Quantum jump processes for decoherence

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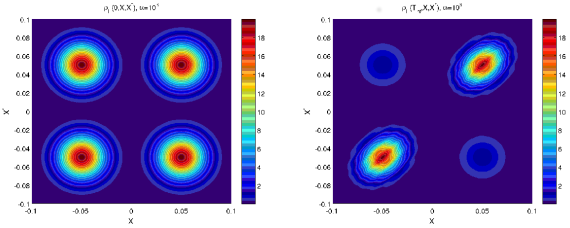
Nice, December 2017, PSPDE VI

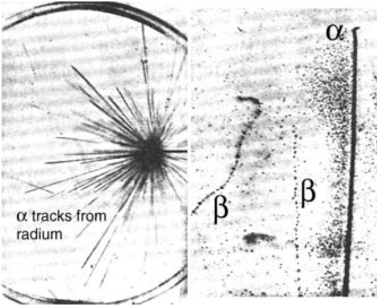


Content

Section 1

Three physical experiments

Wilson's cloud chamber



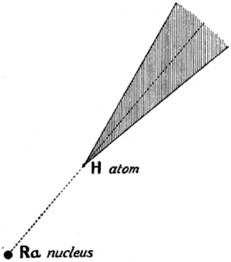
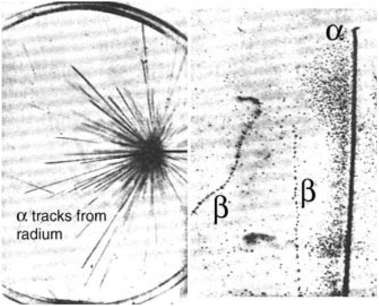


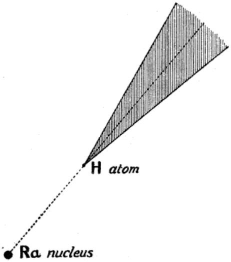
FIG. 1.

Photography of Wilson's cloud chamber (PRLS '12), and Mott's original paper.

- **Question :** Why the spherical wave function of an α particles gives straight *ionization line* in the cloud chamber? [Darwin (grandson), Heisenberg et Mott]

Wilson's cloud chamber





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Photography of Wilson's cloud chamber (PRLS '12), and Mott's original paper.

- **Question :** Why the spherical wave function of an α particles gives straight *ionization line* in the cloud chamber? [Darwin (grandson), Heisenberg et Mott]
- Answer given by Mott [Mott, PRLS '29].
- Recently re-examined mathematically [Dell'Antonio, Figari & Teta, JMP '08] and [Teta, EJP '10] and [Carlone, Figari & Negulescu ,preprint]

The two slit experiment of Young revisited

The decrease of interference fringes is observed experimentally in a two slit experiment near vacuum: [Hornberger & coll., PRL '03]

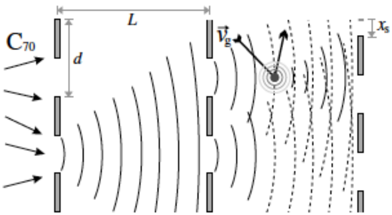

FIG. 1. Schematic setup of the near-field interferometer for C_{70} fullerenes. The third grating uncovers the interference pattern by yielding an oscillatory transmission with lateral shift x_s. Collisions with gas molecules localize the molecular center-of-mass wave function leading to a reduced visibility of the interference pattern.

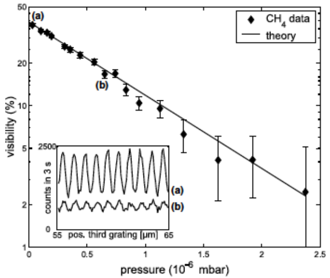

FIG. 2. Fullerene fringe visibility vs methane gas pressure on a semilogarithmic scale. The exponential decay indicates that each collision leads to a complete loss of coherence. The solid line gives the prediction of decoherence theory; see text. The inset shows the observed interference pattern at (a) $p = 0.05 \times 10^{-6}$ mbar and (b) $p = 0.6 \times 10^{-6}$ mbar.

