# Particles approximation for Vlasov equation with singular interaction

M. Hauray, in collaboration with P.-E. Jabin.

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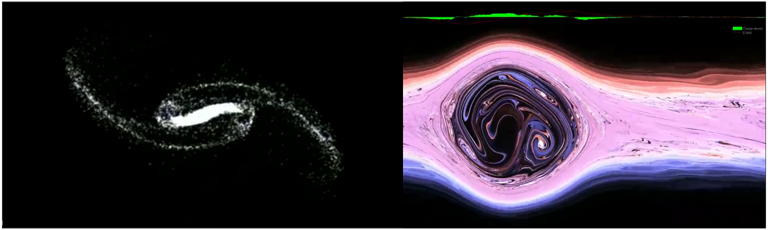

# Outline

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1. Introduction of the problem
2. A toy model: the 1D Vlasov-Poisson system.
3. The convergence of particles systems in 3D
4. Some ingredients of the proof.
5. The related problem of stability

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# Particle systems with singular forces.

$N$ particles with masses (or charges) $a_i/N$, positions $X_i$ et speed $V_i$ in $\mathbb{R}^{2d}$ $[Z_i = (X_i, V_i)]$ interacting through force $F$

$$\forall i \leq N, \quad \begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N}\sum_{j\neq i} a_j F(X_i - X_j) + \qquad 0\, dB_i. \end{cases}$$

**Singular forces :** Satisfying for some $0 < \alpha < d-1$, $F \in C_b^1(\mathbb{R}^d \backslash \{0\})$ and :

$$F(x) \underset{x\to 0}{\sim} \frac{x}{|x|^{\alpha+1}} \qquad \text{precisely} \quad |F(x)| \leq \frac{C}{|x|^\alpha}, \quad |\nabla F| \leq \frac{C}{|x|^{\alpha+1}} \quad (S^\alpha\text{-condition})$$

**About the resolution**

- Repulsive case : OK (No collisions).
- Attractive case : For $\alpha = d-1 \Rightarrow N$-body problem.
  True collisions are rare, but does non *non-collisions* singularities are? (Xia) and (Saary)
- $\alpha < 1$ : OK by DiPerna-Lions theory.

**For $N$ large**, particles systems should converge towards...

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# An example : Antennae galaxies.

## The Vlasov-"Poisson" equation

$f(t,x,v)$ is the density of particles and satisfies :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = 0 \\ E(t,x) = \int_\Omega F(x-y)\rho(t,y)\,dy, \quad \rho(t,x) = \int f(t,x,v)\,dv \end{cases} \tag{1}$$

+ initial condition: $f(0,x,v) = f^0(x,v)$.

**Two particular cases :** $F(x) = \pm c\frac{x}{|x|^d} \Rightarrow E = -\nabla V,\ \Delta V = \pm\rho$,
$-$: gravitationnal case , $+$: Coulombian one.

**About the Resolution**

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# The case of regular interaction forces.

**Important remark :** Under the assumption $F(0) = 0$, The empirical distribution

$$\mu_Z^N(t) = \frac{1}{N} \sum_{i=1}^{N} a_i \delta_{Z_i(t)}$$

of the particle system is a solution of the Vlasov eq. (1).

$\Rightarrow$ For smooth $F$, a theory of **measure solutions** of the Vlasov eq. is possible
Stability of meas. sol $\Rightarrow$ Convergence of part. systems

Theorem (Braun & Hepp '77, Neunzert & Wick '79, Dobrushin)

*Two measures solution $\mu$ and $\nu$ of the Vlasov eq. satisfy*

$$W_1(\mu(t), \nu(t)) \leq e^{(1+2\,\|\nabla F\|_\infty)t} W_1(\mu^0, \nu^0)$$

*Also CLT available using linearisation of VP, ...*

$W_1$ is the order one Monge-Kantorovich-Wasserstein distance.

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# The quantic equivalent is “better” understood.

