Mean Field Limit and Propagation of chaos for particle systems quasi-neutral and gyro-kinetic limits for plasmas

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1. Limits of particle systems
   - Some definitions about particles systems
   - Stability analysis and quantitative estimates
   - Propagation of chaos via Functional Analysis

2. Quasi-neutral and gyrokinetic limits
   - Quasi-neutral limit alone
   - Merging quasi-neutral and gyro-kinetic limit

3. Decoherence via a toy model
   - Some pictures
What is a particle system?

- $N$ particles described by their position $X_i^N$ and velocity $V_i^N$, and possibly a parameter $a_i^N$,
- Satisfying Newton’s second law with an interaction force $F(X_i^N - X_j^N)$:

$$\frac{d}{dt} X_i^N(t) = V_i^N(t),$$

$$\frac{d}{dt} V_i^N(t) = \frac{1}{N} \sum_{j \neq i}^{N} a_j^N F(X_i^N(t) - X_j^N(t)) + \sigma \frac{d B_i(t)}{dt}$$

- The factor $\frac{1}{N}$ appears if time and position scale fit well.
- The $B_i$ are (independent) Brownian motions:
  - $\sigma = 0 \Rightarrow$ deterministic, $\sigma > 0 \Rightarrow$ stochastic.
- The $a_i^N$ are parameters (mass, charge, ...). Here for simplicity, all $a_i^N = 1$.

Examples

Stars in a galaxy, Galaxies in a cluster, ions or electrons in a plasma, insects in swarm,...
A complex particle system: Antennae Galaxies.

Credit: NASA, ESA, and the Hubble Heritage Team (STScI/AURA)-ESA/Hubble Collaboration – Under “Public domain” licence
Andromeda and Milky Way collision

Credit: NASA; ESA; Z. Levay and R. van der Marel, STScI; T. Hallas, and A. Mellinger – Under “Public domain” licence
Particle systems of order one

The previous system was second order (involving positions and velocities), but **first order models** also exist:

- $N$ “particles” with positions $X_i^N$ (and one parameter $a_i^N$)
- Satifying a system of ODE with interaction kernel $K$

$$
\frac{d}{dt} X_i^N(t) = \frac{1}{N} \sum_{j \neq i}^N a_j^N K(X_i^N(t) - X_j^N(t)) + \sigma \frac{dB_i(t)}{dt}
$$

**Examples**

Vortex in fluids, bacterias and chemotaxis, ions in a homogeneous plasma...

**Major issues**

In all the above mentioned examples, $N$ is too large to understand the behaviour of the system. Typically, $10^6 \leq N \leq 10^{23}$.

In almost all these example, the system can even not be numerically simulated.
Large scale structure in the universe

From the millenium run, done at the Max Planck Institute für Astrophysik.
Credit: Springel et al. (2005)
Limit for large $N$ leads to mean-field equations

Replace the $N$ particles by a distribution $f$ of particles:

A time $t$, in a small volume $dx \times dv$ around $(x, \nu)$, you will find roughly $f(t, x, \nu)dx dv$ particles.

Then a typical limit trajectory satisfies

$$\frac{d}{dt} X(t) = V(t), \quad \frac{d}{dt} V(t) = E[f](t, X(t)) + \sigma \frac{dB}{dt}$$

with $E[f] = \int F(x - y) f(t, y, w) dy dw$

In particular, $f$ satisfies the collisionless Boltzmann equation (a.k.a. Jeans-Vlasov)

$$\frac{df}{dt} + \nu \cdot \frac{df}{dx} + E[f] \frac{df}{dv} = \frac{\sigma^2}{2} \Delta_v f$$

Mean field equation (MFE)

It is called a mean-field equation, because the force field $E$ is a kind of average of the interaction force $F$, with the weight $f$. 
Mean-field equations for order one models

In order one systems, the limit trajectories are

\[
\frac{d}{dt} X(t) = E[f](t, X(t)) + \sigma \frac{dB}{dt}, \quad E[f] = \int F(x - y)f(t, y) \, dy
\]

or equivalently (with probabilistic notations)

\[
dX(t) = \mathbb{E}_Y [K(X(t) - Y(t))] \, dt + \sigma dB_t
\]

where \( Y(\cdot) \) is an indep. copy of \( X(\cdot) \).