A scheme of the experimental protocol and the results of Hornberger & all.

The quantum measurement problem

The postulate of Quantum mechanics

- (P1) The phase space is a **Hilbert space** $\mathcal{H} = L^2(D)$ (for us $D = \mathbb{R}^d$),
- (P2) A quantum observable is a **self-adjoint operator** on $\mathcal{H}$:

$$A = \sum_{i \in \mathbb{N}} \lambda_i \phi_i \otimes \phi_i = \sum_{i \in \mathbb{N}} \lambda_i |\phi_i\rangle\langle\phi_i|,$$

- (P3-4) For a system in the state ψ, the **measurement** of A gives

$$\lambda_i \quad \text{avec proba} \quad p_i := \left|\langle\phi_i|\psi\rangle\right|^2,$$

- (P5) **Wave packet collapse.** After a measurement which result is λ_i, the system is in the state

$$\psi_+ = \phi_i \quad \text{ou} \quad \psi_+ = \frac{1}{\|P_i \psi_-\|} P_i \psi_i,$$

- The free evolution of a quantum system is driven by a **Schrödinger equation** :

$$i\partial_t \psi_t = H\psi_t.$$

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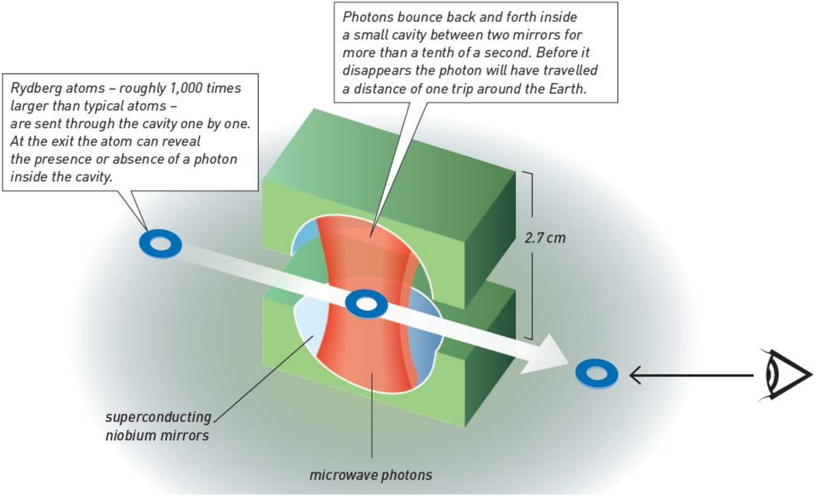
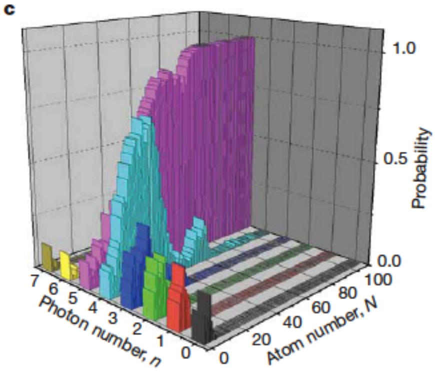
Recurrent question: Why a postulate to describe a measurement ?

Haroche experiment: Following experiment along time



Experimental set-up of Haroche & all. [Nature '07].

Haroche experiment: Following experiment along time



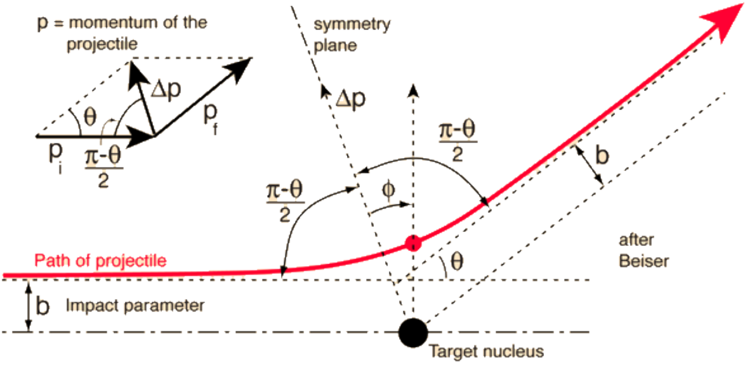
Experimental results by Haroche & all. [Nature '07].