- Convergence of Hartree-Fock towards Schordinger-Poisson already obtained by (Bardos, Golse, ... ’90), Erdös-Yau, Nier (’12), Pickl.
- Formalism more complex, but the non exact localization of particles may act like a cut-off.

# Numerical approximation with soften forces : PIC methods

Particle-in-Cell methods : introduce **virtual "large" particles** to solve the VP equation.

The Poisson or gravitational force is **cut off** at a length $\varepsilon(N)$ : $F_\varepsilon(x) = \frac{x}{(|x|+\varepsilon)^d}$.

Two possibilities for the computation of the field :

- **PM :** Compute it at the **nodes of a mesh** with the appropriate solver (*plasma*).
- **PP :** Use only binary interaction (*astrophysics*).
  Problem : PP requires normally $N^2$ operations, except if you use a tree code (cost reduced to $N \ln N$).

Theorem (Cottet-Raviart '91, Victory & all '89)

*Assume that*

- *$f$ is a smooth solution of the VP equation, with initial data $f^0$.*
- *The $Z_i^N(0)$ at the node of a mesh of size $\beta \approx N^{1/2d}$, and $a_i = f^0(Z_i^N(0))$.*
- $\varepsilon \approx \beta^r$ **for some r < 1.**

*Then, if the $\bar{Z}_i^N(t)$ are transported by the flow of the VP eq. $(\bar{Z}_i^N(0) = Z_i^N(0))$*

$$\|Z^N(t) - \bar{Z}^N(t)\|_p \le CN^{-s}, \quad \textit{for some } s > 0.$$

*$s$ depends on the regularity of $f$, and the cut-off used.*

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# The mean-field limit for VP1D.

In **1D**, the interaction is not very singular: $F(x) = \mathrm{sign}(x)$.

$\Rightarrow$ the problem is simpler.

In fact there is a weak-strong stability principle for the 1D VP equation

**Theorem (H. 2013)**

*Assume that:*

- *$f$ is a solution to VP1D with bounded density $\rho$,*
- *$\mu$ is a weak measure solution.*

*Then, for some $c > 0$ and all $t \geq 0$*

$$W_1(f(t),\mu(t) \leq e^{c\int_0^t \|\rho(s)\|_\infty \, ds} W_1(f^0,\mu^0).$$

But $\mu = \mu_N$ is allowed. It implies

**Theorem (Mean-field limit, Trocheris '86)**

*If $\mu_N^0 \rightharpoonup f^0$, then for any time $t \geq 0$*

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# The propagation of molecular chaos.

The notion goes back to L. Boltzmann and its famous "Stosszahl Ansatz". Formalized by Sniztmann

## Definition (Chaotic sequences of particle distribution.)

*A sequence of symmetric probabilities $(F^N)$ of $\mathcal{P}(\mathbb{R}^{2dN})$ is $f$-chaotic if (equivalent conditions)*

1. $\mu^N \rightharpoonup f$ *in law in* $\mathcal{P}(\mathbb{R}^{2d})$,
2. *For all $k$ the sequence of $k$ marginals* $F_k^N \rightharpoonup f^{\otimes k}$,
3. $F_2^N \rightharpoonup f^{\otimes 2}$.

It is also possible to quantify that notion of convergence: (Mischler & Mouhot) or (H. & Mischler).

$$W_1(F_2^N, f^{\otimes 2}) \leq \frac{1}{N} W_1(F^N, f^{\otimes N}) \leq C \left[ W_1(F_2^N, f^{\otimes 2}) \right]^{\alpha}.$$

# The propagation of molecular chaos for VP1D.

Just take the expectation of the mean-field result.

**Theorem (Prop of chaos for VP1D)**

*If $f$ is a solution to VP1D with bounded density: Then, for some $c > 0$ and all $t \geq 0$*

$$\mathbb{E}\big[W_1(f(t), \mu(t))\big] \leq e^{c \int_0^t \|\rho(s)\|_\infty \, ds} \mathbb{E}\big[W_1(f^0, \mu^0)\big].$$

- Obtain also **large deviation upper bound** in the same way.
- Results are also obtained on the trajectories.