The associated MFE is

\[
\frac{df}{dt} + E[f] \frac{df}{dx} = \frac{\sigma^2}{2} \Delta_x f
\]

Some examples.

In 2D, \( K(x) = \frac{x^\perp}{2\pi|x|^2} \) leads to **Navier-Stokes** (or Euler if \( \sigma = 0 \) equation).

In 2D, \( K(x) = -\frac{x}{2\pi|x|^2} \), leads to the **Keller-Segel** model for bacteria aggregation.
Proving rigorously the limit

Main question

If the $N$ particles are initially roughly distributed according to $f(0)$, does they are still roughly distributed according to $f(t)$ at time $t$.

Definition (Empirical measures)

For a given configuration $Z^N = (X^N_i, V^N_i)_{i \leq N}$ of particles, the associated empirical measure is

$$\mu^N_{Z} := \frac{1}{N} \sum_{i=1}^{N} \delta(X^N_i, V^N_i), \text{ where } \delta \text{ denotes the Dirac mass.}$$

New formulation: Is the following diagram commutative?

$$\mu^N_{Z}(0) \xrightarrow{\text{cvg}} f(0)$$

$$N\text{part} \downarrow \quad \text{Mean-field} \quad \downarrow$$

$$\mu^N_{Z}(t) \xrightarrow{\text{cvg ?}} f(t)$$
A good distance to quantify the limit

One interest of the empirical measures

Empirical measure $\mu^N_Z$ and distribution $f \, dx \, dv$ are both measures on the phase space. We can compare them.

Definition (Monge Kantorovicth-Wasserstein distance of order one.)

The MKW distance of order one $W_1$ between two probabilities $\mu$ and $\nu$ is defined by

$$W_1(\mu, \nu) := \inf_{\Pi} \int |x - y| \, \Pi(dx, dy)$$

where the infimum is taken on the probability $\Pi$ with first (resp. second) marginal $\mu$ (resp. $\nu$).

Or equivalently with probabilistic notations

$$W_1(\mu, \nu) := \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[|X - Y|].$$
Convergence for smooth interaction when $\sigma = 0$

A second interest of empirical measures

Empirical measures are “weak” solutions of the limit equation, when $\sigma = 0$ and $(F(0) = K(0) = 0)$.

Very useful when the force $F$ is Lipschitz and $\sigma = 0$ in view of

Theorem (Unconditionnal stability of mean-field equation)

If $F$ is Lipschitz (and any $\sigma \geq 0$), any measure solutions $\mu$ and $\nu$ of the MFE satisfies

$$W_1(\mu(t), \nu(t)) \leq e^{2\|\nabla F\|_\infty t} W_1(\mu^0, \nu^0).$$

Apply it to $f(t) \, dx \, dv$ and $\mu^N_Z$ and obtain:

Corollary (deterministic Mean-Field limit for smooth forces)

If $F$ is Lipschitz and $\sigma = 0$, the convergence of PS towards MFE holds and

$$W_1(f(t), \mu^N_Z(t)) \leq e^{2\|\nabla F\|_\infty t} W_1(f^0, \mu^N_Z, 0).$$

A result due to Braun & Hepp, Dobrushin, Neunzert & Wick.
Problem

In the stochastic case, the empirical measures are not solutions of the limit equation.