- Uses **non-demolishing** measurement: only the wave-packet of the probe collapse.
- The wave packet collapse of the photons follows, but only after (many) **repeated interactions**.
- Mathematical explanation given by Bauer and Bernard [Phys. Rev. A 84, 2011] with a interesting Markov process.

Section 2

Super-operator describing quantum collisions

Classical collisions



!allows to deals with **instantaneous collisions** :

$$(\mathbf{x}, \mathbf{v}) \stackrel{\text{coll}}{\longmapsto} (\mathbf{x}, \mathbf{v}') \qquad \text{avec} \quad \mathbf{v}' = \mathbf{g}(\mathbf{v}, \theta)$$

Describing a quantum collision

- Using the quantum scattering operator S_V (V interaction potential)

$$S_V := \lim_{t \to +\infty} e^{itH_0} e^{-2itH_V} e^{itH_0}, \quad \text{with} \quad H_0 = -\frac{1}{2}\Delta, \quad H_V = -\frac{1}{2}\Delta + V$$

- **Quantum scattering with a massive particle**
  The two particle wave-function $\psi(t, X, x)$ evolves according to the Schrödinger eq.:

$$i\partial_t \psi = -\frac{1}{2m}\Delta_x \psi - \frac{1}{2M}\Delta_X \psi + V(x - X)\psi$$

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An instantaneous quantum collision

$$\psi_{in} = \phi \otimes \chi \xrightarrow{\text{Collision}} \psi_{out} \simeq \phi(X)\left[S^X \chi\right](x),$$

where S^X is the scattering operator of the light particle with a center in X.
Obtained rigorously in [Adami, H., Negulescu, CMS 2016]

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- **Problem :** After the collision, the two particles are entangled: no wave function for one particle anymore.
- **Partial answer :** The density matrix formalism introduced by *von Neumann*.

Density matrix or operator

- Quantum systems now described by **compact self-adjoint positive operators with unit trace** on $H = L^2(\mathbb{R}^d)$.
- **Pure states:** If a state has a wave function :

$$\rho = |\psi\rangle\langle\psi| \quad (= \psi \otimes \psi)$$

- **Mixed state**: The general case, after diagonalization

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

- The **partial trace** : allows to average on "degrees of freedom".

$$\rho_L(X, X') = \int \rho(X, x, X', x)\, dx$$

A super-operator describing instantaneous collisions

According to [Joos-Zeh, Z. Phys. B '85], the effect of one interaction on the massive particle is describe by a super-operator $\mathcal{S}_1^+ \subset \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$:

$$\mathcal{S}_1^+ := \big\{\rho \text{ sym. positive}, \ \mathrm{Tr}\,\rho < +\infty\big\}$$

Definition (Instantaneous collision operator)

defined on $\mathcal{S}_1^+$ *with* $I_V^\chi(X, X') := \langle S^X \chi, S^{X'} \chi \rangle$

$$\begin{array}{rccc} & \rho & \overset{\mathcal{I}_V^\chi}{\longmapsto} & \mathcal{I}_V^\chi[\rho] \\ \textit{with kernel} & \rho(X, X') & \mapsto & \rho(X, X') I_V^\chi(X, X'), \end{array}$$

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General properties

- **Contractive**: $|I_V^\chi(X,X')| \leq 1$,
- **Trace preserving**: $I_V^\chi(X,X') = 1$,
- **Completely positive** (see the Stinespring dilatation theorem).

References: Davies CMP 78, Diosí, Europhys. Lett. '95 ; AltenMüller, Müller, Schenzle, Phys. Rev. A '97 ; Dodd, Halliwell, Phys. Rev. D '03 ; Hornberger, Sipe, Phys. Rev. A '03 ; Adler, J. Phys. A '06, Attal & Joye, JSP 07.