# A more usual viewpoint: The Vlasov Hierarchy.

The **Liouville Equation** for the time marginals of the N "indistinguishable" particles

$$\partial_t F^N + \sum_{i=1}^{2} v_i \cdot \nabla_{x_i} F^N + \frac{1}{N} \sum_{i \neq j} \nabla V(x_i - x_j) \cdot \nabla_{v_i} F^N = 0,$$

satisfies in the limit $N \to +\infty$ the Vlasov Hierarchy

$$\partial_t F_1 + v_1 \cdot \nabla_{x_1} F_1 \quad + \quad \int \nabla V(x_1 - x_2) \cdot \nabla_{v_1} F_2(v_1, v_2)\, dv_2 = 0,$$

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# A strong result: Propagation of entropic chaos.

**Definition (Entropy chaotic sequences.)**

*A sequence of symmetric probabilities $(F^N)$ of $\mathcal{P}(\mathbb{R}^{2dN})$ is $f$-chaotic if*

- *it is $f$-chaotic,*
- $\frac{1}{N} H(F^N) \to H(f)$.

A stronger notion: $\Rightarrow$ strong convergence of the marginals

$$\lim_{N \to +\infty} \left\| F_k^N - f^{\otimes k} \right\|_1 = 0$$

**Theorem (Prop. of entropic chaos.)**

*The propagation of entropic chaos holds for the VP1D equation.*

It is a “simple” consequence of the preservation of entropy in VP1D and Liouville equation.

What about Fisher chaos?

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# The "‘mean-field” convergence result (compact support).

In the sequel, we set $a_i = 1$ for all $i$ (all the particles have the same mass).

**Theorem (H., Jabin '11)**

*Assume that $F$ satisfies a $S^\alpha$-condition with*

$$\alpha < d - 1,$$

*and that $f$ is a strong bounded sol. of VP', and $\gamma \in (0,1)$. For each $N$, choose the initial positions $(Z_i)$ such that*

$$(i) \quad \sup_{z \in \mathbb{R}^{2d}} N^{-1} \mu\big(B(z, N^{-\frac{\gamma}{2d}})\big) \leq C$$

$$(ii) \quad \inf_{i \neq j} |X_i(0) - X_j(0)| \geq C\, N^{-\frac{\gamma(1+r)}{2d}},$$

*for some $r < \frac{d-1}{\alpha+1}$. Then for some $\kappa > 0$*

$$W_1(\mu_z^N(t), f(t)) \leq e^{\kappa t} \Big( W_1(\mu_z^N(0), f_0) + 2\, N^{-\frac{\gamma}{2d}} \Big)$$

The $r$ may be chosen larger than 1 only for $d > 3$. It implies the next result.

# Chaos propagation for singular interactions.

In the sequel, we set $a_i = 1$ for all $i$ (all the particles have the same mass).

**Theorem (H., Jabin '11)**

*Assume that $F$ satisfies a $S^\alpha$-condition with*

$$\alpha < 1 \text{ if } d \geq 3, \qquad \alpha < \frac{1}{2} \text{ if } d = 2$$

*For each $N$, choose the initial positions $Z_i$ independently according to the continuous and compact profile $f^0$. Then propagation of chaos holds and precisely for $\gamma < 1$ (but close enough) there exists $\kappa$ (almost as before) and $\beta > 0$ (but small) s.t.*

$$\mathbb{P}\left( W_1(\mu_z^N(t), f(t)) \geq \frac{e^{\kappa t}}{N^{\frac{\gamma}{2d}}} \right) \leq \frac{C}{N^\beta}$$

**Roughly :** For independent initial conditions with profile $f^0$, we have with large probability

$$W_1(\mu_z^N(0), f^0) \leq \varepsilon := N^{-\frac{1}{2d}}$$

which propagates in time

$$W_1(\mu_z^N(t), f(t)) \leq e^{\kappa t}\varepsilon.$$

# The first scale : Average distance between particles.

Precisely : Average distance between a particle and its closest neighbour in phase space.