Solution: Couple solutions \((X_i^N, V_i^N)\) of the particle system to \(N\) copies of the limit equation with the same noises and same initial conditions:

\[
\begin{align*}
    dY_i^N(t) &= W_i^N(t) \, dt, \\
    dW_i^N(t) &= E[f](t, Y_i^N(t)) \, dt + \sigma \, dB_i
\end{align*}
\]

Then, the same estimate than previously (the noises disappear) leads to

Theorem (Propagation of chaos for smooth interaction by McKean)

When \(F\) is Lipschitz, for some constant \(C_t\)

\[
E \left[ W_1 \left( \mu_Z^N(t), \mu_Z^N(t) \right) \right] \leq \frac{C_t}{\sqrt{N}} e^{2 \| \nabla F \|_\infty t}.
\]

where \(Z^N = (Y_i^N, W_i^N)_{i \leq N}\).
A first definition of propagation of molecular chaos
Important if there is some randomness, because of noise or/and initial conditions.

Definition (Propagation of chaos)
It holds when
\[
\text{if } \mathbb{E}\left[ W_1(\mu_{\mathbb{Z}}^N, f^0) \right] \to 0, \text{ then } \mathbb{E}\left[ W_1(\mu_{\mathbb{Z}}^N(t), f(t)) \right] \to 0
\]

The previous theorem implies the propagation of chaos if the \((X_i^{N,0}, W_i^{N,0})\) are i.i.d. with law \(f^0\) because of

Theorem (Ajtai-Komlós-Tusnády '84, Fournier-Guillin '14)
In dimension \(d\) if \((\mathbb{Z}^N') = f^\otimes N\) (and technical assumptions), then
\[
\mathbb{E}\left[ W_1(\mu_{\mathbb{Z}'}^N, f) \right] \lesssim C N^{-\frac{1}{d}}
\]

In fact, all in all
\[
\mathbb{E}\left[ W_1(\mu_{\mathbb{Z}}^N(t), f(t)) \right] \leq \frac{C_t}{\sqrt{N}} e^{2\|\nabla F\|_\infty t} + C_t N^{-\frac{1}{d}}.
\]
Physical interactions are often singular

**Problem**

The previous results apply to smooth interaction: but in many physical situations, the interaction is singular

**Examples:**

- Gravitational or Coulombian force: \( \pm c \frac{x}{|x|^{d-1}} \) in dimension \( d \);
- The Navier-Stokes (or Euler) equation where \( K(x) = c \frac{x}{|x|^2} \) in 2D;
- In Chemotaxis where \( K(x) = -c \frac{x}{|x|^2} \) in 2D; ...

**How to handle singularities?**

**A general strategy to go further**

Study the stability of the limit MFE in MKW distance.
A first example: Vlasov-Poisson in 1D.

Here $d = 1$, the force $\mathbf{F}(x) = \text{sign } x$ (not Lipschitz) and $\sigma = 0$. The stability result is

**Theorem (Weak-strong stability, H. '13, Sem X)**

If $f_t$ solves VP1D with bounded density $\rho_t = \int f_t \, dv$. Then, any measure solution $\nu$ of VP1D satisfies for all $t \geq 0$

$$W_1(\nu_t, f_t) \leq e^{a(t)} W_1(\nu_0, f_0), \quad \text{with} \quad a(t) := \sqrt{2} t + 8 \int_0^t \|\rho_s\|_\infty \, ds.$$ 

In short, the stability holds provided that one solution is a strong one.

**Consequences**

- Apply it to $\nu_t = \mu_t^N$: $W_1(\mu_t^N, f_t) \leq e^{a(t)} W_1(\mu_0^N, f_0) \Rightarrow$ Mean-Field Limit
- Taking expectation: $\mathbb{E}[W_1(\mu_t^N, f_t)] \leq e^{a(t)} \mathbb{E}[W_1(\mu_0^N, f_0)] \Rightarrow$ Prop. of Chaos

Further: Also Propagation of entropic chaos, See [Hauray & Mischler, JFA '14].
Second example: Vlasov-Poisson-Fokker-Planck in 1D

Here \( d = 1 \), the force \( \mathbf{F}(x) = \text{sign } x \) (not Lipschitz) and \( \sigma > 0 \).