A simpler form in 1D

The scattering theory in 1D is simpler:

$$S\left(e^{ikx}\right) = t_k e^{ikx} + r_k e^{-ikx}, \quad \text{et} \quad S^X\left(e^{ikx}\right) = t_k e^{ikx} + e^{2ikX} r_k e^{-ikx}$$

Particular case : Collision super-operator in 1D.

$$I_V^\chi(X, X') = 1 - \Theta_V^\chi(X - X') + i\,\Gamma_V^\chi(X) - i\,\Gamma_V^\chi(X')$$

with $\Theta_V^\chi \in \mathbb{C}$ and $\Gamma_V^\chi \in \mathbb{R}$ defined with the help of reflexion and transmission amplitudes (r_k, t_k)

$$\Theta_V^\chi(Y) := \int_{\mathbb{R}} \left(1 - e^{2ikY}\right) |r_k|^2 |\widehat{\chi}(k)|^2 \, dk,$$

$$\Gamma_V^\chi(X) := -i \int_{\mathbb{R}} e^{2ikX}\, r_k \overline{t_{-k}}\, \widehat{\chi}(k) \overline{\widehat{\chi}(-k)} \, dk.$$

Θ_V^χ est la partie **"décoherente"**, et $\Gamma_V^\chi \in [-1, 1]$ la partie **"potentielle"**.

The general decomposition

A similar decomposition exists in larger dimension, when $S = I + iT$.

General form of the collision super-operator

$$I_V^\chi(X,X') = 1 - \Theta_V^\chi(X,X') + i\,\Gamma_V^\chi(X) - i\,\Gamma_V^\chi(X')$$

with $\Theta_V^\chi \in \mathbb{C}$, et $\Gamma_V^\chi \in \mathbb{R}$, defined T by

$$\begin{aligned}\Theta_V^\chi(X,X') &:= \Im\langle\chi, T^X\chi\rangle + \Im\langle\chi, T^{X'}\chi\rangle - \langle T^X\chi, T^{X'}\chi\rangle \\ \Gamma_V^\chi(X) &:= \Re\langle\chi, T^X\chi\rangle.\end{aligned}$$

Θ_V^χ is the **"decoherent"** part, while $\Gamma_V^\chi \in [-1,1]$ is **"the potential"** one.

Remark: The optical theorem ensures that $\Theta_V^\chi(X,X) = 0$.

Simplifications in 1D: GWP et "quasi"-scattering

- If χ is a **Gaussian Wave Packet** (GWP) with parameters $(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{\sigma})$

$$\chi(x) = \frac{1}{(2\pi\boldsymbol{\sigma}^2)^{1/4}} e^{-\frac{(x-\boldsymbol{x})^2}{4\boldsymbol{\sigma}^2} + i\boldsymbol{p}x};$$

- **"Freeze"** the reflection and transmission amplitude

$$r_k = \alpha \in [0,1] \qquad \text{and} \qquad t_k = \pm i\sqrt{1-|\alpha|^2}.$$

Important: This approximation preserves all the important properties : unitarity of the scattering, complete positivity, commutation with the free evolution...

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A simple explicit approximation.

$$I_\alpha^{\boldsymbol{p},\boldsymbol{\sigma}}(X,X') = 1 - \Theta_\alpha^{\boldsymbol{p},\boldsymbol{\sigma}}(X-X') + i\,\Gamma_\alpha^{\boldsymbol{p},\boldsymbol{\sigma}}(X-x) - i\,\Gamma_\alpha^{\boldsymbol{p},\boldsymbol{\sigma}} i(X'-x),$$

$$\text{with} \quad \Theta_\alpha^{\boldsymbol{p},\boldsymbol{\sigma}}(Y) = \alpha^2\left(1 - e^{2i\boldsymbol{\sigma}\boldsymbol{p}\frac{Y}{\boldsymbol{\sigma}} - \frac{Y^2}{2\boldsymbol{\sigma}^2}}\right),$$