**Heuristic :** Pick all $Z_i$ uniformly in $[0,1]^{2d}$. Average distance of order $N^{-\frac{1}{2d}}$.

**Precise results :**

Proposition (Peyre '07,Boissard '11)

*For $N$ independant r.v. $Z_i$ with law $f$ compact and $d \geq 2$, there exists a constant $L_0$ such that*

$$\mathbb{P}\left(W_1(\mu_z^N, f) \geq \frac{L}{N^{\frac{1}{2d}}}\right) \leq e^{-N^\alpha (L-L_0)} \qquad \alpha = \frac{d-1}{2d}$$

Remark : $W_1(\mu_z^N, f) \geq \frac{c}{\|f\|_\infty} N^{-\frac{1}{2d}}$

Theorem (Gao '03)

*If $\nu_N = \mu_N * \frac{\chi_{B_\varepsilon}}{|B_\varepsilon|}$ with $\varepsilon = N^{-\frac{\gamma}{2d}}$, then*

$$\limsup_{N \to +\infty} \frac{1}{N^{1-\gamma}} \ln \mathbb{P}\left(\|\nu_N\|_\infty \geq 2\|f\|_\infty\right) \leq c\|f\|_\infty, \qquad \text{with } c = |B_1|(2\ln 2 - 1)$$

# The first scale : Average distance between particles.

Precisely : Average distance between a particle and its closest neighbour in phase space.

**Heuristic :** Pick all $Z_i$ uniformly in $[0,1]^{2d}$. Average distance of order $N^{-\frac{1}{2d}}$.

**Precise results :**

Proposition (Peyre '07,Boissard '11)

*For N independant r.v. $Z_i$ with law f compact and $d \geq 2$, there exists a constant $L_0$ such that*

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# An unphysical scale : the minimal inter-particle distance.

$$d_z^N := \min_{i \neq j}(|Z_i - Z_j|)$$

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Proposition (H. '07)

*For $Z_i$ uniformly distributed with profile $f$ bounded, then*

$$\mathbb{P}\left(d_z^N \geq \frac{l}{N^{1/d}}\right) \geq e^{-c_{2d}\|f^0\|_\infty l^d}.$$

Important : It is a very weak deviation result. (Ineq. in bad sense).

$$\text{In fact,} \quad \mathbb{P}\left(d_z^N \leq \frac{l}{N^{1/d}}\right) \leq 1 - e^{-c_{2d}\|f^0\|_\infty l^{-d}} \leq c_{2d}\|f^0\|_\infty l^d.$$

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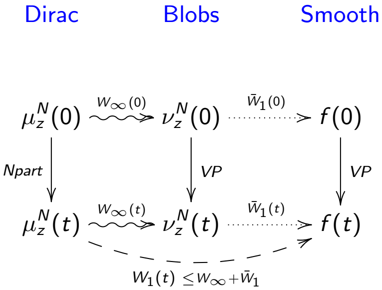
# Sketch of the proof (for $d = 3$).