**First idea:**
Apply the same strategy than previously, but to the coupling with \( N \) i.i.d. of the limit McKean-Vlasov diffusion. It leads roughly to

\[
\mathbb{E}\left[ W_1(\mu^N_Z(t), \mu^N_Z(0)) \right] \leq \frac{C_t}{\sqrt{N}} \mathbb{E}\left[ e^{a(t)} \right], \quad a(t) := \sqrt{2} t + 8 \int_0^t \| \mu^N_Z(s) \|_{\infty} ds.
\]

Useless because \( \| \mu^N_Z(s) \|_{\infty} = +\infty \).

**Solution:**

- Introduce **discrete infinite norms** \( \| \mu \|_{\infty, \varepsilon} := \sup_{(x, \nu)} \frac{\mu(B_\varepsilon(x, \nu))}{\text{Vol}(B_\varepsilon(x, \nu))} \),
- use a **relaxed version** of the weak-strong stability estimate for VPFP1D

\[
\frac{d}{dt} W_1(\nu_t, \mu_t) \leq (\sqrt{2} + \| \nu_t \|_{\infty, \varepsilon}) (W_1(\nu_t, \mu_t) + \varepsilon).
\]
Second example: Vlasov-Poisson-Fokker-Planck in 1D

- Look at the literature on deviation upper bounds for \( \| \mu_N^Z(t) \|_{\infty, \epsilon} \) at fixed time,

- Deduce some deviation upper bounds on \( \int_0^t \| \mu_N^Z(s) \|_{\infty, \epsilon} \, ds \)

All in all, obtain a good deviation upper bound for the PS

**Theorem (H. & Salem, WiP)**

*If \( f \) is a strong solution of the VPFP1D, then for some \( C_t \),

\[
\mathbb{P} \left( \mathcal{W}_1(\mu_N^Z(t), \mu_N^Z(t)) \geq \epsilon \right) \leq C_t N^4 e^{-\frac{1}{2} N \epsilon^2}
\]

and for some \( c_t' \), \( C'_t \)

\[
\mathbb{P} \left( \mathcal{W}_1(\mu_N^Z(t), f(t)) \geq \epsilon \right) \leq C'_t N^4 e^{-c'_t N \epsilon^2}
\]

which implies propagation of chaos.
Homogeneous Landau equation in dimension 3

The HL equation for moderately soft potentials ($\gamma \in (-1, 0)$) reads

$$\frac{d}{dt} f(t, v) = \text{div} \left( \int a(v - v') \left[ f(t, v) \frac{df}{dv}(t, v') - f(t, v') \frac{df}{dv}(t, v) \right] dv' \right)$$

where $a(w) = |w|^{2+\gamma} \left( \text{Id} - \frac{w \otimes w}{|w|^2} \right)$

An associated particle system contains:

- A non Lipschitz interaction kernel $K = \text{div} b$,
- A diffusion with $\sigma$ (non Lipschitz) dependent of the $V_i^N$.

The strategy used for the VPFP1D also work

**Theorem (H. Fournier, WiP)**

For any $t \geq 0$, any strong solution $f$, there exists a constant $C_{t, \gamma}$

$$\mathbb{E}[W_2^2(\mu_t^N, f_t)] \leq C_{t, \gamma} N^{-\alpha}.$$  

for some $\alpha > 0$ depending explicitly on $\gamma$. 
Homogeneous Landau equation in dimension 3

Additional difficulties:

- Since $\sigma \neq \text{cst}$, a non trivial coupling is necessary to get a weak-strong stability estimate,
- For the same reason, it is more difficult to get deviation upper bound.
4th example: Jeans-Vlasov in 3D, weakly singular case

Here $\sigma = 0$, and $F(x) = \frac{c}{|x|^\alpha}$, $\alpha \in (0, 1)$.
Always deals with solution with compact support in $\nu$ [Pfaffelmoser '95].