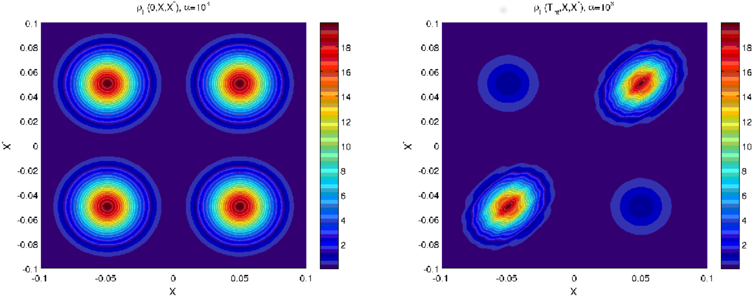
$$\Gamma_\alpha^{\boldsymbol{p},\boldsymbol{\sigma}}(X) = \pm\alpha\sqrt{1-|\alpha|^2}\, e^{-2\boldsymbol{\sigma}^2\boldsymbol{p}^2 - \frac{X^2}{2\boldsymbol{\sigma}^2}}$$

A simulation.

Initially, a massive particle in a superposed state

$$\phi(0) = \frac{1}{\sqrt{2}}\left(|\phi_-\rangle + |\phi_+\rangle\right) := \frac{1}{\sqrt{2}}\left|GWP\left(-\boldsymbol{X}, \boldsymbol{P}, \boldsymbol{\Sigma}\right)\right\rangle + \frac{1}{\sqrt{2}}\left|GWP\left(\boldsymbol{X}, -\boldsymbol{P}, \boldsymbol{\Sigma}\right)\right\rangle$$

$$\rho^M(0) = |\phi(0)\rangle\langle\phi(0)|$$

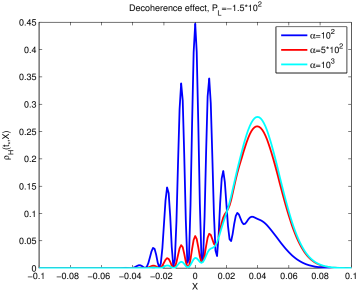


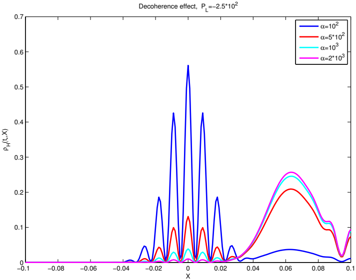
Density operator $\rho^M(0)$ (modulus of the kernel) before and after collision.

From [Adami, H., Negulescu, CMS 2016].

Simulation of the effect of the interaction on the interference fringes.

- Without interaction: the two bumps superposes at a time T^* with interference fringes.
- with interaction, the density is $\rho_a^M(T^*, X, X)$:





Density profil $\rho^M(T^*, X, X)$ for different interaction strength α, and velocity $\boldsymbol{p}$.

- Observation :
  - Damped interference fringes,
    linked to the transmission of the light particle,
  - A bump on the right without fringes,
    linked to the réflexion of the light particle. $\Rightarrow$ Moment exchange.

Section 3

Generators of quantum semi-groups: Lindblad super-operators

Lindblad equations and super-operators

Importance of the **complete positivity**, see [Kossakowsky, RMP 72] and [Lindblad, CMP 76].

Definition (Lindblad super-operator)

It is the generator of a quantum semi-group, that preserves complete positivity.

$$\partial_t \rho = L^* \rho := \sum_i \Big(V_i \rho V_i^* - \frac{1}{2}\big(V_i^* V_i \rho + \rho V_i^* V_i\big)\Big)$$

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- Poisson semi-group with unitary V_i:

$$\partial_t \rho = \sum_i \alpha_i V_i \rho V_i^* - \Big(\sum \alpha_i \Big) \rho$$

- Gaussian semi-group with self-adjoint V_i

$$\begin{aligned} \partial_t \rho &= \sum_i \Big(V_i \rho V_i - \frac{1}{2} \big(V_i^2 \rho + \rho V_i^2 \big) \Big) \\ &= - \sum_i \big[V_i, [V_i, \rho] \big] \end{aligned}$$

Classical and quantum Poisson semi-group.