1. (Probabilistic) Eliminate **bad** initial conditions.
2. (Deterministic) Estimate the distance $\bar{W}_1(t) := W_1(\nu_z^N(t), f(t))$.
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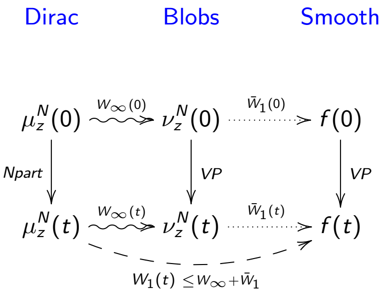


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## Step 1 and 2

Step 1 : Choose $r$ and $\gamma$ such that

$$1 < r < \frac{2}{1+\alpha}, \qquad \frac{2}{1+r} < \gamma < 1.$$

Define our reference scale $\varepsilon = N^{-\frac{\gamma}{2d}}$. Then with large probability we have,

- $\|\nu_z^N(0)\|_\infty \leq 2\|f(0)\|_\infty$,
- $d_z^N \geq \varepsilon^{1+r}$,
- $\bar{W}_1(0) \leq C\varepsilon$.

Step 2 :

- Prove propagation of the compact support : Supp $f(t), \nu_z^N(t) \subset [-R(t),, R(t)]^6$.
- Then bound $\|\rho(t)\|_\infty \leq 2\|f(0)\|_\infty R(t)^d$.
- Use the following proposition

**Proposition (Loeper '06)**

*For two solutions of Vlasov-"Poisson" with an $S^\alpha$-condition, $\alpha < d-1$*

$$W_1(f(t), g(t)) \leq e^{\kappa t} W_1(f(0), g(0)), \qquad \text{where } \kappa = C \sup_{t\in[0,T]}(\|\rho_f(t)\|_\infty + \|\rho_g(t)\|_\infty)$$

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# Step 3

- Choose the simplest coupling between $\mu_z^N(0)$ and $\nu_z^N(t)$.
- Integrate the evolution on a small interval of tim $[t - \varepsilon^{r'}, t]$, ( $r' > r$).
- Compare the two mean fields with a partition of phase space

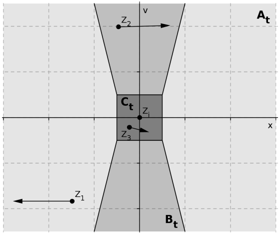


Figure : The partition of phase space.

and obtain the estimates...

## The estimates of step 3.

$$\begin{aligned}
\frac{\tilde{W}_\infty(t) - \tilde{W}_\infty(t-\varepsilon^{r'})}{\varepsilon^{r'}} &\leq C_2 \left( \underbrace{\tilde{W}_\infty(t)}_{A_t} + \underbrace{\varepsilon^{\lambda_1} \tilde{W}^d_\infty(t)}_{B_t} + \underbrace{\varepsilon^{\lambda_2} \tilde{W}^{2d}_\infty(t)\, \tilde{d}_N^{-\alpha}(t)}_{C_t} \right), \\
|\nabla^N E|_\infty(t) &\leq C_2 \left( 1 + \varepsilon^{\lambda_3} \tilde{W}^d_\infty(t) + \varepsilon^{\lambda_4} \tilde{W}^{2d}_\infty(t)\, \tilde{d}_N^{-\alpha}(t)) \right) \\
\tilde{d}_N(t) + \varepsilon^{r'-r} &\geq [\tilde{d}_N(t-\tau) + \varepsilon^{r'-r}] e^{-\tau(1+|\nabla^N E|_\infty(t))}.
\end{aligned}$$

Where $\lambda_i > 0$, and the minimum is $\lambda_3 = d - 1 - (1+\alpha)r'$.
$\varepsilon$ sufficiently small $\Rightarrow$ the system is almost linear $\Rightarrow$ No **Explosion**.

# More singular but with cut-off.

We may use cuted-off forces $S_m^\alpha$

$$|F(x)| \leq \frac{C}{(|x|+\varepsilon^m)^\alpha}, \qquad |\nabla F| \leq \frac{C}{(|x|+\varepsilon^m)^{\alpha+1}}$$

and get a similar result for $\alpha \geq 1$.