Theorem (Strong-strong stability, reformulation of [Loeper '05])

For two strong solutions $f$ and $g$ of the Jeans-Vlasov equation

$$\frac{d}{dt} W_p(f_t, g_t) \leq C \max(\|f_t\|_\infty, \|g_t\|_\infty) W_p(f_t, g_t)$$

for any MKW distance of order $p \in [1, +\infty]$.

Same strategy: Relax the stability estimate with one discrete infinite norm. Tedious calculation leads to

$$\frac{d}{dt} W_p(f_t, \mu^N_t) \leq C \max(\|f_t\|_\infty, \|\mu^N_t\|_\infty, \varepsilon)(W_p(f_t, \mu^N_t) + R(\varepsilon))$$
4th example: Jeans-Vlasov in 3D, weakly singular case

A new improvement

Take $p = \infty$ and $\varepsilon = W_\infty(f_t, \mu^N_t) =: W(t)$, because $\|\mu^N_t\|_\infty, W(t) \leq 2^d \|f_t\|_\infty$.

End up with a relaxed weak-strong stability estimate

$$\frac{d}{dt} W_\infty(f_t, \mu^N_t) \leq 2^d C \|f_t\|_\infty \left( W_\infty(f_t, \mu^N_t) + R(\varepsilon) \right)$$

Implies Mean Field limit and Propagation of Chaos. [H. & Jabin '15]

Some comments:

- Many difficulties hidden in $R(\varepsilon)$, which depends on

$$\min_{i \neq j} (|X_i^N - X_j^N| + |V_j^N - V_i^N|),$$

- The argument fails for $p \neq \infty$,
- The Prop. of Chaos is also entropic.
Dissipation of entropy for order one models

The entropy $H$ and the Fisher information $I$ are defined for a R.V. $X$ or its law $f$ by

$$H(X) = H(f) = \int f \ln f \, dx \quad I(X) = I(f) = \int \left| \frac{df}{dx} \right|^2 f \, dx.$$  

When $\sigma > 0$ in order one models, entropy (or free energy) is dissipated:

For instance for the vortex systems (linked to NS2D)

$$\frac{d}{dt} X_i^N(t) = \frac{1}{N} \sum_{j \neq i}^N a_j^N \frac{(X_i^N(t) - X_j^N(t))}{|X_i^N(t) - X_j^N(t)|^2} + \sigma \frac{dB_i(t)}{dt}$$

we get

$$H(X_t^N) + \frac{\sigma^2}{2} \int_0^t I(X_s^N) \, ds = H(X_0^N).$$

But, how to use it?
Numerical applications.
A simulation by Chorin in the ’70.
Fisher information and Prop. of chaos

Properties of the Fisher information (same for entropy).

1. \( I \) is convex and lower semi-continuous,
2. \( I \) is super-additive: \( I(X_1, X_2) \leq I(X_1) + I(X_2) \) with equality only if \( X_1, X_2 \) are independent.
3. \( I \) controls some \( L^p \)-norms thanks to GNS inequalities (good in low dim.),
4. "\( I \) go through the limit": if the sequence of R.V. \( (\mu_N^X) \) goes in law towards a R.V. \( g \), then

\[
\mathbb{E}[I(g)] \leq \liminf_{N \to \infty} \frac{1}{N} I(X^N). \quad \text{[H. Mischler '14]}
\]

Then, apply Sznitman martingale method [St-Flour lecture notes '84]:
- Property 2 and 3 imply tightness and consistency,
- Property 4 implies the uniqueness of the limit.

Theorem (Osada '86; Fournier, H. & Mischler, '14)

The (entropic) Prop. of chaos holds for the stochastic vortex model, towards the NS2D equation, for any \( \sigma > 0 \).
Further applications of that qualitative method

Some comments:

- Here the results are qualitative (compactness technics), but allow to handle stronger singularities.
- In the previous part, the noise was a problem. Here it really helps.
- Work only for full noise, or almost full as in homogeneous Landau equation with $\gamma \in (-2, -1)$ [Fournier & H., WiP]
- Work also for relative entropy (or free energy) dissipation, as for sub-critical Keller-Segel model [Godinho & Quininao ’14]
An important scale in Plasma, the Debye length.

