sphere S, which is embedded in M. The circle C is essential in the surface S. The ergocity result in the case of a 4-holed sphere shows that a Γ_S-invariant function is almost everywhere equal to a function that does not depend on the trace $\mathrm{tr}(\rho(C))$. The function f is Γ_S-invariant, and hence is almost everywhere equal to a function depending only on the traces of the boundaries $\mathcal{C} = (k_1, \ldots, k_m)$, which proves ergodicity in this case.

7.2. **In genus 1.** . Let M be the non-orientable surface $N_{1,m}$ with $m > 3$, and let C be a separating circle such that one of the two subsurfaces is the surface $A = \mathbb{N}_{1,3}$ and the other is a $(m+1)$-holed sphere $B = \Sigma_{0,m+1}$.

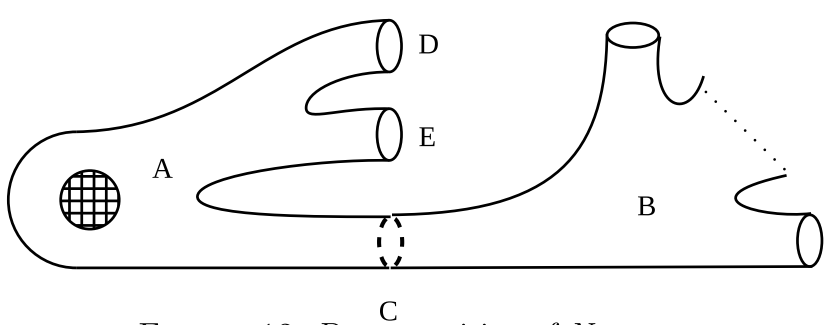

FIGURE 12. Decomposition of $N_{1,m}$

The proof uses the same arguments as in the previous case, the only difference is that we have to keep track of the two other boundary components of the subsurface A. Let $f : \mathfrak{X}(M) \to \mathbb{R}$ be a measurable Γ_M-invariant function. The action of the Dehn twist about the curve C is ergodic on almost all fibers of $j : \mathfrak{X}(M) \to \mathfrak{X}(M; A, B, C)$. The mapping class group Γ_A acts ergodically on almost every fibers of the map

$$\begin{aligned} \phi' : \mathfrak{X}(M; A, B, C) &\longrightarrow \mathfrak{X}(B) \times [-2, 2]^2 \\ ([\rho_A], [\rho_B]) &\longmapsto ([\rho_B], \mathrm{tr}(\rho_A(D)), \mathrm{tr}(\rho_A(E))). \end{aligned}$$

The mapping class group Γ_B acts ergodically on almost every fiber of the map

$$\begin{aligned} \widetilde{\partial^\sharp} : \mathfrak{X}(B) \times [-2, 2]^2 &\longrightarrow [-2, 2]^{m+3} \\ ([\rho], d, e) &\longmapsto (\mathrm{tr}(\rho(k_1)), ..., \mathrm{tr}(\rho(k_m)), \mathrm{tr}(\rho(C)), d, e). \end{aligned}$$

Hence there exists a measurable function $F : [-2, 2]^{m+3} \to \mathbb{R}$ such that $f = F \circ \widetilde{\delta^\sharp} \circ \phi' \circ j = F(\mathrm{tr}(\rho(k_1)), ..., \mathrm{tr}(\rho(k_m)), \mathrm{tr}(\rho(C)), \mathrm{tr}(\rho(D)), \mathrm{tr}(\rho(E)))$ almost everywhere. As in the previous case we can find a 4-holed sphere S embedded in M such that C is an essential circle in S. Hence the function F does not depend on $\mathrm{tr}(\rho(C))$ and the proof of Theorem 2 is complete. $\square$

References

[1] Freed D., *Reidemeister torsion, spectral sequences, and Brieskorn spheres*, J. Reine Angew. Math. , **429** (1992), 75–89.

[2] Frobenius G., Schur I., *Über die reellen Darstellung der endlichen Gruppen*, Sitzungsberichte der königlich preussischen Akademie der Wissenschaften (1906), 186–208.

[3] Furstenberg H., *Recurence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press (1981)

[4] Gendulphe M., *Paysage Systolique Des Surfaces Hyperboliques Compactes De Caracteristique -1*, arXiv:math/0508036.

[5] Goldman W., *The symplectic nature of fundamental groups of surfaces*, Adv. Math. , **85** (1984),200–225.

[6] Goldman W., *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. **85** (1986), no. 2, 263–302.

[7] Goldman W., *Topological components of spaces of representations*, Invent. Math. **93** (1988), 557–607.

[8] Goldman W., *Ergodic theory on moduli spaces*, Ann. of Math. (2) **146** (1997), no. 3, 475–507.

[9] Goldman W., *The complex-symplectic geometry of $SL(2, \mathbb{C})$-characters over surfaces*, Algebraic groups and arithmetic, 375–407, Tata Inst. Fund. Res., Mumbai, 2004.

[10] Goldman W., *Mapping class group dynamics on surface group representations* In: Problems in Mapping Class Groups and Related Topics. Proceedings of Symposia in Pure Math. Amer. Math. Soc. (to appear).

[11] Jeffrey L., Weitsman J., *Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula*, Commun. Math. Phys. **150** (1992), 595–620.

[12] Ho N., Jeffrey L., *The volume of the moduli space of flat connections on a nonorientable 2-manifold*, Commun. Math. Phys. **256** (2005), 539-564.

[13] Klein D., *Goldman flows on non-orientable surfaces*, arXiv:0710.5265.

[14] Korkmaz M., *Mapping class group of nonorientable surfaces*, Geom. Dedicata **89** (2002) 109–133.

[15] Lickorish W.B.R., *Homeomorphisms on non-orientable two-manifolds*, Math. Proc. Cambridge Philos. Soc. **59** (1963), 307–317.

[16] Lickorish W.B.R., *A finite set of generators for the homeotopy group of a 2-manifold*, Math. Proc. Cambridge Philos. Soc. **60** (1964), 769–778.

[17] Magnus W., *Rings of Fricke characters and automorphism group of free groups*, Math. Zeitschrift **170** (1980), 91–103.

[18] Mangler W., *Die Klassen topologisher Abbildungen einer geschlossenen Flache auf sich*, Math. Zeitschrift **44** (1939), 541–554.

[19] Mulase M., Penkava M., *Volume of representation varieties*, arXiv:math/0212012

[20] Nielsen J., *Die isomorphismen der allgemeinen unendlichen Gruppen mit zwei Erzeugenden*, Math. Ann. **78** (1918) , 385–397.

[21] Pickrell D., Xia E., *Ergodicity of mapping class group actions on representation variety, I. Closed surfaces*, Comment. Math. Helv. **77** (2001), 339–362.

[22] Pickrell D., Xia E., *Ergodicity of mapping class group actions on representation variety, II. Surfaces with boundaries*, Transformation Groups **8** (2003), 397–402.

[23] Previte J., Xia E., *Topological dynamics on Moduli Spaces I*, Pac. J. Math. **193** (2000), 397–417.

[24] Previte J., Xia E., *Topological dynamics on Moduli Spaces II*, Trans. Amer. Math. Soc. **354** (2002), 2475–2494.

[25] Previte J., Xia E., *Exceptional discrete mapping class group orbits in moduli spaces*, Forum Math. **15** (2003), 949-954.

[26] Stukow M., *The twist subgroup of the mapping class group of a nonorientable surface*, arXiv:0709.2798

[27] Witten E., *On quantum gauge theory in two dimensions*, Commun. Math. Phys. **141** (1991), 153–209.

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