**Theorem (H., Jabin '11)**

*Assume that $F$ satisfies a $S_m^\alpha$-condition with*

$$m < \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha}\right)$$

*For each $N$ initial independant positions with $Z_i$ law $f^0$ (continuous and compact). Then propagation of chaos holds and precisely for $\gamma < 1$ (but close enough) there exists $\kappa$ (almost as before) and $\beta > 0$ (but small) s.t.*

$$\frac{1}{N^\beta} \ln \mathbb{P}\left(W_1(\mu_z^N(t), f(t)) \geq \frac{e^{\kappa t}}{N^{\frac{\gamma}{2d}}}\right) \leq -C < 0.$$

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# Perspectives : Towards more singular interaction.

**An interesting question :** Can we get some estimate on the second marginal $F_2^N$ of the N particles Law?

$\rightarrow$ A difficult question since everything is correlated.

**Near a gaussian equilibrium**, good stability properties can be shown even for singular forces ($1 < \alpha < 2$). Work with P.-E. Jabin and J. Barré, '10.

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# Stability of Vlasov Equilibrium.

Vlasov equation admits many equilibrium :

- Gravitationnal case : **spherical galaxies**.
  $\Rightarrow$ They are non-linearly stable (Méhats, Lemou, Raphael '10-11).
- Plasma in a periodic domain : **Stationary profiles** ($f(x, v) = g(v)$).
  $\Rightarrow$ If decreasing they are non-lineraly stable (Marchioro & Pulvirenti, Batt & Rein '93).
  $\Rightarrow$ Penrose criteria : some double-humped profile are non-linearly unstable (Guo-Strauss '95)

# Stability of N particles system around Vlasov equilibrium.

For the Hamiltonian Mean Field (HMF) model : $x \in \mathbb{R}/\pi\mathbb{Z}$ and $F = -\nabla V$ with

$$V(x) = \frac{1 - \cos x}{2}$$

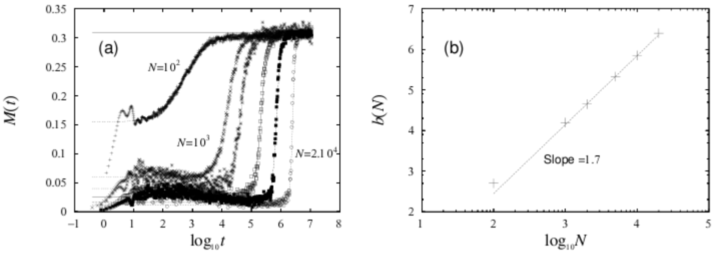


Fig. 11. Panel (a) presents the temporal evolution of the magnetization $M(t)$ for different particles numbers: $N = 10^2(10^3)$, $10^3(10^2)$, $2.10^3(8)$, $5.10^3(8)$, $10^4(8)$ and $2.10^4(4)$ from left to right, the number between brackets corresponding to the number of samples. The horizontal line represents the equilibrium value of $M$. Panel (b) shows the logarithmic timescale $b(N)$ as a function of $N$, whereas the dashed line represents the law $10^{b(N)} \sim N^{1.7}$.

Figure : The stability law for QSS (from Yamaguchi, Barré, Bouchet, Dauxois & Ruffo '03 )

## Stability of N particles system : rigourous results.

Going back to the convergence result in the regular interaction case, we get

$$W(\mu_N(t), g_{eq}) \leq e^{\|\nabla F\|_\infty t} W(\mu_N(t), g_{eq}) \approx \frac{e^{\|\nabla F\|_\infty t}}{N^{1/2d}}$$

The N system stay close to $f_{eq}$ at least till $T = \ln N$.
This has been improved

**Theorem (Caglioti & Rousset '07-08)**

*Assume that $g_{eq}(|v|)$ is a smooth decreasing equilibrium, and the force is repulsive ($\hat{V} \leq 0$). Then, in dimension N for almost all initial configuration, we have*

$$\|\mu_N(t) - g_{eq}\|_{LipN} \leq \frac{C}{\sqrt{N}}(1 + Mt)^2$$

*for all* $\mathbf{t} \leq \mathbf{CN^{1/8}}$.