After a nondimensionalization, the Vlasov-Poisson equation on the 1D torus $\mathbb{T}$ reads:

$$\frac{df_\varepsilon}{dt} + v \frac{df_\varepsilon}{dx} - \frac{dV_\varepsilon}{dx}(t, x) \frac{df_\varepsilon}{dv} = 0,$$

with 

$$\varepsilon^2 \frac{d^2 V_\varepsilon}{dx^2} = \rho_\varepsilon - 1 = \int f_\varepsilon dv - 1$$

$\varepsilon$ is here the ratio between the Debye length and the typical length of the system.

Langmuir waves or plasma oscillations.

An interesting wave phenomena is observed in plasma. 

$J_\varepsilon = \int fv dv$ and $\varepsilon V_\varepsilon$ oscillate with frequency $\frac{1}{\varepsilon}$:

$$\partial_t [\varepsilon V_\varepsilon] = - \frac{J_\varepsilon}{\varepsilon}$$

$$\partial_t J_\varepsilon = \frac{\varepsilon V_\varepsilon}{\varepsilon} + \left( \frac{d}{dx} \right)^{-1} \left( \varepsilon \nabla_x V_\varepsilon \right)^2 - \int f_\varepsilon v^2 dv \right) + \frac{1}{2} \left( \varepsilon \frac{dV_\varepsilon}{dx} \right)^2$$
A very small scale in most situations.

Usually, the Debye length is much smaller than the typical scale of the system.

<table>
<thead>
<tr>
<th>Plasma</th>
<th>Density $n_e$ (m$^{-3}$)</th>
<th>Electron temperature $T$ (K)</th>
<th>Magnetic field $B$ (T)</th>
<th>Debye length $\lambda_D$ (m)</th>
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</thead>
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<tr>
<td>Solar core</td>
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<td>$10^7$</td>
<td>--</td>
<td>$10^{-11}$</td>
</tr>
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<td>Tokamak</td>
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<td>$10^8$</td>
<td>10</td>
<td>$10^{-4}$</td>
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<td>Gas discharge</td>
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<td>$10^4$</td>
<td>--</td>
<td>$10^{-4}$</td>
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<td>$10^{-5}$</td>
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<td>Intergalactic medium</td>
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<td>$10^6$</td>
<td>--</td>
<td>$10^5$</td>
</tr>
</tbody>
</table>

From a course by Kip Thorne at Caltech.

**Problem:**

Then the Langmuir waves are very fast.
An experimental observation of Langmuir waves in ionosphere


**Figure 2.** An example of narrow band Langmuir waves. The upper panel contains the waveform and the lower panel contains the power spectrum.
Another experimental observation of Langmuir waves

From Matlis, Downer & all, University of Texas and Michigan, Nature Phys 2006.

Figure 3 Strongly driven wake with curved wavefronts. 

(a) Probe phase profile $\Delta \phi_{\text{p}}(r, \zeta)$ for an $\sim$30 TW pump, $n_{\text{e}}^{\text{max}} = 2.2 \times 10^{18}$ cm$^{-3}$ in the He$^{3+}$ region. 

(b) Simulated density profile $n_{\text{e}}(r, \zeta)$ near the jet centre. 

(c) Same data as in (a), with the background $n_{\text{e}}$ subtracted to highlight the wake. 