Wigner transform

It is "almost" a **position velocity distribution**, associated to ρ :

$$f(x,v) := \int \rho\Big(x - \frac{k}{2}, x + \frac{k}{2}\Big) e^{ikv}\, dk,$$

Problème : $f \in \mathbb{R}$ but not necessarily $f \geq 0$. But Husimi transform $\tilde{f} = e^{\frac{1}{4}\Delta_{x,v}} f \geq 0$.

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- quantum Poisson semi-group (θ probability on $\mathbb{R}$) :

$$\boxed{\partial_t \rho = \int_{\mathbb{R}} \big(e^{ikx} \rho e^{-ikx} - \rho\big)\theta(dk).}$$

- After Wigner transform

$$\boxed{\partial_t f(x,v) = \int_{\mathbb{R}} \big(f(x, v-k) - f(x,v)\big)\theta(dk),}$$

it is the Fokker-Planck equation for a jump process on the velocities.

classical quantum Gaussian semi-groups

- Quantum Gaussian semi-group:

$$\boxed{\partial_t \rho = X\rho X - \frac{1}{2}(X^2\rho + \rho X^2)}$$

After Wigner transform

$$\boxed{\partial_t f(x,v) = \frac{1}{2}\Delta_{\mathrm{v}} f(x,v),}$$

which is the Fokker-Planck equation for a Langevin process(Brownian motion on velocities).

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- And for the Gaussian quantum semi-group :

$$\partial_t \rho = (i\partial)\rho(i\partial) - \frac{1}{2}((i\partial)^2\rho + \rho(i\partial)^2)$$

After Wigner

$$\partial_t f(x,v) = \frac{1}{2}\Delta_x f(x,v).$$

Section 4

The “weak coupling” limit.

An environment modeled by a thermal bath.

With N interaction par time unit

At random time T_i, the massive particle interact with GWP of parameter $(\boldsymbol{x}_i, \sigma_i = 1, \boldsymbol{p}_i)$ where $(T_i, \boldsymbol{x}_i, \boldsymbol{p}_i)$ are given by a Poisson Random Measure of intensity

$$N \, \frac{1}{2R_N} \, dt \otimes \frac{1}{2R_N} \mathbf{1}_{[-R_N, R_N]} d\boldsymbol{x} \otimes \frac{1}{\sqrt{2\pi}\bar{\sigma}} e^{-\frac{1}{2\bar{\sigma}^2} p^2} d\boldsymbol{p}.$$

This is a **thermic bath** at temperature $T = 1 + \bar{\sigma}^2$.

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- R_N is a truncature parameter, necessary in $1D$.
- $(T_{i+1} - T_i)$ are i.i.d. with exponential law $\mathcal{E}(1/2N)$.
- $\boldsymbol{x}_i$ are i.i.d. with uniform law $\mathcal{U}\big([-R_N, R_N]\big)$.
- $\boldsymbol{p}_i$ are i.i.d normal law $\mathcal{N}(0, \bar{\sigma}^2)$.
- everything is independent of everything.

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- everything is independent of everything.

Question: N interactions by time unit:
$\Rightarrow$ How to scale the interaction force to get a **finite effect** in the limit?

The appropriate scaling for the interaction strentgh

- The super-operator $\mathcal{I}_\alpha^{p,x}$ multiply the kernel by

$$I_\alpha^{\boldsymbol{p},\boldsymbol{x}}(X,X') = 1 - \alpha^2\left(1 - e^{2i\boldsymbol{p}Y - \frac{1}{2}Y^2}\right) \pm i\alpha\sqrt{1-\alpha^2}e^{-2\boldsymbol{p}^2}\left(e^{-\frac{1}{2}(X-\boldsymbol{x})^2} - e^{-\frac{1}{2}(X'-\boldsymbol{x})^2}\right)$$

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- The "decoherent" term with α^2/N has a bounded expectation by time unit.
- The "potential" term with $\alpha/\sqrt{N}$ has uniform expectation, and bounded fluctuations.