(d) Evolution of the reciprocal radius of wavefront curvature behind the pump (data points), compared with calculated evolution (dashed lines) for indicated wake potential amplitudes. Each data point (except at $\zeta = 0$) averages over three adjacent periods. The horizontal error bars extend over the three periods averaged, and the vertical error bars extend over the range of fitted curvature values averaged.
Quasi-neutrality: the formal limit when $\varepsilon = 0$.

$$\frac{df}{dt} + \nu \frac{df}{dx} - \frac{dV}{dx}(t, x) \frac{df}{d\nu} = 0,$$
with $\rho = 1$

**Problems:**

- That model is probably ill-posed (mathematically). Very few results about it:
  - existence of solutions for short time and analytical initial data [Grenier '96, Jabin, Tallay & all '13, Jabin & Nouri '13],
- The equation on $V$ is implicit,
- What about the Langmuir waves?

However, some rigorous result of convergence when $\varepsilon \to 0$, in the zero temperature limit: when $f_\varepsilon^0 \to \rho^0 \delta_{v^0}$

- Use well-prepared initial data [Bernier '00],
- Filtrate the plasma oscillation [Grenier '96, Masmoudi '01].

Then $f_\varepsilon(t)$ converges to $\delta_{v(t)}$, where $v$ is a solution of the incompressible Euler equation.
The case of general initial data

A note by Grenier [JEDP '96] explains two phenomena:

- Possible instabilities are instantaneous in the quasi-neutral limit,
- Around solutions of VQN with always “one bump in $v$”, the limit should hold.

In a joint work [H. Han-Kwan '15], we try to rigorously prove the above statements.

**Theorem (Instantaneous instability)**

Let $\mu(v)$ be a smooth profile satisfying the Penrose instability criterion. For any $N > 0$ and $s > 0$, there exists $f_\varepsilon^0$ such that

$$\|f_\varepsilon^0 - \mu\|_{W^{s,1}_{x,v}} \leq \varepsilon^N,$$

and for any $r \in \mathbb{Z}$, we have

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon]} \|f_\varepsilon(t) - \mu\|_{W^{r,1}_{x,v}} > 0.$$
The case of general initial data

Theorem (Stability after filtration of oscillations)

Let $\mu$ be a stable stationary profile. For any smooth potential $V_0$, we define an associated “modulated free energy”

$$\mathcal{L}_\varepsilon^O(t) := H_Q \left[ f_\varepsilon \left( t, x, v - \partial_x V_0(x - \bar{v} t) \sin \frac{t}{\varepsilon} \right) \right]$$

$$+ \frac{1}{2} \int \left[ \varepsilon \partial_x V_\varepsilon - \partial_x V_0(x - \bar{v} t) \cos \frac{t}{\varepsilon} \right]^2 dx.$$

Then,

$$\mathcal{L}_\varepsilon^O(t) \leq e^{2\|V_0''\|_{L^\infty} t} \left[ \mathcal{L}_\varepsilon^O(0) + K\varepsilon \right],$$

where $H_Q$ is a functional generalizing the entropy. It controls in most case somes $L^p$ norm ($p = 1, 2, \ldots$):

$$\|f - f^0\|_p \leq C H_Q(f).$$
Merging quasi-neutral and gyro-kinetic limit

Seems more complicated but in fact may help.

For instance, the **quasi-neutral gyro-system**

\[
\frac{df}{dt} + (J_0^u \nabla_x V) \cdot \nabla_x f = \beta u \partial_u f + 2\beta f + \nu \left( \Delta_x f + \frac{1}{u} \partial_u (u \partial_u f) \right),
\]

\[\Phi = \rho,\]

\[\rho(t, x) = \int (J_0^u f(t, x, u) 2\pi u du),\]

where \(J^0\) is the gyro-average operator, is well posed [H. & Nouri ’11].

In fact, as far as regularity is concerned it is very similar to the NS2D equation.
Decoherence: a basic model and its simulations

No time for much: only pictures from [Adami, H., Negulescu 2017].
Decoherence: a basic model and its simulation

Two other pictures.

The density $\rho^M(T^*, X, X)$ for different values of $\alpha$, and $p$.

To do

A simple model for one collision. Get the limit master equation in a many interactions regime [Gomez & H. WiP]
Thanks!

Thanks for your attention!
Merci pour votre attention!
¡ Gracias por su atención!