A quantum jump process

The "weak coupling" model

$$i\partial_t \rho_t^N = \left[H_0, \rho_t^N\right] \quad \text{on} \; [T_i, T_{i+1}), \qquad \text{avec} \quad H_0 = -\frac{1}{2}\Delta$$

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$$\rho_t^N \quad = \quad \rho_0^N \; - \; i\int_0^t \left[H_0, \rho_s^N\right] ds \; + \; i \iiint_0^t \left[\mathcal{I}_{\alpha,N}^{p,x}\rho_s^N \; - \; \rho_s^N\right] P_N(ds, dx, dp),$$

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- Or with the compensated PRM $\tilde{P}_N$, $\theta_\infty(Y) = e^{-\frac{2}{T}Y^2}$:

$$\rho_t^N = \rho_0^N - i\int_0^t \left[H_0 + \gamma_N, \rho_s^N\right] ds + \alpha^2 \int_0^t \left(\theta_\infty\left[\rho_s^N\right] - \rho_s^N\right) ds + i\iiint_0^t \left[\mathcal{I}_{\alpha,N}^{p,x}\rho_s^N - \rho_s^N\right] \tilde{P}_N(ds, dx, dp).$$

A convergence result: low density environement

Theorem (Gomez & H., arXiv 2016, rough version)

If the cut-off parameter $R_N \to \infty$, then the solution converges in $\mathcal{S}_p$ $(p > 1)$ towards the unique solution of the **Lindblad** *equation*

$$i\partial_t \rho_t^\infty = \left[H_0, \rho_t^\infty\right] + i\alpha^2\left(\theta_\infty\left[\rho_t^\infty\right] - \rho_t^\infty\right),$$

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If $R_N \leq N$, then fluctuations $Z_t^N = \sqrt{R_N}(\rho_t^N - \rho_t^\infty)$ converge in law in $\mathcal{S}_2$ towards the unique solution of

$$i\, dZ_t^\infty = \left[H_0\, dt, Z_t^\infty\right] + i\alpha^2\left(\theta_\infty\left[Z_t^\infty\right] - Z_t^\infty\right) dt + \alpha^2\left[dW_t, \rho_t^\infty\right],$$

where W_t is a cylindrical BM with covariance

$$\mathbb{E}\left[W_t(X)W_s(X')\right] = c(s \wedge t)\alpha^2 e^{-\frac{1}{4}(X-X')^2}.$$

A convergence result: dense environnement

Theorem (Gomez & H., arXiv 2016, rough version)

If $R_N = R$, then ρ^N converges in law in $\mathcal{S}_p$ (pour $p > 1$) towards the unique solution of a **stochastic Lindblad** *equation*

$$i\, d\rho_t^\infty = \left[H_0\, dt + dW_t, \rho_t^\infty\right] + i\alpha^2\left(\theta_\infty\left[\rho_t^\infty\right] - \rho_t^\infty\right) dt,$$

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$$\mathbb{E}\left[W_t(X)W_s(X')\right] = c(s \wedge t)\frac{\alpha^2}{R} e^{-\frac{1}{4}(X-X')^2} g_R(X,X').$$

where $g_R(X,X') \simeq 1$ when $|X|, |X'| << R$.

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or in Stratonovitch formulation

$$i\, d\rho_t^\infty = \left[H_0\, dt + dW_t \circ, \rho_t^\infty \right] + i\alpha^2 \left(\theta_\infty \left[\rho_t^\infty \right] - \rho_t^\infty \right) dt + ic \int_{-R}^{R} \left[\gamma(\cdot - x), \left[\gamma(\cdot - x), \rho_t^\infty \right] \right] dx dt$$

Effect of the stochastic potential on the decoherence

Stratonovich formulation separates the dynamics in:

- A **reversible** part:

$$i\, d\rho_t^\infty = \left[H_0\, dt + dW_t \circ, \rho_t^\infty \right]$$

- A **dissipative** part:

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Remark: The brown term decreases decoherence.

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Remarque

The Heisenberg-Ito equation

$$i\partial_t \rho = \left[H_0 dt + dW_t, \rho_t\right]$$

increases coherence, because in Stratonovich formulation

$$i\partial_t \rho_t = \left[H_0 dt + dW_t, \rho_t\right] + 2i\left(g(0) - g(X - X')\right)\rho_t$$

where g is the correlation function of the BM W.